Vedic Mathematics

Lecture Notes – 3

Equation

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VEDIC MATHEMATICS
OR SIXTEEN SIMPLE MATHEMATICAL FORMULAE

Sixteen Sutras and Their Corollaries

Sūtras

1. एकाधिकेन पूर्वेण
Ekādhikena Pūrveṇa (also a corollary)
2. निखिलं नवतश्चरमं दशतः
Nikhilaṃ Navataścaramaṃ Daśataḥ
3. ऊर्ध्वतिर्यग्भ्याम्
Ūrdhva-tiryagbhyām
4. परावर्त्य योजयेत्
Parāvartya Yojayet
5. शून्यं साम्यसमुच्चये
Śūnyaṃ Sāmyasamuccaye
6. (आनुरूप्ये) शून्यमन्यत्
(Ānurūpye) Śūnyamanyat
7. संकलनव्यवकलनाभ्याम्
Saṅkalana-vyavakalanābhyam (also a corollary)
8. पूरणापूरणाभ्याम्
Pūraṇāpūraṇābhyām
9. चलनकलनाभ्याम्
Calana-Kalanābhyām
10. यावदूनम्
Yāvadūnam
11. व्यष्टिसमष्टिः
Vyaṣṭisamaṣṭiḥ
12. शेषाण्यङ्केन चरमेण
Śeṣāṇyaṅkena Carameṇa
13. सोपान्त्यद्वयमन्त्यम्
Sopāntyadvayamantyam
14. एकन्यूनेन पूर्वेण
Ekanyūnena Pūrveṇa
15. गुणितसमुच्चयः
Guṇitasamuccayaḥ

Sub-Sūtras or Corollaries

1. आनुरूप्येण
Ānurūpyeṇa
2. शिष्यते शेषसंज्ञः
Śiṣyate Śeṣasaṃjñaḥ
3. आद्यमाद्येनान्त्यमन्त्येन
Ādyamādyenāntya-mantyena
4. केवलैः सप्तकं गुण्यात्
Kevalaiḥ Saptakaṃ Guṇyāt
5. वेष्टनम्
Veṣṭanam
6. यावदूनं तावदूनम्
Yāvadūnaṃ Tāvadūnam
7. यावदूनं तावदूनीकृत्य वर्गं च योजयेत्
Yāvadūnaṃ Tāvadūnīkṛtya Vargañca Yojayet
8. अन्त्ययोर्दशकेऽपि
Antyayordaśake'pi
9. अन्त्ययोरेव
Antyayoreva
10. समुच्चयगुणितः
Samuccayaguṇitaḥ
11. लोपनस्थापनाभ्याम्
Lopanasthāpanābhyām
12. विलोकनम्
Vilokanam
13. गुणितसमुच्चयः समुच्चयगुणितः
Guṇitasamuccayaḥ
Samuccayaguṇitaḥ

(Editor of the original book on Vedic Mathematics)

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Eg.(iii) Solve (x + 2)(x – 12) = (x – 3)(x +8)

Current Method

$(x+2)(x-12) = (x-3)(x+8)$
$x^2 - 10x - 24 = x^2 + 5x - 24$
$-10x - 5x = -24 + 24$
$-15x = 0$
$x = 0$

Vedic Method

$(x+2)(x-12) = (x-3)(x+8)$
$cd - ab = -24 + 24 = 0$
Therefore, $x = 0$.

3. If the equation is in the form of $\frac{ax+b}{cx+d} = \frac{p}{q}$ **(standard form 3)**

Proof:

$\frac{ax+b}{cx+d} = \frac{p}{q}$ (On Cross-multiplication)
$q(ax+b) = cpx + pd$
$aqx - cpx = pd - bq$
$(aq - cp)x = pd - bq$

$x = \frac{pd - bq}{aq - cp}$

Vedic Method

By Paravartya q will result in multiplication. Similarly (cx + d) result in multiplication.

Paravartya results in (cross multiplication)
$(ax + b)q = (cx + d)p$
$aqx + bq = cpx + pd$

Again by Paravartya cpx becomes –cpx and bq becomes –bq
$aqx - cpx = pd - bq$
$x(aq - cp) = pd - bq$

Again by Paravartya aq – cp results in division. Hence, $x = \frac{pd - bq}{aq - cp}$

Eg.(i) Solve $\frac{2x+3}{6x+5} = \frac{3}{2}$

Current Method

$\frac{2x+3}{6x+5} = \frac{3}{2}$

$2(2x+3) = 3(6x+5)$
$4x + 6 = 18x + 15$
$4x - 18x = 15 - 6$
$-14x = 9$
$x = -9/14$

Vedic Method

$\frac{ax+b}{cx+d} = \frac{p}{q}$ standard form 3

$x = \frac{pd - bq}{aq - cp}$

Formula is $x = \frac{15-6}{4-18} = \frac{9}{-14} = \frac{-9}{14}$

Eg.(ii) Solve $\frac{8x+3}{4x+7}=\frac{9}{5}$

Current Method

$\frac{8x+3}{4x+7}=\frac{9}{5}$

$5(8x + 3) = 9(4x + 7)$
$40x + 15 = 36x + 63$
$40x - 36x = 63 - 15$
$4x = 48$
$x = 48 / 4 = 12$

Vedic Method

$\frac{8x+3}{4x+7}=\frac{9}{5}$

By using the standard Formula:

$x=\frac{63-15}{40-36}=\frac{48}{4}=12$

4. If the equation is in the form of $\frac{m}{x+a}+\frac{n}{x+b}=0$ <u>standard form 4</u>

<u>Proof:</u>

$\frac{m}{x+a}+\frac{n}{x+b}=0$

$m(x + b) + n(x + a) = 0$
$mx + mb + nx + na = 0$
$(m + n)x + mb + na = 0$
$(m + n)x = - mb - na$

$$x=\frac{-mb-na}{m+n}$$

Vedic Method

1. By Paravartya of terms the equation becomes $\frac{m}{x+a}=\frac{-n}{x+b}$

Again by Paravartya (x + a) and (x + b) result in multiplication.
(Cross multiplication)
As $m(x + b) = -n(x + a)$
$mx + mb = -nx - na$

Again by Paravartya mb becomes –mb and –nx becomes nx.
$mx + nx = -na - mb$
$(m + n)x = - na - mb$

Again by Paravartya (m + n) results in division. Hence, $x=\frac{-mb-na}{m+n}$

2. We can equate the numerators and apply sunyam Samyam method II

$\frac{m}{x+a}+\frac{n}{x+b}=0$

$\frac{mn}{n(x+a)}+\frac{mn}{m(x+b)}=0$

$\Rightarrow nx + na + mx + mb = 0$ (sunyam Samya Samuccaye)

$(m + n)x = - mb - na$

$$x=\frac{-mb-na}{m+n}$$

Eg.(i) Solve $\frac{2}{x+1}+\frac{5}{x+3}=0$

Current Method

$\frac{2}{x+1}+\frac{5}{x+3}=0$

$\frac{2(x+3)+5(x+1)}{(x+1)(x+3)}=0$

$2x+6+5x+5=0$
$7x+11=0$
$x=-11/7$

Vedic Method

$\frac{m}{x+a} \quad \frac{\ }{x+b}=0$

$\frac{2}{x+1}+\frac{5}{x+3}=0$

Formula is $x=\frac{-mb-na}{m+n}$

$x=(-6-5)/7=-11/7$

We can also do the above problem by using Sunyam Samya Samuccaye Sutram (II) by equating the numerators.

Eg.(ii) Solve $\frac{1}{x+4}+\frac{4}{x+5}=0$

Current Method

$\frac{\ }{x+4} \quad \frac{\ }{x+5}=0$

$\frac{x+5+4(x+4)}{(x+4)(x+5)}=0$

$x+5+4x+16=0$
$5x+21=0$
$x=-21/5$

Vedic Method

$\frac{1}{x+4}+\frac{4}{x+5}=0$

By using the Vedic Formula:

$x=\ -5-16 \quad -21$

<u>An extension to three factors</u>

$\frac{m}{x+a} \quad \frac{\ }{x+b} \quad \frac{p}{x+c}=0$ and if m + n + p = 0 (x^2 coefficient), as otherwise it becomes quadratic. (See the proof). By Paravartya method, it can easily be shown that

$x=\frac{-mbc-nca-pab}{m(b+c)+n(a+c)+p(a+b)}$ as follows.

Proof:

1. $\frac{m}{x+a}+\frac{n}{x+b}+\frac{p}{x+c}=0, m+n+p=0$

where m + n+ p is the coefficient of x^2 term

By LCM of three factors,

$$\frac{m(x+b)(x+c)+n(x+a)(x+c)+p(x+a)(x+b)}{(x+a)(x+b)(x+c)}=0$$

$m(x + b)(x + c) + n(x + a)(x + c) + p(x + a)(x + b) = 0$

$m(x^2 + bx + cx + bc) + n(x^2 + ax + cx + ac) + p(x^2 + ax + bx + ab) = 0$

$\therefore mbx + mcx + mbc + nax + ncx + nac + pax + pbx + pab = 0$

$x(mb + mc + na + nc + pa + pb) = -pab - mbc - nac$

$\therefore x[m(b + c) + n(a + c) + p(a + b)] = -pab - mbc - nac$

$$\therefore x = \frac{-pab-mbc-nac}{m(b+c)+n(a+c)+p(a+b)}$$

2. $\frac{m}{x+a}+\frac{n}{x+b}+\frac{p}{x+c}=0, m+n+p=0$

$$\frac{m}{x+a}+\frac{n}{x+b}=\frac{-p}{x+c}$$

$$\frac{m(x+b)+n(x+a)}{(x+a)(x+b)}=\frac{-p}{x+c}$$

$[m(x + b) + n(x + a)](x + c) = -p(x + a)(x + b)$

$(mx + mb + nx + na)(x + c) = -p(x + a)(x + b)$

$x(mb + na + mc + nc + pb + pa) = -nac - mbc - pab$

$\therefore x[m(b+c) + n(a+c) + p(a+b)] = -nac - mbc - pab$

$$\therefore x = \frac{-mbc-nca-pab}{m(b+c)+n(a+c)+p(a+b)}$$

Vedic Method

$\frac{m}{x+a}\ \frac{n}{x+b}+\frac{p}{x+c}=0$,(general form)

$m + n + p = 0$ (Condition for simple equation)

On Paravartya

$$\frac{m}{x+a}\ \frac{n}{x+b}=\frac{-p}{x+c}$$

By taking LCM of two factors on left hand side,

$$\frac{m(x+b)+n(x+a)}{(x+a)(x+b)}=\frac{-p}{x+c}$$

Again by Paravartya, or cross multiplication

$[m(x + b) + n(x + a)](x + c) = -p(x + a)(x + b)$

Again by Paravartya,

$m(x + b)(x + c) + n(x + a)(x + c) + p(x + a)(x + b) = 0$

$x^2 (m + n+ p) + x [(m (b + c) + n (c + a) + p(a + b)] + (mbc + nca + pab) = 0$

On simplification:

$mbc + nca + pab + x[m(b + c) + n(a + c) + p(a + b)] = 0$ when $m + n + p = 0$

At this stage, solution for x can be easily written down as $\frac{N}{D}$ where N is a specific combination of all independent terms under Paravartya and D is sum of coefficients of x under Paravartya. $\therefore$One can write down the answer.

$$x = \frac{-mbc-nca-pab}{m(b+c)+n(c+a)+p(a+b)}$$

It can also be clearly noticed that there is a cyclic symmetry order in the final answer.

This is noticed to be extremely elegant.

This is extendable to any number of such fractions. Provided it satisfies the relations that all higher order terms of x vanish thus yielding to a simple equation

Eg.(iii) Solve $\frac{5}{x}+\frac{3}{x+1}-\frac{8}{x-4}=0$

Current Method

$$\frac{5}{x}+\frac{3}{x+1}-\frac{8}{x-4}=0$$

$$\frac{5(x+1)(x-4)+3x(x-4)-8(x+1)x}{x(x+1)(x-4)}=0$$

$5(x^2-3x-4)+3x^2-12x-8x^2-8x=0$
$5x^2-15x-20+3x^2-12x-8x^2-8x=0$
$-35x-20=0$
$x=-20/35=-4/7$

Vedic Method

$$\frac{m}{x+a}+\frac{n}{x+b}+\frac{p}{x+c}=0$$ standard form 5

$$\frac{5}{x}+\frac{3}{x+1}\ \frac{8}{x-4}=0$$

$m+n+p=5+3-8=0$

Therefore, by using the Vedic Formula:

$$x=\frac{-mbc-nca-pab}{m(b+c)+n(c+a)+p(a+b)}$$

$$x=\frac{20}{-15-12-8}\ \frac{-20}{35}=\frac{-4}{7}$$

Vedic method is definitely simpler than the current method as is very clear from the operations shown in each case.

In all the above problems one has to identify the form and conditions and then simply put the answer.

(II) Application of Sunyam Samya Samuccaye Sutra:

The sutram is applied to solve the equations. This means that if, in a total expression there exists some similarity (common), then the problem can be solved by equating that similarity to zero. The similarity is described in a number of disguised forms, which are explained below:

Different Forms of Samyam:

1. If there is a common in a totality, then that common is Samyam and is zero.

Eg.(i) Solve 8x + 7x = 2x + 3x

Current Method

$8x + 7x = 2x + 3x$
$8x+7x-2x-3x = 0$
$15x-5x = 0$
$5x = 0,\ x = 0$

Vedic Method

$8x + 7x = 2x + 3x$
x is Samyam (common) and hence is zero.
$x = 0$
$\therefore x = 0$ In case of identity for example $6x + 3x = 4x + 5x$, $x = 0$ is one solution but x can take any value as an exception.

Eg.(ii) Solve 2(x – 3) = - 9(3 – x)

Current Method

$2(x-3) = 9(3-x)$
$2x - 6 = 27 - 9x$
$2x + 9x = 27 + 6$
$11x = 33$
$x = 3$
$x^2 + 12x + 32 = x^2 + 18x + 32$
$x = 0$

Vedic Method

$2(x-3) = 9(3-x) = -9(x-3)$
x – 3 is Samyam (common) and hence is zero.
$\therefore x - 3 = 0$
$x = 3$

2. The word 'Samuccaya' has, as its second meaning the product of the independent terms.
Here 8 x 4 = 16 x 2
Therefore x = 0

3. If the equation is in the form of $\frac{N_1}{D_1} + \frac{N_2}{D_2} = 0$ where N_1, N_2 and D_1, D_2 are expressions and if the numerators are equal then $D_1 + D_2$ is 'Samuccaaya' and is zero – (See Eg-i)
If N_1, N_2 are also functions, then it may lead to higher orders depending on the degree of the fractions. (See Eg-ii) (Sum of the denominators)

Eg.(i) Solve $\frac{1}{x+2} + \frac{1}{x+3} = 0$

Current Method

$$\frac{1}{x+2} + \frac{1}{x+3} = 0$$

$$\frac{x+3+x+2}{(x+2)(x+3)} = 0$$

Vedic Method

$$\frac{1}{x+2} + \frac{1}{x+3} = 0$$

Numerators are same.

$2x + 5 = 0$
$x = -5/2$

When Numerators are same, Samyam is identified as the sum of the denominators and hence sum is zero.
Therefore
$2x+5 = 0$
$x = -5/2$

Eg.(ii) Solve $\frac{x+1}{x+2} + \frac{x+1}{x+5} = 0$

Current Method

$$\frac{x+1}{x+2} + \frac{x+1}{x+5} = 0$$

$$(x+1)[\frac{1}{(x+2)} + \frac{1}{(x+5)}] = 0$$

$$(x+1)[\frac{x+5+x+2}{(x+2)(x+5)}] = 0$$

$(x+1)(x+5+x+2) = 0$
$x+1 = 0$ or $2x+7 = 0$
$x = -1$ or $x = -7/2$

Vedic Method

$$\frac{x+1}{x+2} + \frac{x+1}{x+5} = 0$$

x + 1 is Samyam (common) and hence by (one disguise) Sunyam Samya Samuccaye

$x + 1 = 0; \quad x = -1$

Since numerators are same, Samyam is identified as the sum of the denominators (another disguise) and hence is zero.

$\therefore x + 2 + x + 5 = 0$
$2x + 7 = 0$ or $x = -7/2$

(This is a Quadratic Equation). In this problem we used two different disguises of Samyam.

If numerators are different

$$\frac{m}{ax+b} + \frac{n}{cx+d} = 0$$

$mcx + md + nax + nb = 0$
$x(mc + na) = -nb - md$

$$x = \frac{-nb-md}{mc+na} \text{ (if } mc + na \neq 0)$$

If Numerators are different

i.e., $\frac{m}{ax+b} + \frac{n}{cx+d} = 0$ <u>standard form 6</u>

Then the numerators can be equalised, to identify with the form 2

$$\frac{mn}{nax+nb} + \frac{mn}{mcx+md} = 0$$

Then by Sunyam Samya Samuccaye, sum of denominators is Samuccaya and hence is zero.

$D_1 + D_2 = 0$
$nax + nb + mcx + md = 0$

$$x = \frac{-(md+nb)}{na+mc}$$ (If na + mc $\neq$ 0) for a finite value

Eg.(iii) Solve $\frac{2}{3x+1} + \frac{3}{x+5} = 0$

Current Method

$$\frac{2}{3x+1} + \frac{3}{x+5} = 0$$

$$\frac{2(x+5)+3(3x+1)}{(3x+1)(x+5)} = 0$$

$2x + 10 + 9x + 3 = 0$
$11x + 13 = 0$
$x = -13/11$

Vedic Method

$$\frac{2}{3x+1} + \frac{3}{x+5} = 0$$

Numerators are different, so we make numerators equal to apply Sunyam Samya Samuccaye Sutram.

$$9x+3 \quad 2x+10 = 0$$

Then Samuccaya is sum of the denominators and hence is zero.
$D_1 + D_2 = 0$; $11x + 13 = 0$
$x = -13/11$
This can be solved also using standard form 3

Eg (iv) $\frac{x}{3x-8} + \frac{2x}{7x+10} = 0$

Current Method

$$\frac{x}{3x-8} \cdot \frac{2x}{7x+10} = 0$$
$$\frac{x(7x+10)+2x(3x-8)}{21x^2+30x-56x-80} = 0$$
$$\frac{7x^2+10x+6x^2-16x}{21x^2+30x-56x-80} = 0$$

$13x^2 - 6x = 0$
$x(13x - 6) = 0$
$13x = 6$
$x = 6/13$
$x = 0 \quad x = 6/13$

Vedic Method

$$\frac{x}{3x-8} + \frac{2x}{7x+10} = 0$$

Numerators are different, so we make numerators equal to apply sunyam Samya Samuccaya sutram

$$\frac{2x}{6x-16} + \frac{2x}{7x-10} = 0$$

x is common and hence

The sum of the denominators is zero.

$D_1 + D_2 = 0$
$13x - 6 = 0$
$13x = 6$
$x = 6/13$
As x is common, x = 0 is one solutions

Eg (v) $\frac{x+3}{8x+9}+\frac{2x+6}{7x+12}=0$

Current Method

$$\frac{x+3}{8x+9}+\frac{2x+6}{7x+12}=0$$

$$\frac{(x+3)(7x+12)+(2x+6)(8x+9)}{56^2+96x+63x+108}$$

$(7x^2 + 12x + 21x + 36) + (16x^2 + 18x + 48x + 54) = 0$

$7x^2 + 33x + 36 + 16x^2 + 66x + 54 = 0$

$23x^2 + 99x + 90 = 0$

$$x = \frac{-99 \pm \sqrt{9801-8280}}{46}$$

$$x = \frac{-99 \pm \sqrt{1521}}{46} \quad \frac{-99 \pm 39}{46}$$

$$= -3, \frac{-30}{23}$$

Vedic Method

$$\frac{x+3}{8x+9}+\frac{2x+6}{7x+12}=0$$

$$\frac{x+3}{8x+9}+\frac{2(x+3)}{7x+12}=0$$

From numerators $N_1 = 2N_2$

$x + 3 = 0 \; x = -3$

$$\left(\frac{1}{8x+9}+\frac{2}{7x+12}\right)=0$$

Equating the numerators

$$\frac{2}{16x+18}+\frac{2}{7x+12}=0$$

Then Samuccaya is sum of the denominators and hence is zero.

$D_1 + D_2 = 0$

$23x + 30 = 0$

$23x = -30$

$$x = \frac{-30}{23}$$

Consider $\frac{N_1}{D_1}+\frac{N_2}{D_2}=0$

If $N_1 = N_2$ then (numbers)

$D_1 + D_2 = 0$ By samyam

But if $N_1 \neq N_2$ they can be made equal by suitable operation. In such a case $D'_1 + D'_2 = 0$ when the equality is achieved by numbers.

If $N_1 = N_2$ as linear function of x then $(D_1 + D_2)$ may not be zero one should confirm the non–existence of higher order terms to see that it comes under the above rule.

For example $\frac{5x-38}{x^2-17x+52}+\frac{5x-38}{x^2-17x+70}=0$

$5x - 38$ is common $\quad 5x - 38 = 0$

$$x = \frac{38}{5}$$

But simplification shows a cubic nature hence the other two solution have to be obtained solving the cubic equation.

4. If the equation is in the form of $\frac{N_1}{D_1} = \frac{N}{D_2}$

where N_1, N_2, D_1, D_2 are expressions.

Combinations or total

Vedic Method

The following relations define Samyam

a) $N_1 + N_2 = D_1 + D_2$
b) $N_1 \sim N_2 = D_1 \sim D_2$

} In all these cases a multiple is also valid in equality.

$\frac{x+8}{2x+6} \quad \frac{x+9}{2x+5}$

Cross Multiplication

$\cancel{2x^2} + 11x + 40 = \cancel{2x^2} + 24x + 54$

$-3x = 14$

$x = -\frac{14}{3}$

Then that equality is Samuccaya and is zero.

$N_1 + D_1 = N_2 + D_2$

$\frac{x+8}{2x+6} = \frac{x+9}{2x+5}$

$3x + 14 = 3x + 14 = 0$

$x = -\frac{14}{3}$

Eg.(i) Solve $\frac{2x+5}{2x+6} = \frac{3x+7}{3x+6}$

Current Method

$\frac{2x+5}{2x+6} = \frac{3x+7}{3x+6}$

$(2x + 5)(3x + 6) = (3x + 7)(2x + 6)$

$6x^2 + 27x + 30 = 6x^2 + 32x + 42$

$27x - 32x = 42 - 30$

$-5x = 12$ or $x = -12 / 5$

Vedic Method

$\frac{2x+5}{2x+6} = \frac{3x+7}{3x+6}$

$N_1 + N_2 = 5x + 12$

$D_1 + D_2 = 5x + 12$

$N_1 + N_2 = D_1 + D_2$.

$\therefore$ By Sunyam Samya Samuccaye Sutram, this equality is Samyam and is zero.

Therefore, $5x + 12 = 0$ or $x = -12 / 5$

Eg. (ii) Solve $\frac{2x+3}{8x+5} = \frac{x+1}{4x+11}$

Current Method

$\frac{2x+3}{8x+5} = \frac{x+1}{4x+11}$

$(2x + 3)(4x + 11) = (x + 1)(8x + 5)$

$8x^2 + 34x + 33 = 8x^2 + 13x + 5$

$34x - 13x = 5 - 33$

$21x = -28$

$x = -28 / 21 = -4 / 3$

Vedic Method

$\frac{2x+3}{8x+5} = \frac{x+1}{4x+11}$

$N_1 + N_2 = 3x + 4$

$D_1 + D_2 = 12x + 16 = 4(3x + 4)$

$D_1 + D_2$ is multiple of $N_1 + N_2$.

$\therefore$ By Sunyam Samya Samuccaye, $N_1 + N_2$ is the Samyam and hence is zero.

Therefore, $3x + 4 = 0$

$x = -4 / 3$

Eg.(iii) Solve $\frac{10x-7}{5x-3} = \frac{4x+5}{2x+3}$

Current Method

$$\frac{10x-7}{5x-3} = \frac{4x+5}{2x+3}$$

$(10x - 7)(2x + 3) = (4x + 5)(5x - 3)$
$20x^2 + 16x - 21 = 20x^2 + 13x - 15$
$16x - 13x = -15 + 21$
$3x = 6$
$x = 2$

Vedic Method

$$\frac{10x-7}{5x-3} \quad \frac{4x+5}{2x+3}$$

$N_1 \sim N_2 = 6x - 12 = 6(x - 2)$
$D_1 \sim D_2 = 3x - 6 = 3(x - 2)$
$N_1 \sim N_2 = D_1 \sim D_2$
By Sunyam Samya Samuccaye, this relation is Samyam and hence it is zero.
$\therefore x - 2 = 0$ or $x = 2$

5th type where the quadratic term is non vanishing

Eg.(i) Solve $\frac{3x+7}{9x+10} = \frac{7x+9}{x+6}$ (Quadratic)

Current Method

$$\frac{3x+7}{9x+10} \quad \frac{7x+9}{x+6}$$

$(3x + 7)(x + 6) = (7x + 9)(9x + 10)$
$3x^2 + 25x + 42 = 63x^2 + 151x + 90$
$60x^2 + 126x + 48 = 0$
$30x^2 + 63x + 24 = 0$
$10x^2 + 21x + 8 = 0$
$10x^2 + 5x + 16x + 8 = 0$

$5x(2x + 1) + 8(2x + 1) = 0$
$(2x + 1)(5x + 8) = 0$

$2x + 1 = 0$; $5x + 8 = 0$
$x = -1/2$; $x = -8/5$

Vedic Method

$$\frac{3x+7}{9x+10} \quad \frac{7x+9}{x+6}$$

x^2 coefficient on both sides is different. Therefore, it is a quadratic Equation.

$N_1 + N_2 = 10x + 16$
$D_1 + D_2 = 10x + 16$
$N_1 + N_2 = D_1 + D_2$
$\therefore$ By Sunyam Samya Samuccaye, this relation is Samyam and hence is zero, giving one solution.

$\therefore 10x + 16 = 0$
$x = -8/5$
$N_1 \sim N_2 = 4x + 2 = 2(2x + 1)$
$D_1 \sim D_2 = 8x + 4 = 4(2x + 1)$
$N_1 \sim N_2 = D_1 \sim D_2$
$\therefore$ By Sunyam Samya Samuccaye, this relation is Samyam and hence is zero, giving another solution. Therefore, $2x + 1 = 0$
$x = -1/2$

Eg (ii) Solve $\frac{4x-8}{9x+7} \quad \frac{3x+12}{5x+1}$

It is notice that $N_1 + N_2 = 7x + 4$ and
$D_1 + D_2$ = two times $N_1 + N_2$
$D_1 + D_2 = 14x + 8 \longrightarrow 2(N_1 + N_2)$

Hence using Samya Samuccaya sutram $7x + 4 = 0$

$$x = -\frac{4}{7}$$

But the square component i.e. x^2 is not canceled, hence it has one more root as it is a quadratic equation one can try combinations of numerators and denominators to get the second solution. A trial of combinations N_1, N_2, D_1 & D_2 shows the $2N_1 - D_1 = 2N_2 - D_2$

$2N_1 \sim D_1 = x + 23$

$2N_2 \sim D_2 = x + 23$

$x + 23 = 0$

$x = -23$

The two solutions of the above problem are $-\frac{4}{7}, -23$ is to be noted that one can try the combinations of numerators and denominators with multiples of numerators and denominators as well, so that the given equation is not disturbed.

4(a) If the equation is in the form of $\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{N_3}{D_3} + \frac{N_4}{D_4}$

where N_1, N_2, N_3, N_4 are numbers and D_1, D_2, D_3, D_4 are expressions.

Vedic Method

If numerators are equal and also if $D_1 + D_2 = D_3 + D_4$, then this relation is Samyam and is equal to zero.

If numerators are different, then first equate them by L.C.M and then test for the above relation and if it is satisfied then apply the sutram. (See 6)

If LHS and RHS do not have same number of terms then try for the merging method (explained later). (See Merger 1)

Eg.(i) Solve $\frac{1}{x-13} + \frac{1}{x-4} = \frac{1}{x-9} + \frac{1}{x-8}$

Current Method

$$\frac{1}{x-13} - \frac{1}{x-9} = \frac{1}{x-8} - \frac{1}{x-4}$$

$$\frac{x-9-x+13}{(x-13)(x-9)} = \frac{x-4-x+8}{(x-8)(x-4)}$$

$$\frac{4}{(x-13)(x-9)} = \frac{4}{(x-8)(x-4)}$$

$(x-13)(x-9) = (x-8)(x-4)$

$x^2 - 22x + 117 = x^2 - 12x + 32$

$117 - 32 = 22x - 12x$

$85 = 10x$

$x = 85/10 = 17/2$

Vedic Method

$$\frac{1}{x-13} \quad \frac{1}{x-4} \quad \frac{1}{x-9} \quad \frac{1}{x-8}$$

Numerators are equal.

$D_1 + D_2 = 2x - 17$

$D_3 + D_4 = 2x - 17$

$\therefore$ By Sunyam Samya Samuccaye this relation is Samyam and hence is zero.

$\therefore 2x - 17 = 0$

$x = 17/2$

Eg (ii) Solve $\frac{3}{x-13}+\frac{2}{x-4}=\frac{4}{x-10}+\frac{1}{x-7}$

Current Method

$$\frac{3}{x-13}+\frac{2}{x-4}=\frac{4}{x-10}+\frac{1}{x-7}$$

$$\frac{3(x-4)+2(x-13)}{(x-13)(x-4)}=\frac{4(x-7)+1(x-10)}{(x-10)(x-7)}$$

$$\frac{3x-12+2x-26}{x^2-4x-13x+52}=\frac{4x-28+x-10}{x^2-7x-10x+70}$$

$$\frac{5x-38}{x^2-17x+52}=\frac{5x-38}{x^2-17x+70}$$

By cross multiplication

$(5x-38)(x^2-17x+70)=(5x-38)(x^2-17x+52)$
$5x^3-85x^2+350x-38x^2+646x-2660$
$=5x^3-85x^2+260x-38x^2+646x-1976$
$350x-2660-260x+1976=0$
$90x-684=0$
$90x=684$

$x=\frac{684}{90}=\frac{38}{5}$

Vedic Method

When numerators are not equal

$$\frac{3}{x-13}+\frac{2}{x-4}=\frac{4}{x-10}+\frac{1}{x-7}$$

$$\frac{3}{1}+\frac{2}{1}=\frac{4}{1}+\frac{1}{1}$$

Equating the numerators

$$\frac{12}{4x-52}+\frac{12}{6x-24}=\frac{12}{3x-30}+\frac{12}{12x-84}$$

Numerators are equal

$D_1+D_2=10x-76=2(5x-38)$
$D_3+D_4=15x-114=3(5x-38)$

$5x-38=0$
$5x=38$

$x=\frac{38}{5}$

Eg (iii) Solve $\frac{3}{x-13}+\frac{2}{x-4}=\frac{4}{x-8}+\frac{1}{x-7}$

$\frac{3}{1}+\frac{2}{1}=\frac{4}{1}+\frac{1}{1}$ (yes)

Equating the numerators

$$\frac{12}{4x-52}+\frac{12}{6x-24}=\frac{12}{3x-24}+\frac{12}{12x-84}$$

Numerators are equal

$D_1+D_2=10x-76=2(5x-38)$
$D_3+D_4=15x-108=3(5x-36)$

As both are not equal

$N_1D_2+N_2D_1=3(x-4)+2(x-13)$
$=3x-12+2x-26=5x-38$

$N_3D_4+N_4D_3=4(x-7)+1(x-8)$
$=4x-28+x-8$
$=5x-36$

As both are not equal, this method is not applicable for solving this equation This may turn out to be either a quadratic or cubic equation.

4 b) If the equation is in the form of $\frac{N_1}{D_1} - \frac{N_2}{D_2} = \frac{N_3}{D_3} - \frac{N_4}{D_4}$

Vedic Method

Transpose to get the standard form then the sutram Samyam is worked out accordingly.

Eg.(ii) Solve $\frac{1}{x+16} - \frac{1}{x+9} = \frac{1}{x+12} - \frac{1}{x+5}$ Disguised form

Current Method

$$\frac{1}{x+16} - \frac{1}{x+9} = \frac{1}{x+12} - \frac{1}{x+5}$$

$$\frac{(x+9)-(x+16)}{(x+16)(x+9)} = \frac{(x+5)-(x+12)}{(x+12)(x+5)}$$

$$\frac{-7}{(x+16)(x+9)} = \frac{-7}{(x+12)(x+5)}$$

$(x + 16)(x + 9) = (x + 12)(x + 5)$

$x^2 + 25x + 144 = x^2 + 17x + 60$

$25x - 17x = -144 + 60$

$8x = -84$

$x = -84 / 8 = -21 / 2$

Vedic Method

$$\frac{1}{x+16} - \frac{1}{x+9} = \frac{1}{x+12} - \frac{1}{x+5}$$

Transpose the negative terms suitably to identify the problem with the standard form

$$\frac{1}{x+16} + \frac{1}{x+5} = \frac{1}{x+12} + \frac{1}{x+9}$$

Numerators are equal on both sides

$D_1 + D_2 = 2x + 21$

$D_3 + D_4 = 2x + 21$

$D_1 + D_2 = D_3 + D_4$

By Sunyam Samya Samuccaye, this relation is Samyam and hence is zero

$2x + 21 = 0$

$x = -21 / 2$

Eg.(ii) Solve $\frac{1}{x+b} - \frac{1}{x+b+2d} = \frac{1}{x+c-2d} - \frac{1}{x+c}$

Vedic Method

$$\frac{1}{x+b} + \frac{1}{x+c} = \frac{1}{x+b+2d} + \frac{1}{x+c-2d}$$

Numerators being same

$D_1 + D_2 = D_3 + D_4 = 2x + b + c = 0x = \frac{-b-c}{2}$

5. If the equation is in the form $\frac{x+a}{x+b}+\frac{x+c}{x+d}=\frac{x+e}{x+f}+\frac{x+g}{x+h}$ (Medium disguises)

Vedic Method

Coefficients of x are same both in the numerator and denominator in each term such that the sum of ratios $(\frac{N}{D})$ of the coefficients of x on the LHS = sum of similar ratios on the RHS. After this test, one has to proceed to the division by Paravartya to convert it into the form 4(a). Then the sutram Samyam is worked out accordingly.

Eg.(i) Solve $\frac{x+3}{x+6}-\frac{x+6}{x+9}=\frac{x+2}{x+5}-\frac{x+5}{x+8}$.

Current Method

$$\frac{x+3}{x+6}-\frac{x+6}{x+9}=\frac{x+2}{x+5}-\frac{x+5}{x+8}$$

$$\frac{(x+3)(x+9)-(x+6)^2}{(x+6)(x+9)}=\frac{(x+2)(x+8)-(x+5)^2}{(x+5)(x+8)}$$

$$\frac{x^2+12x+27-x^2-12x-36}{x^2+15x+54}=\frac{x^2+10x+16-x^2-10x-25}{x^2+13x+40}$$

$$\frac{-9}{x^2+15x+54}=\frac{-9}{x^2+13x+40}$$

$x^2 + 15x + 54 = x^2 + 13x + 40$

$15x - 13x = 40 - 54$

$2x = -14$ or $x = -7$

Vedic Method

$$\frac{x+3}{x+6}-\frac{x+6}{x+9}=\frac{x+2}{x+5}-\frac{x+5}{x+8}$$

Transpose the negative terms suitably to identify the problem with the standard form 5.

$$\frac{x+3}{x+6}+\frac{x+5}{x+8}=\frac{x+2}{x+5}+\frac{x+6}{x+9}$$

$$\frac{1}{1}+\frac{1}{1}=\frac{1}{1}+\frac{1}{1}$$

Then by Paravartya division*,

$$1-\frac{3}{x+6}+1-\frac{3}{x+8}=1-\frac{3}{x+5}+1-\frac{3}{x+9}$$

$$\frac{3}{x+6}+\frac{3}{x+8}=\frac{3}{x+5}+\frac{3}{x+9}$$

Numerators being equal, test if

$D_1 + D_2 = D_3 + D_4$

$D_1 + D_2 = D_3 + D_4 = 2x + 14$

By Sunyam Samya Samuccaye this relation is Samyam and hence is zero.

Therefore, $2x + 14 = 0$

$x = -7$

* Refer lecture notes on division.

Eg. (ii) Solve $\frac{x-a+b}{x-a}+\frac{x-b}{x-2b}=\frac{x}{x-b}+\frac{x-a}{x-a-b}$

Current Method

$$\frac{x-a+b}{x-a}+\frac{x-b}{x-2b}=\frac{x}{x-b}+\frac{x-a}{x-a-b}$$

$$\frac{(x-a+b)(x-2b)+(x-a)(x-b)}{(x-a)(x-2b)}=\frac{x(x-a-b)+(x-a)(x-b)}{(x-b)(x-a-b)}$$

$$\frac{x^2-ax+bx-2bx+2ab-2b^2+x^2-ax-bx+ab}{x^2-ax-2bx+2ab}=\frac{x^2-ax-bx+x^2-ax-bx+ab}{x^2-bx-ax+ab-bx+b^2}$$

$$\frac{2x^2-2ax-2bx+3ab-2b^2}{x^2-ax-2bx+2ab}=\frac{2x^2-2ax-2bx+ab}{x^2-ax-2bx+ab+b^2}$$

$(2x^2-2ax-2bx+3ab-2b^2)(x^2-ax-2bx+ab+b^2)$

$=(2x^2-2ax-2bx+ab)(x^2-ax-2bx+2ab)$

$2x^4-2ax^3+3abx^2-2b^2x^2-2ax^3+2a^2x^2+2abx^2+2abx^2-2a^2bx-2ab^2x+3a^2b^2-2ab^3+$
$2b^2x^2-2ab^2x-2b^3x+3ab^3-2b^4$

L.H.S= $2x^4-2ax^3-2bx^3+abx^2-2ax^3+2a^2x^2-2abx^2-a^2bx-4bx^3+4abx^2+4b^2x^2-2ab^2x+$
$4abx^2-4a^2bx-4ab^2x+2a^2b^2$

$445abx^2-5a^2bx-8ab^2x+2b^3x+3ab^2b^2+ab^3-2b^4=5abx^2-5a^2bx-6ab^2x+2a^2b^2$

$2ab^2x-2b^3x-a^2b^2-ab^3+2b^4=0$

$2ax-2bx-a^2-ab+2b^2=0$

$(2a-2b)x=a^2+ab-2b^2$

$$x=\frac{a^2+ab-2b^2}{2(a-b)}=\frac{a(a+b)-2b^2}{2(a-b)}=\frac{(a+2b)}{2}$$

Vedic Method

$$\frac{x-a+b}{x-a}+\frac{x-b}{x-2b}=\frac{x}{x-b}+\frac{x-a}{x-a-b}$$

$$\frac{1}{1}+\frac{1}{1}=\frac{1}{1}+\frac{1}{1}$$

Applying Paravartya Division

$$\frac{b}{x-a}+\frac{b}{x-2b}=\frac{b}{x-b}+\frac{b}{x-a-b}$$

Numerators are equal

$D_1+D_2=2x-a-2b$

$D_3+D_4=2x-a-2b$

$D_1+D_2=D_3+D_4$

By Sunyam Samya Samuccaye, this relation is Samyam and hence is zero.

Therefore, $2x-a-2b=0$

$x=(a+2b)/2$

The ease with which the problem can be tackled by the Vedic Method is excellent.

Even when the coefficients of x are different in any term, then also test if the sum of the ratios ($\frac{N}{D}$) of the coefficients of x on the LHS = RHS. If this condition is satisfied then Paravartya Division is applied to convert it into the form 4(a). The solution is worked out accordingly.

Eg.(ii) Solve $\frac{2(3x-1)}{x-1}+\frac{x-2}{x-6}=\frac{2(x-2)}{x-4}+\frac{5x-11}{x-3}$

Current Method

$$\frac{2(3x-1)}{x-1}+\frac{x-2}{x-6}=\frac{2(x-2)}{x-4}+\frac{5x-11}{x-3}$$

$$\frac{6x-2}{x-1}-\frac{5x-11}{x-3}=\frac{2(x-2)}{x-4}-\frac{x-2}{x-6}$$

$$\frac{(6x-2)(x-3)-(5x-11)(x-1)}{(x-1)(x-3)}$$
$$=\frac{(x-2)[2(x-6)-(x-4)]}{(x-4)(x-6)}$$

$$\frac{6x^2-20x+6-5x^2+16x-11}{x^2-4x+3}$$
$$=\frac{(x-2)(x-8)}{x^2-10x+24}$$

$$\frac{x^2-4x-5}{x^2-4x+3}=\frac{x^2-10x+16}{x^2-10x+24}$$

$(x^2-4x-5)(x^2-10x+24)$
$=(x^2-10x+16)(x^2-4x+3)$
$x^4-14x^3+59x^2-46x-120$
$=x^4-14x^3+59x^2-94x+48$
$-46x+94x=48+120$
$48x=168$
$x=168/48=7/2$

Vedic Method

$$\frac{2(3x-1)}{x-1}+\frac{x-2}{x-6}=\frac{2(x-2)}{x-4}+\frac{5x-11}{x-3}$$

$$\frac{6x-2}{x-1}+\frac{x-2}{x-6}=\frac{2x-4}{x-4}+\frac{5x-11}{x-3}$$

1. $\frac{6}{1}+\frac{1}{1}=\frac{2}{1}+\frac{5}{1}$
2. By Paravartya Division after multiplying the denominator with the coefficient of x of the corresponding numerator.

$$6+\frac{4}{x-1}+1+\frac{4}{x-6}=2+\frac{4}{x-4}+5+\frac{4}{x-3}$$

$$\frac{4}{x-1}+\frac{4}{x-6}\quad\frac{4}{x-4}+\frac{4}{x-3}$$

Numerators are equal.
$D_1+D_2=D_3+D_4=2x-7$

By Sunyam Samya Samuccaye this relation is Samyam and hence is zero.

Therefore, $2x-7=0$
$x=7/2$

6. If the equation is in the form of $\frac{N_1}{D_1}+\frac{N_2}{D_2}=\frac{N_3}{D_3}+\frac{N_4}{D_4}$ where numerators are only numbers.

Vedic Method

If numerators are not equal then LCM can be considered to make numerators equal. This will modify the equation.

In the modified equation if $D_1' + D_2' = D_3' + D_4'$ then Samyam is applied.

Sometimes the LCM method may be a bit cumbersome and the relation $D_1 + D_2 = D_3 + D_4$ may not satisfy with the modified denominators ($'$) which is noticed only at the end. Hence another method is suggested which makes use of certain preliminary test to see that the sum of the ratios ($\frac{N}{D}$) of the coefficients of x on the LHS = RHS. After this test, one has to proceed to another relation namely to see if $N_1D_2 + N_2D_1 = N_3D_4 + N_4D_3$ which is the Samyam and hence is zero. Even if L.H.S. is a multiple of R.H.S and vice versa sunyam can be applied as explained in the problem below

Eg. (i) Solve $\frac{2}{2x+3}-\frac{3}{3x-1}=\frac{1}{x+2}-\frac{6}{6x+1}$

Current Method

$$\frac{2}{2x+3}-\frac{3}{3x-1}=\frac{1}{x+2}-\frac{6}{6x+1}$$

$$\frac{2(3x-1)-3(2x+3)}{(2x+3)(3x-1)}=\frac{(6x+1)-6(x+2)}{(x+2)(6x+1)}$$

$$\frac{6x-2-6x-9}{6x^2+7x-3}=\frac{6x+1-6x-12}{6x^2+13x+2}$$

$$\frac{-11}{6x^2+7x-3}=\frac{-11}{6x^2+13x+2}$$

$6x^2 + 7x - 3 = 6x^2 + 13x + 2$
$6x + 5 = 0$
$x = -5/6$

Vedic Method

$$\frac{2}{2x+3}-\frac{3}{3x-1}=\frac{1}{x+2}-\frac{6}{6x+1}$$

Method 1: Transpose the negative terms suitably to identify the problem with the standard form 4a.

$$\frac{2}{2x+3}+\frac{6}{6x+1}=\frac{1}{x+2}+\frac{3}{3x-1} \quad \text{---(1)}$$

Since Numerators are different on both sides, we make numerators equal by taking LCM.

$$\frac{6}{6x+9}+\frac{6}{6x+1}=\frac{6}{6x+12}+\frac{6}{6x-2}$$

$$\frac{1}{6x+9}+\frac{1}{6x+1}=\frac{1}{6x+12}+\frac{1}{6x-2}$$

$D_1' + D_2' = D_3' + D_4' = 12x + 10$

$\therefore$By Sunyam Samya Samuccaye this relation is Samyam and hence zero.

$12x + 10 = 0$ or $x = -10/12 = -5/6$

Method 2: (From stage 1)

Preliminary test

$\frac{2}{2} + \frac{6}{6} = \frac{1}{1} + \frac{3}{3}$ (yes)

$N_1D_2 + N_2D_1 = 24x + 20 = 4(6x + 5)$

$N_3D_4 + N_4D_3 = 6x + 5$

$N_1D_2 + N_2D_1$ is multiple of $N_3D_4 + N_4D_3$.

$\therefore$ By Sunyam Samya Samuccaye

$N_3D_4 + N_4D_3$ is Samyam and hence is zero.

$\therefore 6x + 5 = 0$

$x = -5/6$

In the form

$$\frac{N_1}{D_1} + \frac{N_2}{D_2} \quad \frac{N_3}{D_3} + \frac{N_4}{D_4}$$

If the numerators of the terms also contain x, then the test of sum of ratios $(\frac{N}{D})$ of x coefficients on both sides should be equal to apply Paravartya Division. The equation has to be converted to the standard form given in (6) and then the sutram is worked out accordingly or after the coefficient ratios test is carried out, one can even at this stage test for $N_1D_2 + N_2D_1 = N_3D_4 + N_4D_3$ of the original equation. If it is satisfied then it is Samyam and hence is zero by Sunyam Samya Samuccaye.

The following examples illustrates this

Eg.(ii) Solve $\frac{4x+12}{2x+5} \quad \frac{15x-17}{3x-4} \quad \frac{5x+6}{x+1} + \frac{12x+8}{6x+1}$

Current Method

$$\frac{4x+12}{2x+5}+\frac{15x-17}{3x-4}=\frac{5x+6}{x+1}+\frac{12x+8}{6x+1}$$

$$\frac{4x+12}{2x+5}-\frac{12x+8}{6x+1}=\frac{5x+6}{x+1}-\frac{15x-17}{3x-4}$$

$$\frac{(4x+12)(6x+1)-(12x+8)(2x+5)}{(2x+5)(6x+1)}$$

$$=\frac{(5x+6)(3x-4)-(15x-17)(x+1)}{(x+1)(3x-4)}$$

$$\frac{24x^2+76x+12-(24x^2+76x+40)}{12x^2+32x+5}$$

$$\frac{15x^2-2x-24-(15x^2-2x-17)}{3x^2-x-4}$$

$$\frac{-28}{12x^2+32x+5} \quad \frac{-7}{3x^2-x-4}$$

$$12x^2+32x+5 \quad 3x^2-x-4$$

$12x^2-4x-16=12x^2+32x+5$

$-21=36x$

$x=-21/36=-7/12$

Vedic Method

$$\frac{4x+12}{2x+5}+\frac{15x-17}{3x-4} \quad \frac{5x+6}{x+1} \quad \frac{12x+8}{6x+1}$$

$$\frac{4x}{2x}+\frac{15x}{3x}=\frac{5x}{x}+\frac{12x}{6x} \text{ (Yes)}$$

Applying Paravartya Sutra

$$2+\frac{2}{2x+5}+5+\frac{3}{3x-4}=5+\frac{\cdot}{x+1}+2+\frac{\cdot}{6x+1}$$

$$\frac{}{2x+5} \quad \frac{}{3x-4} \quad \frac{1}{x+1} \quad \frac{}{6x+1} \quad \text{—(1)}$$

This is in the form (6)

Method 1: By taking LCM we make numerators equal.

$$\frac{6}{6x+15}+\frac{6}{6x-8}=\frac{6}{6x+6}+\frac{6}{6x+1}$$

$D_1+D_2=D_3+D_4=12x+7$

∴ By Sunyam Samya Samuccaye, this relation is Samyam and hence zero.

$12x+7=0,\ x=-7/12$

Method 2:

From stage (1): (Form 6)

Preliminary test

1. $\frac{2}{2}+\frac{3}{3}=\frac{1}{1}+\frac{6}{6}$ (Yes)
2. $N_1D_2+N_2D_1=12x+7$
   $N_3D_4+N_4D_3=12x+7$
   $N_1D_2+N_2D_1=N_3D_4+N_4D_3$

∴ By Sunyam Samya Samuccaye, this relation is Samyam and hence is zero.

$12x+7=0$, or $x=-\frac{7}{12}$

(2) (3) (4)

Solve $\frac{x^2+3x+3}{x-2}+\frac{x^2-15}{x-4}=\frac{x^2+7x+11}{x+5}+\frac{x^2-4x-20}{x-7}$

Current Method

$$\frac{x^2+3x+3}{x+2}+\frac{x^2-15}{x-4}=\frac{x^2+7x+11}{x+5}+\frac{x^2-4x-20}{x-7}$$

$$\Rightarrow \frac{(x^2+3x+3)(x-4)+(x^2-15)(x+2)}{(x+2)(x-4)}$$

$$=\frac{(x^2+7x+11)(x-7)+(x^2-4x-20)(x+5)}{(x+5)(x-7)}$$

$$\frac{(x^3-4x^2+3x^2-12x+3x-12)+(x^3+2x^2-15x-30)}{x^2-4x+2x-8}$$

$$\frac{(x^3-7x^2+7x^2-49x+11x-77)+(x^3+5x^2-4x^2-20x-20x-100)}{x^2+5x-7x-35}$$

$$\Rightarrow \frac{6x^3+x^2-24x-42}{x^2-2x-8}=\frac{6x^3+x^2-78x-177}{x^2-2x-35}$$

$\Rightarrow (6x^3+x^2-78x-177)(x^2-2x-8)$

$=(x^2-2x-35)(6x^3+x^2-24x-42)$

$6x^5-12x^4-18x^3+x^4-2x^3-8x^2-78x^3+156x^2+624x-177x^2+354x+1416$

$=6x^5+x^4-24x^3-42x^2-12x^4-2x^3+48x^2+84x-210x^3-35x^2+840x+1470$

$\Rightarrow 108x^3-108x^2+54x-54=0$

$2x^3-2x^2+x-1$

x is a factor

$\therefore$ x = 1

Refer to cubic equation

Vedic Method

$$\frac{x^2+3x+3}{x-2}+\frac{x^2-15}{x-4}=\frac{x^2+7x+11}{x+5}+\frac{x^2-4x-20}{x-7}$$

(1) By Parvartya

x + 2	x^2 + 3x	+ 3
- 2	- 2x	- 2
	x + 1	1

Quotient is x + 1
R = 1

$(x+1)+\frac{1}{(x+2)}$

(2)

x – 4	x^2	- 15
+ 4	+ 4x	+ 16
	x + 4	1

Q = x + 4
R = 1

$(x+4)+\frac{1}{(x+4)}$

(3)

x + 5	x^2 + 7x	+ 11
- 5	- 5x	- 10
	x + 2	1

Q = x + 2
R = 1

$(x+2)+\frac{1}{(x+5)}$

(4)

x - 7	x^2 - 4x	- 20
+ 7	+ 7x	+ 21
	x + 3	1

Q = x + 3
R = 1

$x+3+\frac{1}{x-7}$

By applying Paravartya method the result is

$$x+1+\frac{1}{x+2}+x+4+\frac{1}{x-4}=x+2+\frac{1}{x+5}+x+3+\frac{1}{x-7}$$

$$\frac{1}{x+2}+\frac{1}{x-4} \quad \frac{1}{x+5}+\frac{1}{x-7}$$

Numerators are equal

$D_1 + D_2 = 2x - 2$

$D_3 + D_4 = 2x - 2$

$2x - 2 = 0$

$x = 1$

Proof:

$$\frac{p}{x+a}+\frac{q}{x+b}=\frac{p+q}{x+c}$$

$$\frac{p}{x+a}+\frac{q}{x+b}=\frac{p}{x+c}+\frac{q}{x+c}$$

$$\frac{p}{x+a}-\frac{p}{x+c}=\frac{q}{x+c}-\frac{q}{x+b}$$

$$\frac{p(x+c-x-a)}{(x+a)(x+c)}=\frac{q(x+b-x-c)}{(x+c)(x+b)}$$

$$\frac{p(c-a)}{x+a}=\frac{q(b-c)}{x+b}$$

$$x[p(c-a)+q(c-b)] = bp(a-c)+aq(b-c)$$

Therefore, $x=\dfrac{bp(a-c)+aq(b-c)}{p(c-a)+q(c-b)}$

7. Merger Type: **(See text (4a))**

(By the Paravartya Method)

1. If the equation is in the form of

$$\frac{p}{x+a}+\frac{q}{x+b}=\frac{p+q}{x+c}$$

The condition for merging is $N_1 + N_2$ (Left Hand Side) = N. (Right hand side)

Merging of the right-hand side term into the left-hand side terms can be done by the following operation. It is also in the first instance noticed that the x coefficients in the denominator are same. Number of terms on the left hand side can be any, but on the right hand side it must be only one which is to be merged into the left hand side.

It has to be merged into all the terms in the left-hand side.

1) The first term is $\frac{\;}{x+a}$. When the right side term, i.e., $\frac{p+q}{x+c}$ is merged into the first term of LHS, then this becomes $\frac{(a-c)p}{x+a}=N_1^1$.

2) Similarly $\frac{q}{x+b}$ becomes $\frac{(b-c)q}{x+b}=N_2^1$. The merged equation is written as $\frac{(a-c)p}{x+a}+\frac{(b-c)q}{x+b}=0$, where the numerators are numericals. On applying the equalization of numerators, we obtain solution by summation of denominators.

Eg.(i) Solve $\frac{1}{x+3}+\frac{6}{x+2}=\frac{7}{x+5}$

Current Method

$$\frac{1}{x+3}+\frac{6}{x+2} \quad \frac{7}{x+5}$$

$$\frac{x+2+6(x+3)}{(x+3)(x+2)}=\frac{7}{x+5}$$

$$\frac{7x+20}{x^2+5x+6}=\frac{7}{x+5}$$

$7x^2 + 20x + 35x + 100$
$= 7x^2 + 35x + 42$
$20x = -58$
$x = -58 / 20 \Rightarrow x = -29 / 10$

Vedic Method

$$\frac{1}{x+3}+\frac{6}{x+2}=\frac{7}{x+5}$$

Test if $N_1 + N_2 = N$, $1 + 6 = 7$ (Yes)
Therefore, we can apply Merging Rule and get the following:

$N_1' = (-5 + 3)\,1 = -2$
$N_2' = (-5 + 2)\,6 = -18$
indicates modified values of Numerators

$$\frac{-2}{x+3}+\frac{-18}{x+2}=0$$

Making Numerators Equal

$$\frac{-18}{9x+27}-\frac{18}{x+2}=0$$

$\therefore$ By Sunyam Samya Samuccaye, the sum of the denominators is Samyam and hence is zero.
$10x + 29 = 0$
$x = -29 / 10$

Eg.(ii) Solve $\frac{9}{x+5}+\frac{4}{x+3}=\frac{13}{x+1}$

Current Method

$$\frac{9}{x+5}+\frac{4}{x+3}=-\frac{13}{x+1}$$

$$\frac{9x+27+4x+20}{x^2+8x+15} \quad \frac{13}{x+1}$$

$$\frac{13x+47}{x^2+8x+15}=\frac{13}{x+1}$$

$13x^2 + 47x + 13x + 47 = 13x^2 + 104x + 195$
$44x + 148 = 0$
$x = -148 / 44$
$x = -37 / 11$

Vedic Method

$$\frac{9}{x+5}+\frac{4}{x+3}=\frac{13}{x+1}$$

Test if $N_1 + N_2 = N$
$9 + 4 = 13$ (Yes)
$\therefore$ We can apply merging method.

$$\frac{36}{x+5}+\frac{8}{x+3}=0$$

Making Numerators Equal

$$\frac{72}{2x+10}+\frac{72}{9x+27}$$

$\therefore$ By Sunyam Samya Samuccaye, the sum of the denominators is Samyam and hence zero.

$11x + 37 = 0$
$x = -37/11$

7(a). If the x coefficients in the denominators are not equal then verify if

$$(LHS)\sum\frac{\text{Numerical value in the numerator}}{\text{x coefficient in the denominator}} = (RHS)\frac{\text{Numerical value in the numerator}}{\text{x coefficient in the denominator}}$$

If this is satisfied, then convert the coefficient of x in the denominators to have a common value. In doing so, the numerators are also multiplied accordingly. At this stage test if the sum of numerators on the left-hand side = numerator on the right hand side. If this test is satisfied, then merger method can be applied.

Eg.(i) Solve $\frac{6}{3x+1}+\frac{7}{x+3}=\frac{18}{2x+5}$

Current Method

$$\frac{6}{3x+1}+\frac{7}{x+3} \quad \frac{18}{2x+5}$$

$$\frac{6x+18+21x+7}{3x^2+10x+3}=\frac{18}{2x+5}$$

$$\frac{27x+25}{3x^2+10x+3}=\frac{18}{2x+5}$$

$54x^2 + 50x + 135x + 125$
$= 54x^2 + 180x + 54$
$185x + 125 = 180x + 54$
$5x + 71 = 0$
$x = -71/5$

Vedic Method

$$\frac{6}{3x+1}+\frac{7}{x+3}=\frac{18}{2x+5}$$

$\frac{6}{3}+\frac{7}{1}=\frac{18}{2}$ (Yes)

Making x coefficients equal in the denominator

$$\frac{12}{6x+2}+\frac{42}{6x+18}=\frac{54}{6x+15}$$

$$\frac{2(N_1')}{6x+2}+\frac{7(N_2')}{6x+18}=\frac{9(N')}{6x+15}$$

Test if $N_1' + N_2' = N'$

$2 + 7 = 9$ (Yes)

∴ We can apply merging method

(Merging) $\frac{-26}{6x+2}+\frac{21}{6x+18}=0$

On further simplification

$$\frac{-13}{3x+1}+\frac{7}{2x+6}=0$$

Making Numerators Equal

$$\frac{-91}{21x+7} \quad \frac{-91}{-26x-78} = 0$$

∴By Sunyam Samya Samuccaye, the sum of the denominators is Samyam and hence is zero.

$5x + 71 = 0$

$x = -71 / 5$

Eg.(ii) Solve $\frac{1}{3x+1} \quad x+5 \quad \frac{-5}{3x+2}$

Current Method

$$\frac{1}{3x+1} - \frac{2}{x+5} = \frac{-5}{3x+2}$$

$$\frac{x+5-6x-2}{(3x+1)(x+5)} \quad \frac{-5}{3x+2}$$

$$\frac{-5x+3}{3x^2+16x+5} = \frac{-5}{3x+2}$$

$-15x^2 + 9x - 10x + 6 = -15x^2 - 80x - 25$

$-x + 6 = -80x - 25$

$79x = -31$ or $x = -31 / 79$

Vedic Method

$$\frac{1}{3x+1} \quad \frac{2}{x+5} \quad \frac{-5}{3x+2}$$

We transpose negative terms suitably to identify the problem with the standard form (7).

$$\frac{1}{3x+1} + \frac{5}{3x+2} = \frac{2}{x+5}$$

$$\frac{1}{3} + \frac{5}{3} = \frac{2}{1}$$

Taking LCM for the coefficients of x in the denominator to make them equal

$$\frac{1}{3x+1} + \frac{5}{3x+2} = \frac{6}{3x+15}$$

Test if $N_1 + N_2 = N$

$1 + 5 = 6$ (Yes)

∴ We can apply merging method.

$$\frac{-14}{3x+1} - \frac{65}{3x+2} = 0$$

Making Numerators Equal

$$\frac{910}{195x+65} + \frac{910}{42x+28} = 0$$

∴ By Sunyam Samya Samuccaye, the sum of the denominators is Samyam and hence is zero.

$195x + 65 + 42x + 28 = 0$

$237x = 93$

$x = -93 / 237 \Rightarrow x = -31 / 79$

Proof:

$$\frac{m}{x+a}+\frac{n}{x+b}+\frac{p}{x+c}\quad\frac{m+n+p}{x+d}$$

$$\frac{m}{x+a}-\frac{m}{x+d}+\frac{n}{x+b}-\frac{n}{x+d}+\frac{p}{x+c}-\frac{p}{x+d}=0$$

$$\frac{m(a-d)}{x+a}+\frac{n(b-d)}{x+b}+\frac{p(c-d)}{x+c}=0\text{----- (1)}$$

$$\frac{m(a-d)(x+b)(x+c)+n(b-d)(x+a)(x+c)+p(c-d)(x+a)(x+b)}{(x+a)(x+b)(x+c)}=0$$

$m(a-d)(x^2+bx+cx+cb)+n(b-d)(x^2+ax+cx+ac)+p(c-d)(x^2+ax+bx+ab)=0$

The condition for the above equation to be a simple equation is x^2 term should vanish.

i.e., $m(a-d)+n(b-d)+p(c-d)=0$

Using this condition at the stage (1)

i.e., substituting $[-m(a-d)-n(b-d)]$ for $p(c-d)$

$$\frac{m(a-d)(a-c)}{x+a}+\frac{n(b-d)(b-c)}{x+b}=0$$

8. Multiple Merging:

A multiple merger can be identified by the general form

$\frac{m}{x+a}+\frac{n}{x+b}+\frac{p}{x+c}=\frac{m+n+p}{x+d}$ where one has to first verify Sum of Numerators on Left-hand side = Sum of the Numerator on the Right hand side.

After this is satisfied, the problem is ready for merging. Merging is of multiple nature in the sense that one has to start from the right hand side term and proceed on to the left hand through the different terms until two terms are left over on the left hand side. The merging has to be carried out only after equalizing the x coefficients in the denominator. The procedure is clearly given below in case of the following examples:

$$\text{In general}\ \frac{m}{x+a}+\frac{n}{x+b}+\frac{p}{x+c}+\frac{q}{x+d}+\frac{r}{x+e}+\ldots\ldots=\frac{m+n+p+q+r+\ldots\ldots}{x+w}$$

$$\frac{m(a-w)(\ldots\ldots)(a-e)(a-d)(a-c)\rightarrow M}{x+a}+\frac{n(b-w)(\ldots\ldots)(b-e)(b-d)(b-c)\rightarrow N}{x+b}=0$$

$$x=\frac{-bm(a-w)(\ldots\ldots)(a-e)(a-d)(a-c)-an(b-w)(\ldots\ldots)(b-e)(b-d)(b-}{m(a-w)(\ldots\ldots)(a-e)(a-d)(a-c)+n(b-w)(\ldots\ldots)(b-e)(b-d)(b-c)}$$

This method is known as Multiple Simultaneous Merger, can be directly applied.

Eg. Solve $\frac{2}{x+} + \frac{5}{x+2} + \frac{3}{x+6} \quad \frac{0}{x+3}$

Current Method

$$\frac{2}{x+1}+\frac{5}{x+2}+\frac{3}{x+6}=\frac{10}{x+3}$$

$$\frac{2}{x+1}+\frac{5}{x+2}=\frac{10}{x+3}-\frac{3}{x+6}$$

$$\frac{2x+4+5x+5}{(x+1)(x+2)}=\frac{10x+60-3x-9}{(x+3)(x+6)}$$

$$\frac{7x+9}{x^2+3x+2}=\frac{7x+51}{x^2+9x+18}$$

$(7x+9)(x^2+9x+18)$
$=(7x+51)(x^2+3x+2)$
$7x^3+9x^2+63x^2+81x+126x+162$
$=7x^3+51x^2+21x^2+153x+14x+102$
$207x+162=167x+102$
$40x+60=0$
$x=-60/40 \Rightarrow x=-3/2$

Vedic Me od

$$\frac{2}{x+1}+\frac{5}{x+2}+\frac{3}{x+6}=\frac{10}{x+3}$$

Test if $N_1+N_2+N_3=N$
$2+5+3=10$ (Yes)
∴ We can apply merging method.

$$-\frac{4}{x+1}-\frac{5}{x+2}+\frac{9}{x+6}=0$$

$$\frac{4}{x+1}+\frac{5}{x+2}=\frac{9}{x+6}$$

Test if $N_1+N_2=N$
$4+5=9$ (Yes)
∴ We can again apply merging method.

$$\frac{-20}{x+1}-\frac{20}{x+2}=0$$

Numerators are Equal; ∴ By Sunyam Samya Samuccaye summation of denominators is Samyam and hence is zero.

$2x+3=0$

$x=-3/2$

A general method called multiple simultaneous merger can replace the above term by term merger in multiple merger.

The general formula can be applied:

Consider the given problem : $\frac{m}{x+a}+\frac{n}{x+b}+\frac{p}{x+c}=\frac{m+n+p}{x+w}$

$$x=\frac{-bm(a-w)(...)(a-e)(a-d)(a-c)-an(b-w)(...)(b-e)(b-d)(b-c)}{m(a-w)(...)(a-e)(a-d)(a-c)+n(b-w)(...)(b-e)(b-d)(b-c)}$$

We can effectively bring out the multiple merger, which, instead of carrying out step by step merger, we can perform simultaneous merging using the above formula.

By using this formula we can solve the given example where a = 1, b = 2, c = 6, w = 3, m = 2, n = 5, p = 3.

Here m + n + p = 10 (Numerators on right hand side)

- bm (a – w) (a – c) – an

$$x=\frac{(-2)(2)(-2)(-5)+(-1)(5)(-1)(-4)}{2(-2)(-5)+5(-1)(-4)}$$

$$x=\frac{-40-20}{20+20}=\frac{-60}{40}=\frac{-3}{2}$$

Eg.(ii) Solve $\frac{2}{x+1}+\frac{2}{2x+1}+\frac{175}{5x+1}=\frac{152}{4x+1}$.

Current Method

$$\frac{2}{x+1}+\frac{2}{2x+1}+\frac{175}{5x+1}=\frac{152}{4x+1}$$

$$\frac{2}{x+1}+\frac{2}{2x+1}=\frac{152}{4x+1}-\frac{175}{5x+1}$$

$$\frac{4x+2+2x+2}{(x+1)(2x+1)}=\frac{760x+152-700x--175}{(4x+1)(5x+1)}$$

$$\frac{6x+4}{2x^2+3x+1}=\frac{.60x-23}{20x^2+9x+1}$$

$120x^3+80x^2+54x^2+36x+6x+4$
$=120x^3-46x^2+180x^2-69x+60x-23$
$42x+4=-9x-23$
$51x=-27$
$x=-27/51=-9/17$

Vedic Method

$$\frac{2}{x+1}+\frac{2}{2x+1}+\frac{175}{5x+1}=\frac{152}{4x+1}$$

$$\frac{2}{1}+\frac{2}{2}+\frac{175}{5}=2+1+35=38$$

$152/4=38$

Coefficients of x in the denominator are different. Therefore, we equate the x coefficients in the denominator.

$$\frac{40}{20x+20}+\frac{20}{20x+10}+\frac{700}{20x+4}=\frac{760}{20x+5}$$

$$\frac{2}{20x+20}+\frac{1}{20x+10}+\frac{35}{20x+4}=\frac{38}{20x+5} \text{ --- (1)}$$

Test if $N_1+N_2+N_3=N$
$2+1+35=38$ (Yes)
∴ We can apply merging method.

$$\frac{30}{20x+20}+\frac{5}{20x+10}-\frac{35}{20x+4}=0$$

$$\frac{30}{20x+20}+\frac{5}{20x+10}=\frac{35}{20x+4}$$

$$\frac{6}{20x+20}+\frac{1}{20x+10}=\frac{7}{20x+4}$$

Test if $N_1+N_2=N$
$6+1=7$ (Yes)

Again by Merging,

$$\frac{96}{20x+20}+\frac{6}{20x+10}=0$$

Equalising numerators

$$\frac{96}{20x+20}+\frac{96}{320x+160}=0$$

∴ By Sunyam Samya Samuccaye summation of denominators is Samyam and hence is zero.

340x + 180 = 0
x = −18 / 34 = −9 / 17

By using simultaneous merging formula at stage (1)

$$20x = \frac{(-10)(2)(15)(16) + (-20)(1)(5)(6)}{(2)(15)(16) + (1)(5)(6)}$$

$$= \frac{-4800 - 600}{480 + 30} = \frac{-5400}{510} = \frac{-180}{17}$$

$$x = \frac{-180}{17 \times 20} = \frac{-9}{17}$$

Eg.(iii) Solve $\frac{3}{3x+1} + \frac{4}{2x+1} + \frac{42}{6x+1} = \frac{40}{4x+1}$

Current Method

$$\frac{3}{3x+1} + \frac{4}{2x+1} + \frac{42}{6x+1} = \frac{40}{4x+1}$$

$$\frac{3}{3x+1} + \frac{4}{2x+1} = \frac{40}{4x+1} - \frac{42}{6x+1}$$

$$\frac{6x + 3 + 12x + 4}{6x^2 + 5x + 1} = \frac{240x + 40 - 168x - 42}{24x^2 + 10x + 1}$$

$$\frac{18x + 7}{6x^2 + 5x + 1} = \frac{72x - 2}{24x^2 + 10x + 1}$$

$432x^3 + 168x^2 + 180x^2 + 70x + 18x + 7$
$= 432x^3 - 12x^2 + 360x^2 - 10x + 72x - 2$

88x + 7 = 62x − 2
26x = −9
x = −9 / 26

Vedic Method

$$\frac{3}{3x+1} + \frac{4}{2x+1} + \frac{42}{6x+1} = \frac{40}{4x+1}$$

$\frac{3}{3} + \frac{4}{2} + \frac{42}{6} = 1 + 2 + 7 = 10$ (L.H.S)

40 / 4 = 10 (R.H.S)

Coefficients of x in the denominator are different. Therefore, making x coefficients equal in the denominator

$$\frac{12}{12x+4} + \frac{24}{12x+6} + \frac{84}{12x+2} = \frac{120}{12x+3}$$

$$\frac{1}{12x+4} + \frac{2}{12x+6} + \frac{7}{12x+2} = \frac{10}{12x+3} \quad \text{--(1)}$$

Test if $N_1 + N_2 + N_3 = N$
1 + 2 + 7 = 10 (Yes)

∴ We can apply merging method.

$$\frac{1}{12x+4} + \frac{6}{12x+6} - \frac{7}{12x+2} = 0$$

$$\frac{1}{12x+4} + \frac{6}{12x+6} = \frac{7}{12x+2}$$

1 + 6 = 7 (Yes)
∴ We can again apply merging.

$$12x+4 \cdot \frac{24}{12x+6} = 0$$

Making Numerators Equal:

$$\frac{24}{144x+48} + \frac{24}{12x+6} = 0$$

By Sunyam Samya Samuccaye, Summation of denominators in Samyam and hence is zero.

156x + 54 = 0

$$x = \frac{-54}{156} \quad \frac{-9}{26}$$

Using the multiple simultaneous merging formula at stage (1):

$$12x = \frac{(-6)(1)(1)(2) + (-4)(2)(3)(4)}{(1)(1)(2) + (2)(3)(4)}$$

$$\frac{-12-96}{2+24} \quad \frac{-108}{26}$$

$$x = \frac{-108}{26 \times 12} = \frac{-9}{26}$$

9. Complex Mergers:

Complex merger is of the form

$\frac{m}{ax+b} + \frac{n}{cx+d} = \frac{p}{ex+f} + \frac{q}{rx+s}$. This can be solved by the following procedure:

1) $\Sigma\left(\frac{\text{Numerator}}{\text{Coefficient of x in the denominator}}\right)$ should be equal on both sides.

$$\frac{m}{a} + \frac{n}{c} = \frac{p}{e} + \frac{q}{r}$$

2) If cross-multiplication on the left hand side m(cx + d) + n(ax + b) is equal to corresponding cross-multiplication on the right hand side p(rx + s) + q(ex + f), ($N_1D_2 + N_2D_1 = N_3D_4 + N_4D_3$), then Sunyam is applied to this relation to get the solution.

3) If the above condition is not satisfied, then equalise x coefficients in the denominator. Then test if the modified (′) numerators are equal, and if sum of the denominators on LHS = sum of the denominators on the RHS

($D_1' + D_2' = D_3' + D_4'$), then Sunyam is applied.

4) If the above condition is not satisfied, then one can solve the problem by transposing one term each from one side to another such that by cross-multiplication the x coefficients get cancelled on both sides.

In this final derived equation, if numerators are equal on both sides, then by the equality of the denominators of the derived equation, x can be solved. Here it can be seen as $D_1D_2 = D_3D_4$.

5) If the numerators in the final derived equation are not equal, then one can try two methods.

a) To equalise the numerators. This is followed by equating the denominators after modification. If x^2 term gets cancelled, then it results in simple equation.
b) To equate x coefficients in the denominator and to check on cross-multiplication if the x^2 term vanishes. If not, this leads to quadratic equation.

6) If the transposition followed by cross-multiplication does not result in getting the x coefficients cancelled on both sides, then one can try the different transposition and proceed further as given previously.

Otherwise the given problem does not come under complex mergers.

The examples dealt with are self explanatory for the above rule

10) **In complex merger** there is another form:

$$\frac{ax+b}{cx+d} + \frac{ex+f}{gx+h} = \frac{px+q}{rx+s} + \frac{lx+m}{nx+p}$$

This can be solved by applying Paravartya division and then bringing it to the first standard form (9) of complex merger and then proceed as explained earlier.

This method can be applied in general for a complex merging of any degree polynomial divided by any degree either equivalent or less than the numerator. By applying Paravartya division as many times as the degree of the Polynomial in the numerator, followed by the remainder method, which is described earlier, it can be solved.

In complex merger, merging is of different type than the one that is already used. But the Paravartya (Transposition) can be taken effectively a merging. The steps followed are as given above

Eg. (i) Solve $\frac{6}{3x+4}-\frac{4}{2x+5}=\frac{3}{3x-10}-\frac{1}{x-1}$

Current Method

$$\frac{6}{3x+4}-\frac{4}{2x+5}=\frac{3}{3x-10}-\frac{1}{x-1}$$

$$\frac{12x+30-12x-16}{6x^2+8x+15x+20}=\frac{3x-3-3x+10}{3x^2-10x-x+10}$$

$$\frac{14}{6x^2+23x+20}=\frac{7}{3x^2-11x+10}$$

$$\frac{2}{6x^2+23x+20}=\frac{1}{3x^2-13x+10}$$

$$6x^2-26x+20=6x^2+23x+20$$

$$49x=0$$

$$x=0$$

Vedic Method

$$\frac{6}{3x+4}-\frac{4}{2x+5}=\frac{3}{3x-10}-\frac{1}{x-1}$$

Or transposing

$$\frac{6}{3x+4}+\frac{1}{x-1}=\frac{4}{2x+5}+\frac{3}{3x-10}$$

$$\frac{6}{3}+\frac{1}{1}=\frac{4}{2}+\frac{3}{3}$$

By taking LCM, we make coefficients of x equal in the denominators.

$$\frac{12}{6x+8}-\frac{12}{6x+15}=\frac{6}{6x-20}-\frac{6}{6x-6}$$

Considering LCM on both sides separately

$$\frac{84}{(6x+8)(6x+15)}=\frac{84}{(6x-20)(6x-6)}$$

Numerators being equal, $D_1D_2=D_3D_4$

$(6x+8)(6x+15)=(6x-20)(6x-6)$

$x=0$ (Since the product of constant terms on both sides is equal, i.e., $8\times15=20\times6$, refer 2 corollary).

Eg.(ii) Solve $\frac{8}{6x+3}+\frac{2}{3x+8}=\frac{2}{3x+1}+\frac{4}{3x+5}$

Current Method

8

6x + 3 3x + 8 3x + 1 3x + 5

$$\frac{8}{6x+3}-\frac{4}{3x+5}=\frac{2}{3x+1}-\frac{2}{3x+8}$$

$$4(\frac{2}{6x+3}\quad\frac{1}{3x+5})=2(\frac{1}{3x+1}\quad\frac{1}{3x+8})$$

$$\frac{4(6x+10-6x-3)}{(6x+3)(3x+5)}=\frac{2(3x+8-3x-1)}{(3x+1)(3x+8)}$$

$$\frac{7\times 4}{18x^2+39x+15}=\frac{2\times 7}{9x^2+27x+8}$$

$$\frac{2}{18x^2+39x+15}=\frac{1}{9x^2+27x+8}$$

$18x^2 + 54x + 16 = 18x^2 + 39x + 15$

$15x = -1$ or $x = -1/15$

Vedic Method

8

6x + 3 3x + 8 3x + 1 3x + 5

$$\frac{8}{6}+\frac{2}{3}=\frac{2}{3}+\frac{4}{3}$$

$N_1D_2 + N_2D_1 \neq N_3D_4 + N_4D_3$ For the cancellation of x terms, transpose one term each from one side to another such that by cross-multiplication x coefficients get cancelled on both sides.

$$\frac{8}{6x+3}-\frac{4}{3x+5}=\frac{2}{3x+1}-\frac{2}{3x+8}$$

By taking LCM, we make coefficients of x equal in the denominators.

8 8

6x + 3 6x + 10 6x + 2 6x + 16

56 56

(6x + 3)(6x + 10) (6x + 2)(6x + 16)

Numerators being equal, $D_1D_2 = D_3D_4$

$(6x + 3)(6x + 10) = (6x + 2)(6x + 16)$ (of form 2).

$$6x = \frac{32-30}{3+10-2-16}\quad\frac{2}{-5}$$

$$x=\frac{2}{-5\times 6}=\frac{-1}{15}$$

Eg.(iii) Solve $\frac{2x-5}{x-3}+\frac{9x+24}{3x+7}=\frac{3x+4}{x+1}+\frac{6x-7}{3x-5}$

Current Method

$$\frac{2x-5}{x-3}-\frac{3x+4}{x+1}=\frac{6x-7}{3x-5}-\frac{9x+24}{3x+7}$$

$$\frac{(2x-5)(x+1)-(3x+4)(x-3)}{(x-3)(x+1)}$$

$$=\frac{(6x-7)(3x+7)-(9x+24)(3x-5)}{(3x-5)(3x+7)}$$

$$\frac{(2x^2-3x-5)-(3x^2-5x-12)}{x^2-2x-3}$$

$$=\frac{(18x^2+21x-49)-(27x^2+27x-120)}{9x^2+6x-35}$$

$$\frac{-x^2+2x+7}{x^2-2x-3}=\frac{-9x^2-6x+71}{9x^2+6x-35}$$

$(-x^2+2x+7)(9x^2+6x-35)=(-9x^2-6x+71)(x^2-2x-3)$

$-9x^4+18x^3+63x^2-6x^3+12x^2+42x+35x^2-70x-245$

$=-9x^4-6x^3+71x^2+18x^3+12x^2-142x+27x^2+18x-213$

$98x^2-28x-245=98x^2-124x-213$

$96x=32$

$x=32/96=1/3$

Vedic Method

$$\frac{2x-5}{x-3}+\frac{9x+24}{3x+7}=\frac{3x+4}{x+1}+\frac{6x-7}{3x-5}\text{ .... E}$$

$$\frac{2}{1}+\frac{9}{3}=\frac{3}{1}+\frac{6}{3}$$

Therefore, By Paravartya Division,

$$2+\frac{1}{x-3}+3+\frac{3}{3x+7}=3+\frac{1}{x+1}+2+\frac{3}{3x-5}$$

$$\frac{1}{x-3}+\frac{3}{3x+7}=\frac{1}{x+1}+\frac{3}{3x-5}\quad\text{——(1)}$$

This equation can be solved by method 1 given below or by complex merging (method 2).

Method 1: for solution of (1)

$N_1D_2+N_2D_1=6x-2$

$N_3D_4+N_4D_3=6x-2$

$N_1D_2+N_2D_1=N_3D_4+N_4D_3$

∴ By Sunyam Samya Samuccaye this relation is Samyam and hence is zero.

Therefore, $6x-2=0$; $x=1/3$.

Method 2: From stage (1) (complex merging by transposing) aiming at cancellation of x. The given equation has to be brought to stag (1) to consider the solution by complex merging.

$$\frac{1}{x-3}-\frac{3}{3x-5}=\frac{1}{x+1}-\frac{3}{3x+7} \text{------ (2)}$$

(L.H.S) (R.H.S)

$\frac{1}{1}=\frac{3}{3}$ $\frac{1}{1}=\frac{3}{3}$ (For cancellation of x)

On taking LCM each side separately in the final derived equation,

$$\frac{4}{(x-3)(3x-5)}=\frac{4}{(x+1)(3x+7)}$$

Numerators on both sides are equal. Therefore, we can proceed directly, as

$D_1D_2 = D_3D_4$

$(x-3)(3x-5) = (x+1)(3x+7)$

$3x^2 - 14x + 15 = 3x^2 + 10x + 7$

$8 = 24x$

$x = 1/3$

Method 3: From the stage (2) obtained by transposition of stage (1) an then aiming at the equality of the coefficient of x in the denominator.

By taking LCM, we make coefficients of x equal in the denominators

$$\frac{3}{3x-9}-\frac{3}{3x-5}=\frac{3}{3x+3}-\frac{3}{3x+7}$$

on taking LCM each side separately,

$$\frac{12}{(3x-9)(3x-5)}=\frac{12}{(3x+3)(3x+7)}$$ Final derived equation

D_1 D_2 D_3 D_4

Now, Numerators are equal. Therefore, $D_1D_2 = D_3D_4$
$(3x - 9)(3x - 5) = (3x + 3)(3x + 7)$

$$3x = \frac{21-45}{-9-5-3-7} = \frac{-24}{-24} = 1$$

$$x = 1/3$$

Eg. (iv) Solve $\frac{0x - 3x - 1}{2x - 3} + \frac{2x \; + 3x}{2x -} \quad \frac{6x^2 - 11x - 4}{3x - 7} + \frac{12x^2 + 10x + 5}{3x + 1}$

Current Method

$$\frac{10x^2 - 13x - 1}{2x - 3} + \frac{2x^2 + 3x}{2x - 1} = \frac{6x^2 - 11x - 4}{3x - 7} + \frac{12x^2 + 10x + 5}{3x + 1}$$

$$\frac{10x^2 - 13x - 1}{2x - 3} - \frac{12x^2 + 10x + 5}{3x + 1} = \frac{6x^2 - 11x - 4}{3x - 7} - \frac{2x^2 + 3x}{2x - 1}$$

$$\frac{(10x^2 - 13x - 1)(3x + 1) - (12x^2 + 10x + 5)(2x - 3)}{(2x - 3)(3x + 1)}$$

$$= \frac{(6x^2 - 11x - 4)(2x - 1) - (2x^2 + 3x)(3x - 7)}{(3x - 7)(2x - 1)}$$

Vedic Method

$$\frac{10x^2 - 13x - 1}{2x - 3} + \frac{2x^2 + 3x}{2x - 1} = \frac{6x^2 - 11x - 4}{3x - 7} + \frac{12x^2 + 10x + 5}{3x + 1}$$

By Paravartya Division

$$5x + 1 + \frac{2}{2x - 3} + x + 2 + \frac{2}{2x - 1} = (2x + 1) + \frac{3}{3x - 7} + (4x + 2) + \frac{3}{3x +}$$

$$\frac{2}{2x - 3} + \frac{2}{2x - 1} = \frac{3}{3x - 7} + \frac{3}{3x + 1} \quad \text{———} (1)$$

This equation can be solved by method (1) or metho (2) or by complex merging (method 3)

$$\frac{(30x^3-39x^2-3x+10x^2-13x-1)-(24x^3+20x^2+10x-36x^2-30x-15)}{6x^2-9x+2x-3}$$
$$=\frac{(12x^3-22x^2-8x-6x^2+11x+4)-(6x^3+9x^2-14x^2-21x)}{6x^2-14x-3x+7}$$

$$\frac{6x^3-13x^2-4x+14}{6x^2-7x-3}=\frac{6x^3-23x^2+24x+4}{6x^2-17x+7}$$

$(6x^3-13x^2+4x+14)(6x^2-17x+7)=(6x^3-23x^2+24x+4)(6x^2-7x-3)$
$6x^5-78x^4+24x^4+84x^2-102x^4+221x^3-68x^2-238x+42x^3-91x^2+28x+98$
$=6x^5-138x^4+144x^3+24x^2-42x^4+161x^3-168x^2-28x-18x^3+69x^2-72x-12$
$-180x^4+287x^3-75x^2-210x+98=-180x^4+287x^3-75x^2-100x-12$
$110=110x$ or $x=110/110=1$

Method 1:
$N_1D_2+N_2D_1=8x-8=8(x-1)$
$N_3D_4+N_4D_3=18x-18=18(x-1)$

$\therefore$ By Sunyam Samya Samuccaye, $x-1$ is the Samyam and hence is zero.

Therefore, $x-1=0$
$x=1$

Method 2: From stage (1)

By taking LCM, we make coefficients of x equal in the denominator

$$\frac{6}{6x-9}+\frac{6}{6x-3}=\frac{6}{6x-14}+\frac{6}{6x+2}$$

Numerators are equal.
$D_1+D_2=D_3+D_4=12x-12$

$\therefore$ By Sunyam Samya Samuccaye this relation is Samyam and hence zero.

$12x-12=0$
$x=1$

Method 3:

Transposing method aiming at cancellation of x. the given equation has to be brought to stage (1) to consider the solution by complex merging.

From stage (1). Transpose the equation

$$\frac{2}{2x-3} - \frac{3}{3x+1} = \frac{3}{3x-7} - \frac{2}{2x-1} \text{------ (2)}$$

$$\frac{2}{2} = \frac{3}{3} \qquad \frac{3}{3} = \frac{2}{2}$$

Numerators are equal on both sides after the final derived equation.

$\therefore (2x-3)(3x+1) = (3x-7)(2x-1)$

$6x^2 - 7x - 3 = 6x^2 - 17x + 7$

$10x = 10$

$x = 1$

Method 4: From stage (2).

By taking LCM, we make coefficients of x equal in the denominator

$$\frac{6}{6x-9} - \frac{6}{6x+2} = \frac{6}{6x-14} - \frac{6}{6x-3}$$

Numerators are equal

Therefore, $(6x-9)(6x+2) = (6x-14)(6x-3)$

$$6x = \frac{42+18}{-9+2+14+3} = \frac{60}{10} = 6$$

$\therefore x = 1$

11. Special Type of Simple Equations:

There are certain special types of equations, which are seeming cubic or biquadratic, but actually they can be reduced to simple equations if certain conditions are satisfied with them.

Special Type of Seeming "<u>Cubics</u>":

1) In case of a special type of seeming cubics of the type $(x - 2a)^3 + (x - 2b)^3 = 2(x - a - b)^3$ then $x = a + b$. One has to verify whether any given cubic equation is of the above form. This is very much helpful also when expressions containing literal coefficients are found to satisfy such a relation as above, as such the answer can be easily written down.

Considering LHS and RHS expression without cubes, if one finds the similarity as,

$(x - 2a) + (x - 2b) = 2(x - a - b)$.

Samyam is $x - a - b$.
Hence by Sunyam Samya Samuccaye Sutram
$x - a - b = 0 \therefore x = a + b$.

Proof:

$(x - 2a)^3 + (x - 2b)^3 = 2(x - a - b)^3$
$x^3 - 6x^2a + 12xa^2 - 8a^3 + x^3 - 6x^2b + 12xb^2 - 8b^3$
$= 2(x^3 - 3x^2a - 3x^2b + 3xa^2 + 3xb^2 + 6xab - a^3 - 3a^2b - 3ab^2 - b^3)$
$2x^3 - 6x^2a - 6x^2b + 12xa^2 + 12xb^2 - 8a^3 - 8b^3$
$= 2x^3 - 6x^2a - 6x^2b + 6xa^2 + 6xb^2 + 12xab - 2a^3 - 6a^2b - 6ab^2 - 2b$
$6xa^2 + 6xb^2 - 12xab = 6a^3 - 6a2^b - 6ab^2 + 6b^3$
$6x(a - b)^2 = 6(a + b)(a - b)^2$
$x = a + b$

Eg. Solve $(x \quad 4)^3 + (x - 6)^3 \quad 2(x - 5)^3$

Current Method

$(x - 4)^3 + (x - 6)^3 = 2(x - 5)^3$
$x^3 - 12x^2 + 48x - 64 + x^3 - 18x^2 + 108x - 216 = 2(x^3 - 15x^2 + 75x - 125$
$2x^3 - 30x^2 + 156x - 280 = 2x^3 - 30x^2 + 150x - 250$
$6x = 30$
$x = 30 / 6 = 5$

Vedic Method

$(x - 4)^3 + (x - 6)^3 = 2(x - 5)^3$
The given equation is in the above standard form where
$(x - 4) + (x - 6) = 2(x - 5)$
Therefore, $x - 5$ is Samyam and hence is zero.
$x = 5$

Eg. Solve $(x + b - a)^3 + (x + c - b)^3 = (2x + c - a)^3$

Current Method

$4[(x + b - a)^3 + (x + c - b)^3] = (2x + c - a)^3$

$4[x^3 + 3x^2(b - a) + 3x(b - a)^2 + (b - a)^3 + x^3 + 3x^2(c - b) + 3x(c - b)^2 + (c - b)^3]$

$= 8x^3 + 12x^2(c - a) + 6x(c - a)^2 + (c - a)^3$

$4(x^3 + 3x^2b - 3x^2a + 3xb^2 + 3xa^2 - 6xab + b^3 - 3b^2a + 3ba^2 - a^3 + x^3 + 3x^2c - 3x^2b + 3xc^2 + 3xb^2 - 6xcb + c^3 - 3c^2b + 3cb^2 - b^3$

$= 8x^3 + 12x^2c - 12x^2a + 6xc^2 + 6xa^2 - 12xca + c^3 - 3c^2a + 3ca^2 - a^3$

$4(2x^3 - 3x^2a + 6xb^2 + 3xa^2 - 6xab - 3b^2a + 3b^2a - a^3 + 3x^2c + 3xc^2 - 6xcb + c^3 - 3c^2b + 3cb^2)$

$= 8x^3 + 12x^2c - 12x^2a + 6xc^2 + 6xa^2 - 12xca + c^3 - 3ca^2 + 3ca^2 - a^3$

$8x^3 - 12x^2a + 24xb^2 + 12xa^2 - 24xab - 12b^2a + 12ba^2 - 4a^3 + 12x^2c + 12xc^2 - 24xcb + 4c^3 - 12c^2b + 12c^2b$

$= 8x^3 + 12x^2c - 12x^2a + 6xc^2 + 6xa^2 - 12xac + c^3 - 3c^2a + 3ca^2 - a^3$

$24xb^2 + 6xc^2 + 6xa^2 - 3a^3 + 3c^3 - 24xab - 24xbc + 12xac - 12b^2a + 12ba^2 - 12c^2b + 12cb^2 + 3c^2a - 3ca^2 = 0$

$x(24b^2 + 6c^2 + 6a^2 - 24ab - 24bc + 12ac) = 3a^3 - 3c^3 + 12b^2a - 12b^2a + 12c^2b - 12cb^2 - 3c^2a + 3ca^2$

$6x(4b^2 + c^2 + a^2 - 4ab - 4bc - 2ac)$

Vedic Method

$4[(x + b - a)^3 + (x + c - b)^3] = (2x + c - a)^3$

$(x + b - a)^3 + (x + c - b)^3 = \dfrac{(2x + c - a)^3}{4}$

$= \dfrac{\left(x + \frac{c - a}{2}\right)^3 \times 2^3}{4}$

$= 2\left(x + \frac{c - a}{2}\right)^3$

Now the given equation in the standard from

LHS = $x + b - a + x + c - b = 2x + c - a$

RHS = $2x + c - a$

$2x + c - a$ is Samyam and hence zero.

Therefore, $2x + c - a = 0$

The elegance of this method is highly striking.

$x = \dfrac{a - c}{2}$

$$= 3(a^3 - c^3 + 4b^2a - 4ba^2 + 4c^2b - 4cb^2 - c^2a + ca^2)$$

$$2x = \frac{a^3 - c^3 + 4b^2a - 4ba^2 + 4c^2b - 4cb^2 - c^2a + ca^2}{4b^2 + c^2 + a^2 - 4ab - 4bc + 2ac}$$

$$= \frac{(a-c)(a^2 + c^2 + ac) + 4b^2(a-c) + (a-c)(-4bc - 4ba) + ca(a-c)}{4b^2 + c^2 + a^2 - 4ab - 4bc + 2ac}$$

$$= \frac{(a-c)[a^2 + c^2 + 2ac + 4b^2 - 4bc - 4ab]}{4b^2 + c^2 + a^2 - 4abc - 4bc + 2ac}$$

$2x = a - c$

or $x = (a - c)/2$

Proof:

$$\frac{(x+a+d)^3}{(x+a+2d)^3} = \frac{x+a}{x+a+3d}$$

$(x + a + d)^3(x + a + 3d)$
$= (x + a)(x + a + 2d)^3$

By expansion on both sides,

$x^4 + x^3(4a + 6d) + x^2(16a^2 + 18ad + 12d^2) + x(4a^3 + 18a^2d + 24ad^2 + 10d^3) + (a^4 + 6a^3d + 12a^2d^2 + 12ad^3 + d^4)$
$= x^4 + x^3(4a + 6d) + x^2(16a^2 + 18ad + 12d^2) + x(4a^3 + 18a^2d + 4ad^2 + 8d^3) + (a^4 + 6a^3d + 12a^2d^2 + 8ad^3)$

Canceling common terms out, we have,
$x(10d^3) + 10ad^3 + 3d^4$
$= x(8d^3) + 8ad^3$
$2d^3x + 2ad^3 + 3d^4 = 0$
$2x + 2a + 3d = 0$
$x = -1/2(2a + 3d)$

Special Type of <u>Seeming Biquadratics:</u>

2) (a) In case of equations which are seeming biquadratics, they can be actually reduced into simple equations if the equation is of the following form:

$$\frac{\overset{N_1}{(x+a+d)^3}}{\underset{D_1}{(x+a+2d)^3}} = \frac{\overset{N_2}{(x+a)}}{\underset{D_2}{(x+a+3d)}}, \text{ where}$$

(i) N_2, N_1, D_1, D_2 are in A.P. which can also be confirmed by the relation $D_2 - N_2 = 3(D_1 - N_1)$ and

(ii) $N_1 + D_1$ (without cubes) $= N_2 + D_2$

Thus if $N_1 + D_1$ (without cubes) $= N_2 + D_2$, then Sunyam is applied to solve the simple equation (Sufficient condition).

$N_1 + D_1$ (without cubes)
$= 2x + 2a + 3d$

$N_2 + D_2 = 2x + 2a + 3d$

$N_1 + D_1 = N_2 + D_2$. This relation is Samyam and hence by Sunyam Samya Samuccaye it is zero.

$2x + 2a + 3d = 0$

$$x = \frac{-(2a+3d)}{2}$$

It can also be seen that the cross-addition of all the terms of N, D gives same value on both sides, at which stage Sunyam can be applied (corollary).

i.e., $(x + a + d) + (x + a + d) + (x + a + d) + (x + a + 3d) = 4x + 4a + 6d = 2(2x+2a + 3d)$

$(x + a + 2d) + (x + a + 2d) + (x + a + 2d) + (x + a) = 4x + 4a + 6d = 2(2x + 2a + 3d)$

Eg.(i) Solve $\frac{(x-6)^3}{(x-9)^3} = \frac{x-3}{x-12}$

Current Method

$$\frac{(x-6)^3}{(x-9)^3} = \frac{x-3}{x-12}$$

$(x-6)^3(x-12) = (x-3)(x-9)^3$
$(x-12)(x^3-18x^2+108x-216)$
$= (x-3)(x^3-27x^2+243x-729)$
$x^4-18x^3+108x^2-216x-12x^3+216x^2$
$-1296x+2592$
$= x^4-27x^3+243x^2-729x-3x^3+$
$81x^2-729x+2187$
$-30x^3+324x^2-1512x+2592$
$= -30x^3+324x^2-1458x+2187$
$405 = 54x$
$x = 405/54 = 15/2$

Vedic Method

$$\frac{(x-6)^3}{(x-9)^3} = \frac{x-3}{x-12}$$

$N_1 + D_1$ (without cubes) $= 2x - 15$
$N_2 + D_2 = 2x - 15$
$D_2 - N_2 = -9$, $3(D_1 - N_1) = -9$
(Test for A.P.)
$N_1 + D_2 = 3(x-6) + (x-12) = 4x - 30$
$= 2(2x-15)$
$N_2 + D_1 = 3(x-9) + (x-3) = 4x - 30$
$= 2(2x-15)$
Therefore, $2x - 15 = 0$
$x = 15/2$

(b) Second type of seeming biquadratics of the form $\frac{(x+a)(x+b)}{(x+e)(x+f)} = \frac{(x+g)(x+h)}{(x+c)(x+d)}$

i.e., $(x+a)(x+b)(x+c)(x+d)$
$= (x+e)(x+f)(x+g)(x+h)$

In Vedic method it is enough to test the following conditions to see if it can be reduced to simple equation. Clearly x^4 term is equal on both sides.

i) If cross-addition gives same total on both sides, i.e.,
$4x + a + b + c + d = 4x + e + f + g + h$
(which is Samyam).
From this it is seen that $a + b + c + d = e + f + g + h$.
The above condition is sufficient to see that x^3 term vanishes.

ii) In order to arrive at the condition that this leads to a simple equation, it should be shown that x^2 term also vanishes. Sum of each pair of binomials on one side is equal to the sum of some pair on the other side.

iii) In addition, one has to show also that $ab + cd = ef + gh$, for vanishing of x^2 term.

If all the above conditions are satisfied, then it leads to a simple equation with the condition that the cross-addition is same on both sides, at which stage Sunyam is applied to solve the equations.

i.e., $4x + a + b + c + d = 0$

$$\therefore x = \frac{-(a+b+c+d)}{4}$$

Eg.(i) Solve $\underset{a}{(x + 7)}\underset{b}{(x + 3)}\underset{c}{(x + 9)}\underset{d}{(x + 11)} = \underset{e}{(x + 4)}\underset{f}{(x + 6)}\underset{g}{(x + 8)}\underset{h}{(x + 12)}$

$$\frac{(x+a)(x+b)}{(x+e)(x+f)} = \frac{(x+g)(x+h)}{(x+c)(x+d)}$$

Current Method

$(x + 7)(x + 3)(x + 9)(x + 11)$
$= (x + 4)(x + 6)(x + 8)(x + 12)$
$(x^2 + 10x + 21)(x^2 + 20x + 99)$
$= (x^2 + 14x + 48)(x^2 + 16x + 48)$
$x^4 + 10x^3 + 21x^2 + 20x^3 + 200x^2 + 420x + 99x^2 + 990x + 2079$
$= x^4 + 14x^3 + 48x^2 + 16x^3 + 224x^2 + 768x + 48x^2 + 672x + 2304$
$30x^3 + 320x^2 + 1410x + 2079$
$= 30x^3 + 320x^2 + 1440x + 2304$
$0 = 30x + 225$
$x = -225 / 30 = -15 / 2$

Vedic Method

$(x + 7)(x + 3)(x + 9)(x + 11)$
$= (x + 4)(x + 6)(x + 8)(x + 12)$
We rewrite the given equation as
$(x + 7)(x + 3)(x + 9)(x + 11) = (x + 4)(x + 6)(x + 8)(x + 12)$ so that $a = 7$, $b = 3$, $c = 9$, $d = 11$, $e = 4$, $f = 6$, $g = 8$, $h = 12$.

(i) $a+b+c+d = e+f+g+h$
$7+3+9 + 11 = 4 + 6 + 8 + 12$

(ii) $(x + 3) + (x + 11) = (x + 6) + (x + 8)$;
$(x + 3) + (x + 7) = (x + 4) + (x + 6)$;
$(x + 3) + (x + 9) = (x + 4) + (x + 8)$;
$(x + 11) + (x + 7) = (x + 6) + (x + 12)$;
$(x + 11) + (x + 9) = (x + 8) + (x + 12)$;
$(x + 7) + (x + 9) = (x + 4) + (x + 12)$;

(iii) $21 + 99 = 24 + 96$

Then by Sunyam Samya Samuccaye,
$4x + a + b + c + d = 4x + e + f + g + h$
is the Samyam and hence zero.
$\therefore 4x + 30 = 0$
$x = -30 / 4 = -15 / 2$

12. Further extension of the Sutram:

There are some forms of the equations, which result finally in the form of a simple equation, if certain slight re-orientations are made in the original equation, such as splitting of the terms, re-distribution, addition of equal values, subtraction of equal values, completing a cyclic form and the like and then compounding of the terms. Some examples are given below.

Eg.(i) Solve $\frac{x+ab}{b}+\frac{x+bc}{c}+\frac{x+ca}{a}=a+b+c$

Current Method

$$\frac{x+ab}{b}+\frac{x+bc}{c}+\frac{x+ca}{a}=a+b+c$$

$$\frac{ac(x+ab)+ab(x+bc)+bc(x+ca)}{abc}=a+b+c$$

$$\frac{acx+a^2bc+abx+ab^2c+bcx+abc^2}{abc}=a+b+c$$

$acx+a^2bc+abx+ab^2c+bcx+abc^2=(a+b+c)abc$

$acx+a^2bc+abx+ab^2c+bcx+abc^2=a^2bc+ab^2c+abc^2$

$(ab+bc+ac)x=0$

$\therefore x=0$

the L.H.S can be written as

$$x\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{ab}{b}+\frac{bc}{c}+\frac{ca}{a}$$

$$=x\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+(a+b+c)$$

$\therefore x=0$

Vedic Method

$$\frac{x+ab}{b}+\frac{x+bc}{c}+\frac{x+ca}{a}=a+b+c$$

Distributing right hand side terms to the left by applying Adyamadyena formula, i.e., subtracting first term of RHS from the first, middle term from the middle and the last term from the last of LHS.

$$\frac{x+ab}{b}-a+\frac{x+bc}{c}-b+\frac{x+ac}{a}-c=0$$

$$\frac{x+ab-ab}{b}+\frac{x+bc-bc}{c}+\frac{x+ac-ac}{a}=0$$

$$\frac{x}{b}+\frac{x}{c}+\frac{x}{a}=0$$

x is Samyam (common)

$\therefore$ By Sunyam Samya Samuccaye Sutra

$x=0$

Eg.(ii) Solve $\frac{x-bc}{b+c}+\frac{x-ac}{a+c}+\frac{x-ab}{a+b}=\frac{x-a^2-bc}{b+c-a}+\frac{x-b^2-ac}{c+a-b}+\frac{x-c^2-ab}{a+b-c}$

Current Method

Taking LCM and working the details is highly complicated

Vedic Method

$$\frac{x-bc}{b+c}+\frac{x-ac}{a+c}+\frac{x-ab}{a+b}$$

$$\frac{x-a^2-bc}{b+c-a}+\frac{x-b^2-ac}{c+a-b}+\frac{x-c^2-ab}{a+b-c}$$

Subtracting 'a' from first term, 'b' from the second term and 'c' from the third term on both sides of the equation, we have

$$\frac{x-bc-ab-ca}{b+c}+\frac{x-ca-ab-bc}{a+c}$$

$$+\frac{x-ab-ca-bc}{a+b}=\frac{x-a^2-bc-ab-ca+a^2}{b+c-a}$$

$$+\frac{x-b^2-ac-bc-ab+b^2}{c+a-b}$$

$$+\frac{x-c^2-ab-ac-bc+c^2}{a+b-c}$$

$\therefore$ x − ab − bc − ac is Samyam (common)

$\therefore$ By Sunyam Samya Samuccaye Sutram, x − ab − bc − ac = 0

$\therefore$ x = ab + bc + ac

A few examples which have certain symmetrical (Cyclic) relations existing in the problem by applying reorientation of certain Symmetrical cyclic nature have been illustrated by Swamiji and they are as follows. These problems using current methods will have complicated working.

1) $$\frac{x+a}{b+c}+\frac{x+b}{c+a}+\frac{x+c}{a+b}=-3$$

Taking − 3 over from the R.H.S to the L.H.S and distributing it amongst the 3 terms there, we have :

$$\frac{x+a}{b+c}+1+\frac{x+b}{c+a}+1+\frac{x+c}{a+b}+1=0$$

On simplifying and by virtue of the Samuccaya rule, this whole working can be done at sight i.e. mentally.

x + a + b + c = 0 $\therefore$ x = − (a + b + c)

2) $$\frac{x+a}{b+c}+\frac{x+b}{c+a}+\frac{x+c}{a+b} \quad \frac{x+2a}{b+c-a}+\frac{x+2b}{c+a-b}+\frac{x+2c}{a+b-c}$$

Add unity to each of the 6 terms and observe the equality of Numerators as x + a + b + c,

$\therefore$ x + a + b + c = 0 $\therefore$ x = − (a + b + c)

3) $$\frac{x-a}{b+c}+\frac{x-b}{c+a}+\frac{x-c}{a+b}=\frac{x+a}{2a+b+c}+\frac{x+b}{2b+c+a}+\frac{x+c}{2c+a+b}$$

Subtract unity form each of the 6 terms; and we have:

x − a − b − c = 0 $\therefore$ x = (a + b + c) = 0

4) $$\frac{x+a^2}{(a+b)(a+c)} \quad \frac{x+b^2}{(b+c)(b+a)} \quad \frac{x+c^2}{(c+a)(c+b)}$$

$$= \frac{x-bc}{a(b+c)} + \frac{x-ca}{b(c+a)} + \frac{x-ab}{c(a+b)}$$

Subtracting 1 from each of the 6 terms, we have:

$x - ab - ac - bc = 0 \quad \therefore \; x = (ab + bc + ca)$

5) $$\frac{x+a^2+2c^2}{b+c} + \frac{x+b^2+2a^2}{c+a} + \frac{x+c^2+2b^2}{a+b} = 0$$

As $(b-c) + (c-a) + (a-b) = 0$, we add $(b-c)$, $(c-a)$ and $(a-b)$ to the first, second and third terms respectively; and we have :

$x + a^2 + b^2 + c^2 = 0 \quad \therefore \; x = -(a^2 + b^2 + c^2)$

6) $$\frac{ax+a(a^2+2bc)}{b-c} + \frac{bx+b(b^2+2ca)}{c-a} + \frac{cx+c(c^2+2ab)}{a-b} = 0$$

as $a(b-c) + b(c-a) + c(a-b) = 0$

$\therefore$ We add $a(b-c)$ to the first term, $b(c-a)$ to the second and $c(a-b)$ to the last; and we have :

$$t_1 = \frac{ax+a(a^2+2bc)}{b-c} + a(b-c) = \frac{a}{b-c}\,[x + (a^2 + b^2 + c^2)]$$

Similarly, $t_2 = \frac{b}{c-a}\,[x + (a^2 + b^2 + c^2)]$

and $t_3 = \frac{c}{a-b}\,[x + (a^2 + b^2 + c^2)] = 0$

$\therefore \; x + a^2 + b^2 + c^2 = 0 \qquad \therefore \; x = -(a^2 + b^2 + c^2)$

7) $$\frac{x+a^3+2b^3}{b-c} + \frac{x+b^3+2c^3}{c-a} + \frac{x+c^3+2a^3}{a-b}$$

$= 2a^2 + 2b^2 + 2c^2 + ab + ac + bc$

Splitting the R.H.S into $(b^2 + bc + c^2) + (c^2 + ca + a^2) + (a^2 + ab + b^2)$, transposing the three parts to the left and combining the first with the first, the second with the second and the third with the third (by way of application of the 'Adyamadyena' formula), we have:

$$t_1 = \frac{x+a^3+2b^3}{b-c} - (b^2+bc+c^2) = \frac{x+a^3+b^3+c^3}{b-c}$$

Similarly, $t_2 = \frac{\textit{the same } N}{c-a}$

and $t_3 = \frac{\textit{the same } N}{a-b}$

$\therefore \; x = -(a^3 + b^3 + c^3)$

Proof:

$$\frac{p}{(x+a)(x+b)}+\frac{q}{(x+b)(x+c)}+\frac{r}{(x+c)(x+a)}$$

$$\frac{p(x+c)+q(x+a)+r(x+b)}{(x+a)(x+b)(x+c)}=0$$

$$x(p+q+r)+pc+qa+rb=0$$

Therefore, $x=\dfrac{-(pc+qa+rb)}{p+q+r}$

13) Miscellaneous Simple Equations:

First Type:

a) If the equation is in the form of sum of series of terms whose denominators are products of expressions in a cyclic form,

$$\frac{p}{(x+a)(x+b)}+\frac{q}{(x+b)(x+c)}+\frac{r}{(x+c)(x+a)}=0$$

The procedure is that each numerator is multiplied by the missing factor from the corresponding denominators.

After multiplying, the sum of the numerators is equated to zero to give the solution.

$$p(x+c)+q(x+a)+r(x+b)=0$$

By Paravartya

$$x=\frac{-pc-qa-rb}{p+q+r}$$

$$x=\frac{\text{Each N multipliedby the absent number reversed}}{N_1+N_2+N_3}$$

Eg.(i) Solve $\dfrac{3}{(x+5)(x-3)}+\dfrac{2}{(x-3)(x+1)}+\dfrac{5}{(x+1)(x+5)}=0$

Current Method

$$\frac{3}{(x+5)(x-3)}+\frac{2}{(x-3)(x+1)}+\frac{5}{(x+1)(x+5)}=0$$

$$\frac{3(x+1)+2(x+5)+5(x-3)}{(x+5)(x-3)(x+1)}=0$$

$$3x+3+2x+10+5x-15=0$$

$$10x-2=0$$

$$x=2/10=1/5$$

Vedic Method

$$\frac{3}{(x+5)(x-3)}+\frac{2}{(x-3)(x+1)}+\frac{5}{(x+1)(x+5)}=0$$

$$x=\frac{-3-10+15}{10}=\frac{1}{5}.$$

b) If the denominator occurs in quadratic expression form, then it can be factorized using Adyamadyena Antyamantyena VII

if it can be of the form given in 13 (a) then the same procedure is adopted for example: if

$$\frac{3}{x^2+2x-15}+\frac{2}{x^2-2x-3}+\frac{5}{x^2+6x+5}=0$$

(Ref. lecture notes on the section of Quadratic equations for the factorization)

Eg.(ii) Solve $\frac{1}{2x^2+3x+1}+\frac{3}{6x^2+5x+1}+\frac{5}{3x^2+4x+1}=0$

Current Method

$$\frac{1}{2x^2+3x+1}+\frac{3}{6x^2+5x+1}+\frac{5}{3x^2+4x+1}=0$$

$$\frac{1}{(x+1)(2x+1)}+\frac{3}{(2x+1)(3x+1)}+\frac{5}{(3x+1)(x+1)}=0$$

$$\frac{3x+1+3(x+1)+5(2x+1)}{(x+1)(2x+1)(3x+1)}=0$$

$3x+1+3x+3+10x+5=0$
$16x+9=0$
$x=-9/16$

Vedic Method

$$\frac{1}{2x^2+3x+1}+\frac{3}{6x^2+5x+1}+\frac{5}{3x^2+4x+1}=0$$

$$\frac{1}{(x+1)(2x+1)}+\frac{3}{(2x+1)(3x+1)}+\frac{5}{(3x+1)(x+1)}=0$$

$3x+1+3x+3+10x+5=0$
$16x+9=0$
$x=-9/16$

Eg.(iii) Solve $\frac{x+1}{(x-1)(x+2)}+\frac{x+5}{(x+2)(x+3)}+\frac{x+2}{(x+3)(x-1)}=\frac{3}{x}$

Current Method

$$\frac{x+1}{(x-1)(x+2)}+\frac{x+5}{(x+2)(x+3)}+\frac{x+2}{(x+3)(x-1)}=\frac{3}{x}$$

$$\frac{(x+1)(x+3)+(x+5)(x-1)+(x+2)^2}{(x-1)(x+2)(x+3)}=\frac{3}{x}$$

Vedic Method

Paravartya of x and Redistribution of 3 in RHS

$$\frac{x+1}{(x-1)(x+2)}+\frac{x+5}{(x+2)(x+3)}+\frac{x+2}{(x+3)(x-1)}=\frac{3}{x}$$

$$\frac{x^2+x}{(x-1)(x+2)}-1+\frac{x^2+5x}{(x+2)(x+3)}-1+\frac{x^2+2x}{(x+3)(x-1)}-1=0$$

$$\frac{x^2+4x+3+x^2+4x-5+x^2+4x+4}{(x^2+x-2)(x+3)}=\frac{3}{x}$$

$$\frac{3x^2+12x+2}{x^3+4x^2+x-6}=\frac{3}{x}$$

$$3x^3+12x^2+2x=3x^3+12x^2+3x-18$$

$$x-18=0$$

$$x=18$$

$$\frac{2}{(x-1)(x+2)}+\frac{-6}{(x+2)(x+3)}+\frac{3}{(x+3)(x-1)}=0$$

$$x=\frac{2\times(-3)+(-6\times1)+(3)(-2)}{N_1+N_2+N_3}$$

$$=\frac{-(6+6+6)}{2-6+3}=18$$

Eg.(iv) Solve $\frac{10x+3}{(2x-1)(5x-1)}+\frac{55x+39}{(5x-1)(11x-1)}+\frac{22x+9}{(11x-1)(2x-1)}=\frac{3}{x-1}$

Current Method

$$\frac{10x+3}{(2x-1)(5x-1)}+\frac{55x+39}{(5x-1)(11x-1)}+\frac{22x+9}{(11x-1)(2x-1)}=\frac{3}{x-1}$$

$$\frac{(10x+3)(11x-1)+(55x+39)(2x-1)+(22x+9)(5x+1)}{(2x-1)(5x-1)(11x-1)}=\frac{3}{x-1}$$

$$\frac{110x^2+23x-3+110x^2+23x-39+110x^2+23x-9}{(10x^2-7x+1)(11x-1)}=\frac{3}{x-1}$$

$$\frac{330x^2+69x-51}{110x^3-87x^2+18x-1}=\frac{3}{x-1}$$

$$\frac{110x^2+23x-17}{110x^3-87x^2+18x-1}=\frac{1}{x-1}$$

$$110x^3-87x^2-40x+17=110x^3-87x^2+18x-1$$

$$18=58x$$

$$x=18/58=9/29$$

Vedic Method

$$\frac{10x+3}{(2x-1)(5x-1)}+\frac{55x+39}{(5x-1)(11x-1)}+\frac{22x+9}{(11x-1)(2x-1)}=\frac{3}{x-1}$$

By Paravartya and redistribution of right hand term

$$\frac{(10x+3)(x-1)}{(2x-1)(5x-1)}-1+\frac{(55x+39)(x-1)}{(5x-1)(11x-1)}-1+\frac{(22x+9)(x-1)}{(11x-1)(2x-1)}-1=0$$

$$\frac{10x^2-7x-3}{(2x-1)(5x-1)}-1+\frac{55x^2-16x-39}{(5x-1)(11x-1)}-1+\frac{22x^2-13x-9}{(11x-1)(2x-1)}-1=0$$

$$\frac{-4}{(2x-1)(5x-1)}+\frac{-40}{(5x-1)(11x-1)}+\frac{-10}{(11x-1)(2x-1)}=0$$

$$\frac{-2}{(2x-1)(5x-1)}+\frac{-20}{(5x-1)(11x-1)}+\frac{-5}{(11x-1)(2x-1)}=0$$

$$x=27/87=9/29$$

13) **Second Type:**

If the equation is in the form of $\frac{1}{AB}+\frac{1}{AC}=\frac{1}{AD}+\frac{1}{BC}$, where A, B, C, D are in A.P.

Here the application of Sopanthyadvayamantyam is considered. In the given equation first identify A, B, C, D and test for existence of A.P. In such a case the answer can be written as L + 2P, where L is the last one and P is penultimate in AP.

Sopantyadvayam means two times the penultimate (2p)

Antyam means Last term (L) here in the AP.

Proof:

$$\frac{1}{AB}+\frac{1}{AC}=\frac{1}{AD}+\frac{1}{BC}$$

$$\frac{1}{A(A+d)}+\frac{1}{A(A+2d)}=\frac{1}{A(A+3d)}+\frac{1}{(A+d)(A+2d)}$$

$$\frac{1}{A(A+d)}-\frac{1}{A(A+3d)}=\frac{1}{(A+d)(A+2d)}-\frac{1}{A(A+2d)}$$

$$\frac{1}{A}[\frac{2d}{(A+d)(A+3d)}]=\frac{1}{A+2d}[\frac{-d}{A(A+d)}]$$

$$\frac{2}{A+3d}=\frac{-1}{A+2d}$$

$$\frac{2}{L}=\frac{-1}{P}$$

Therefore, L + 2P = 0

Eg.(i) Solve $\frac{1}{(x+3)(x+5)}+\frac{1}{(x+3)(x+7)}=\frac{1}{(x+3)(x+9)}+\frac{1}{(x+5)(x+7)}$

Current Method

$$\frac{1}{(x+3)(x+5)}+\frac{1}{(x+3)(x+7)}=\frac{1}{(x+3)(x+9)}+\frac{1}{(x+5)(x+7)}$$

$$\frac{1}{(x+3)(x+5)}-\frac{1}{(x+3)(x+9)}=\frac{1}{(x+5)(x+7)}-\frac{1}{(x+7)(x+3)}$$

$$\frac{x+9-x-5}{(x+3)(x+5)(x+9)}=\frac{x+3-x-5}{(x+3)(x+5)(x+7)}$$

$$\frac{4}{x+9}=\frac{-2}{x+7}$$

Vedic Method

$$\frac{1}{(x+3)(x+5)}+\frac{1}{(x+3)(x+7)}=\frac{1}{(x+3)(x+9)}+\frac{1}{(x+5)(x+7)}$$

Here A = x + 3, B = x + 5, C = x + 7, D = x + 9 are in A.P. Hence by the sutra.

L = D = x + 9 2p = 2(x + 7)

L + 2p = 0; 3x + 23 = 0

$$x=\frac{-23}{3}$$

$4x + 28 = -2x - 18$
$6x = -46$
$x = -46/6 \Rightarrow x = -23/3$

Eg.(ii) Solve $\frac{1}{2x^2+5x+3} + \frac{1}{3x^2+8x+5} = \frac{1}{4x^2+11x+7} + \frac{1}{6x^2+19x+15}$

Current Method

$$\frac{1}{2x^2+5x+3} + \frac{1}{3x^2+8x+5}$$
$$= \frac{1}{4x^2+11x+7} + \frac{1}{6x^2+19x+15}$$

$$\frac{1}{(x+1)(2x+3)} + \frac{1}{(x+1)(3x+5)}$$
$$= \frac{1}{(x+1)(4x+7)} + \frac{1}{(2x+3)(3x+5)}$$

$$\frac{1}{(x+1)(2x+3)} - \frac{1}{(x+1)(4x+7)}$$
$$= \frac{1}{(2x+3)(3x+5)} - \frac{1}{(x+1)(3x+5)}$$

$$\frac{4x+7-2x-3}{(x+1)(2x+3)(4x+7)} = \frac{x+1-2x-3}{(2x+3)(3x+5)(x+1)}$$

$$\frac{2x+4}{4x+7} = \frac{-x-2}{3x+5}$$

$2(x+2)(3x+5) = -1(x+2)(4x+7)$
$2(3x+5) = -(4x-7)$
$10x = -17$
$x = -17/10$

Vedic Method

$$\frac{1}{2x^2+5x+3} + \frac{1}{3x^2+8x+5}$$
$$= \frac{1}{4x^2+11x+7} + \frac{1}{6x^2+19x+15}$$

$$\frac{1}{(x+1)(2x+3)} + \frac{1}{(x+1)(3x+5)}$$

A B A C

$$= \frac{1}{(x+1)(4x+7)} + \frac{1}{(2x+3)(3x+5)}$$

A D B C

$L + 2P = 0$
$(4x + 7) + 2(3x + 5) = 0$
$x = -\frac{17}{10}$

Proof:

$$\frac{AC+D}{BC+E} = \frac{A}{B}$$

$ABC + AE = ABC + BD$
Therefore, $AE = BD$
$$\frac{A}{B} = \frac{D}{E}$$

13) <u>Third Type:</u> (Application of Upasutram Antyayoreva)

If the equation is in the form of $\frac{AC+D}{BC+E} = \frac{A}{B}$, i.e., if the left-hand side expression barring its independent term has the same ratio as the right hand side, then that ratio is equal to ratio of the last terms, independent terms (D/E).

Eg.(i) Solve $\frac{2x^2+3x+6}{x^2+5x+8} \quad \frac{2x+3}{x+5}$

Current Method

$$\frac{2x^2+3x+6}{x^2+5x+8}=\frac{2x+3}{x+5}$$

$(2x^2 + 3x + 6)(x + 5)$
$= (2x + 3)(x^2 + 5x + 8)$
$2x^3 + 3x^2 + 6x + 10x^2 + 15x + 30$
$= 2x^3 + 3x^2 + 10x^2 + 15x + 16x + 24$
$6x + 30 = 16x + 24$
$x = 6 / 10 = 3 / 5$

Vedic Method

$$\frac{2x^2+3x+6}{x^2+5x+8}=\frac{2x+3}{x+5}$$

$$\frac{x(2x+3)+6}{x(x+5)+8}=\frac{2x+3}{x+5}$$

$$\frac{2x+3}{x+5}=\frac{6}{8} \text{or } x=\frac{30-24}{16-6}=\frac{6}{10} \quad \frac{3}{5}$$

Eg.(ii) Solve $\frac{ax^2+bx+ab}{cx^2+dx+cd}=\frac{ax+b}{cx+d}$

Current Method

$$\frac{ax^2+bx+ab}{cx^2+dx+cd}=\frac{ax+b}{cx+d}$$

$(ax^2 + bx + ab)(cx + d)$
$= (ax + b)(cx^2 + dx + cd)$
$acx^3 + bcx^2 + abcx + adx^2 + bdx + abd$
$= acx^3 + bcx^2 + adx^2 + bdx + acdx + bcd$
$abcx + abd = acdx - bcd$
$(abc - acd)x = bcd - abd$

$$x=\frac{bd(c-a)}{ac(b-d)}$$

Vedic Method

$$\frac{ax^2+bx+ab}{cx^2+dx+cd}=\frac{ax+b}{cx+d}$$

$$\frac{x(ax+b)+ab}{x(cx+d)+cd}=\frac{ax+b}{cx+d}$$

Therefore, $\frac{ax+b}{cx+d}=\frac{ab}{cd}$

$$x=\frac{abd-bcd}{acd-abc}=\frac{bd(a-c)}{ac(d-b)}$$

Eg.(iii) Solve $\frac{1-3x}{1-5x} \quad \frac{2+x-3x^2}{7+x-5x^2}$

Current Method

$$\frac{1-3x}{1-5x}=\frac{2+x-3x^2}{7+x-5x^2}$$

$(1 - 3x)(7 + x - 5x^2)$
$= (1 - 5x)(2 + x - 3x^2)$
$7 - 21x + x - 3x^2 - 5x^2 + 15x^3$
$= 2 - 10x + x - 5x^2 - 3x^2 + 15x^3$
$7 - 21x = 2 - 10x$
$5 = 11x$ or $x = 5 / 11$

Vedic Method

$$\frac{1-3x}{1-5x}=\frac{2+x-3x^2}{7+x-5x^2}$$

$$\frac{1-3x}{1-5x}=\frac{x(1-3x)+2}{x(1-5x)+7}$$

$$\frac{1-3x}{1-5x}=\frac{2}{7}$$

$$x=\frac{2-7}{-21+10}=\frac{-5}{-11} \quad 11$$

14) Summation of Series:

Proof: (for the Vedic Method)

$$s_3 = \frac{1}{(x+a)(x+a+d)} \quad \frac{1}{(x+a+d)(x+a+2d)}$$

$$+\frac{1}{(x+a+2d)(x+a+3d)}$$

$$= \frac{1}{x+a+d}\left[\frac{1}{x+a} + \frac{1}{x+a+2d}\right]$$

$$\frac{1}{(x+a+2d)(x+a+3d)}$$

$$= \frac{1}{x+a+d}\left[\frac{2x+2a+2d}{(x+a)(x+a+2d)}\right]$$

$$\frac{1}{(x+a+2d)(x+a+3d)}$$

$$\frac{2(x+a+d)}{(x+a+d)(x+a)(x+a+2d)}$$

$$+\frac{1}{(x+a+2d)(x+a+3d)}$$

$$\frac{}{(x+a)(x+a+2d)} \quad \frac{1}{(x+a+2d)(x+a+3d)}$$

$$= \frac{1}{(x+a+2d)}\left[\frac{2}{x+a} + \frac{1}{x+a+3d}\right]$$

$$x+a+2d \quad \frac{3x+3a+6d}{+a)(x+a+3d)}\Bigg]$$

$$\frac{3(x+a+2d)}{(x+a+2d)(x+a)(x+a+3d)}$$

$$S_3 = \frac{}{(x+3)(x+9)}$$

Vedic Method

In connection with a special type of summation of series addition of fractions, two types are dealt with here, where the Antyayoreva Sutram is applicable.

First Type:

Summation of series where the factors of the denominators are in A.P. and numerators should be equal. An example is given below.

$$\frac{1}{(x+3)(x+5)} \quad \frac{1}{(x+5)(x+7)} \quad \frac{1}{(x+7)(x+9)}$$

It is clearly seen that the factors in the denominators are in A.P., i.e., (x + 3), (x + 5), (x + 7), (x + 9) are in A.P. It is also clear that the cycle of the A.P. of the factors of the different terms is followed. In such a case, subject to the above condition the summation of series is represented by S_3 or S_n in case of n terms = $\frac{\sum N(\text{Sum of Numerators})}{(\text{First Factor})(\text{Last Factor})}$ of the total series, which is by Antyayoreva, i.e., the terms lying only at the ends.

This is the result of application of Antyayoreva Sutram.

In the above example,

The common difference of A.P. may be either a number or in x or both. This can be extendable to numbers as well.

It stands the proof and hence the elegance with which the answer can be simply written down is exemplary. Some examples are given below:

Eg.(i) Find S_3 in the given summation of series: $\frac{1}{(x+2)(x+5)}+\frac{1}{(x+5)(x+8)}+\frac{1}{(x+8)(x+11)}+......$

Current Method

$$S_3=\frac{1}{(x+2)(x+5)}+\frac{1}{(x+3)(x+8)}+\frac{1}{(x+8)(x+11)}$$

$$=\frac{1}{x+5}\left[\frac{1}{x+2}+\frac{1}{x+8}\right]+\frac{1}{(x+8)(x+11)}$$

$$=\frac{1}{(x+5)}\left[\frac{2x+10}{(x+2)(x+8)}\right]+\frac{1}{(x+8)(x+11)}$$

$$=\frac{2(x+5)}{(x+5)(x+2)(x+8)}+\frac{1}{(x+8)(x+11)}$$

$$=\frac{2}{(x+2)(x+8)}+\frac{1}{(x+8)(x+11)}$$

$$=\frac{1}{x+8}\left[\frac{2}{x+2}+\frac{1}{x+11}\right]=\frac{1}{x+8}\left[\frac{2x+22+x+2}{(x+2)(x+11)}\right]$$

$$=\frac{1}{x+8}\left[\frac{3x+24}{(x+2)(x+11)}\right]=\frac{3(x+8)}{(x+8)(x+2)(x+11)}=\frac{3}{(x+2)(x+11)}$$

Vedic Method

$$\frac{1}{(x+2)(x+5)}+\frac{1}{(x+5)(x+8)}+\frac{1}{(x+8)(x+11)}+......$$

$(x+2)(x+5)(x+8)(x+11)$ are in A.P. and numerators are eq

$$\therefore \text{By Antyayoreva } S_3=\frac{\text{Sum of Numerators}}{\text{(First Term)(Last Term)}}$$

$$\therefore S_3=\frac{3}{(x+2)(x+11)}$$

Eg.(ii) Find S_4 in the given summation of series. $\frac{1}{x^2+3x+2}+\frac{1}{x^2+5x+6}+......$

Current Method

$$S_4=\frac{1}{x^2+3x+2}+\frac{1}{x^2+5x+6}+\frac{1}{x^2+7x+12}+\frac{1}{x^2+9x+20}$$

$$=\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x+3)}+\frac{1}{(x+3)(x+4)}+\frac{1}{(x+4)(x+5)}$$

$$=\frac{1}{x+2}\left[\frac{1}{x+1}+\frac{1}{x+3}\right]+\frac{1}{x+4}\left[\frac{1}{x+3}+\frac{1}{x+5}\right]$$

$$=\frac{1}{x+2}\left[\frac{2x+4}{(x+1)(x+3)}\right]+\frac{1}{x+4}\left[\frac{2x+8}{(x+3)(x+5)}\right]$$

$$=\frac{2(x+2)}{(x+2)(x+1)(x+3)}+\frac{2(x+4)}{(x+4)(x+3)(x+5)}$$

$$\frac{2}{(x+1)(x+3)}+\frac{2}{(x+3)(x+5)}$$

$$=\frac{2}{x+3}\left[\frac{1}{x+1}+\frac{1}{x+5}\right]$$

$$=\frac{2}{x+3}\left[\frac{2x+6}{(x+1)(x+5)}\right]$$

$$=\frac{4(x+3)}{(x+3)(x+1)(x+5)}=\frac{4}{(x+1)(x+5)}$$

Vedic Method

$$S_4=\frac{1}{x^2+3x+2}+\frac{1}{x^2+5x+6}+\frac{1}{x^2+7x+12}+\frac{1}{x^2+9x+20}$$

$$=\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x+3)}+\frac{1}{(x+3)(x+4)}+\frac{1}{(x+4)(x+5)}$$

Numerators are equal; $(x+1)(x+2)(x+3)(x+4)(x+5)$ are in A.P.

$\therefore$ By Antyayoreva $S_4=\frac{\text{Sum of Numerators}}{\text{(First Term)(Last Term)}}$

$$\therefore S_4=\frac{4}{(x+1)(x+5)}$$

Eg.(iii) Find S_3 $\frac{1}{(x+3)(2x+4)}+\frac{1}{(2x+4)(3x+5)}+\ldots$

Current Method

$$S_3=\frac{1}{(x+3)(2x+4)}+\frac{1}{(2x+4)(3x+5)}+\frac{1}{(3x+5)(4x+6)}$$

$$=\frac{1}{2x+4}\left[\frac{1}{x+3}+\frac{1}{3x+5}\right]+\frac{1}{(3x+5)(4x+6)}$$

$$=\frac{1}{2x+4}\left[\frac{4x+8}{(x+3)(3x+5)}\right]\frac{1}{(3x+5)(4x+6)}$$

$$=\frac{4(x+2)}{2(x+2)(x+3)(3x+5)}+\frac{1}{(3x+5)(4x+6)}$$

$$=\frac{2}{(x+3)(3x+5)}+\frac{1}{(3x+5)(4x+6)}$$

$$=\frac{1}{3x+5}\left[\frac{2}{x+3}+\frac{1}{4x+6}\right]=\frac{1}{3x+5}\left[\frac{9x+15}{(x+3)(4x+6)}\right].$$

$$=\frac{3(3x+5)}{(3x+5)(x+3)(4x+6)}=\frac{3}{2(x+3)(2x+3)}$$

Vedic Method

$$S_3=\frac{1}{(x+3)(2x+4)}+\frac{1}{(2x+4)(3x+5)}+\frac{1}{(3x+5)(4x+6)}$$

Numerators are equal

(x + 3), (2x + 4), (3x + 5), (4x + 6) are in A.P.

$\therefore$ <u>By Antyayoreva</u> $S_3=\frac{\text{Sum of Numerators}}{\text{(First Term)(Last Term)}}$

$$\therefore S_3=\frac{3}{(x+3)(4x+6)}=\frac{3}{2(x+3)(2x+3)}$$

Eg.(iv) Find S_3 $\frac{1}{(x+a+d)(3x+2a+5d)}+\frac{1}{(3x+2a+5d)(5x+3a+9d)}+\frac{1}{(5x+3a+9d)(7x+4a+13d)}+.....$

Current Method

$$S_3=\frac{1}{(x+a+d)(3x+2a+5d)}+\frac{1}{(3x+2a+5d)(5x+3a+9d)}+\frac{1}{(5x+3a+9d)(7x+4a+13d)}$$

$$\frac{1}{3x+2a+5d}\left[\frac{1}{x+a+d}+\frac{1}{5x+3a+9d}\right]+\frac{1}{(5x+3a+9d)(7x+4a+13d)}$$

$$=\frac{6x+4a+10d}{(3x+2a+5d)(x+a+d)(5x+3a+9d)}+\frac{1}{(5x+3a+9d)(7x+4a+13d)}$$

$$=\frac{2(3x+2a+5d)}{(3x+2a+5d)(x+a+d)(5x+3a+9d)}+\frac{1}{(5x+3a+9d)(7x+4a+13d)}$$

$$=\frac{2}{(x+a+d)(5x+3a+9d)}+\frac{1}{(5x+3a+9d)(7x+4a+13d)}$$

$$=\frac{1}{5x+3a+9d}\left[\frac{2}{x+a+d}+\frac{1}{7x+4a+13d}\right]$$

$$=\frac{1}{5x+3a+9d}\left[\frac{15x+9a+27d}{(x+a+d)(7x+4a+13d)}\right]$$

$$=\frac{3(5x+3a+9d)}{(5x+3a+9d)(x+a+d)(7x+4a+13d)}$$

$$=\frac{3}{(x+a+d)(7x+4a+13d)}$$

Vedic Method

$$\frac{1}{(x+a+d)(3x+2a+5d)}+\frac{1}{(3x+2a+5d)(5x+3a+9d)}+\frac{1}{(5x+3a+9d)(7x+4a+13d)}+$$

Numerators are equal

(x + a + d), (3x + 2a + 5d), (5x + 3a + 9d), (7x + 4a + 13d) are in A.P.

$$\therefore S_3=\frac{1}{(x+a+d)(3x+2a+5d)}+\frac{1}{(3x+2a+5d)(5x+3a+9d)}+\frac{1}{(5x+3a+9d)(7x+4a+13d)}$$

$\therefore$ By Antyayoreva $S_3=\frac{\text{Sum of Numerators}}{\text{(First Term)(Last Term)}}$

$$=\frac{3}{(x+a+d)(7x+4a+13d)}$$

Eg.(v) Find S_5 o: $\frac{1}{5\times7}+\frac{1}{7\times9}+\frac{1}{9\times11}+.....$

Vedic Method

$$S_5=\frac{1}{5\times7}+\frac{1}{7\times9}+\frac{1}{9\times11}+\frac{1}{11\times13}+\frac{1}{13\times15}$$

5, 7, 9, 11, 13, 15 are in A.P. with a common difference 2.

$$\therefore \text{By Antyayoreva } S_5=\frac{\text{Sum of Numerators}}{\text{(First Number)(Last Number)}}$$

$$\therefore S_5=\frac{5}{5\times15}=\frac{1}{15}$$

Second Type:

a) Second type of summation of series is also explained by the same Antyayoreva Sutram. The form is
$$\frac{a-b}{(x+a)(x+b)}+\frac{b-c}{(x+b)(x+c)}+\frac{c-d}{(x+c)(x+d)}+.....(1)$$
Numerator of each term is equal to the difference of the factors in the denominator of that term. The numerators, and denominators are in cyclic order.

b) Denominators may also contain coefficients of x not equal to 1, but should be same and should satisfy the above condition.

$$\frac{a-b}{(px+a)(px+b)}+\frac{b-c}{(px+b)(px+c)}+\frac{c-d}{(px+c)(px+d)}+......(2)$$

The summation is given by Antyayoreva, i.e.,

a) $\dfrac{a-d}{(x+a)(x+d)}$ ----- (1)

b) $\dfrac{a-d}{(px+a)(px+d)}$ -----(2)

Some examples are given below:

Eg. nd S he giv summation of series $\dfrac{1}{(x+2)(x+3)}+\dfrac{3}{(x+3)(x+6)}+\dfrac{4}{(x+6)(x+10)}+\dots$

Current Method

$$S_3=\frac{1}{(x+2)(x+3)}+\frac{3}{(x+3)(x+6)}+\frac{4}{(x+6)(x+}$$

$$\frac{1}{(x+3)}\left[\frac{1}{x+2}+\frac{3}{x+6}\right]+\frac{4}{(x+6)(x+10)}$$

$$\frac{1}{x+3}\left[\frac{4x+12}{(x+2)(x+6)}\right]+\frac{4}{(x+6)(x+10)}$$

$$\frac{4(x+3)}{(x+3)(x+2)(x+6)}+\frac{4}{(x+6)(x+10)}$$

$$\frac{4}{(x+2)(x+6)}+\frac{4}{(x+6)(x+10)}$$

Vedic Method

$$S_3=\frac{1}{(x+2)(x+3)}+\frac{3}{(x+3)(x+6)}+\frac{4}{(x+6)(x+10)}$$

$$=\frac{3-2}{(x+2)(x+3)}+\frac{6-3}{(x+3)(x+6)}+\frac{10-6}{(x+6)(x+10)}$$

It is in the standard form given in (1).
∴ By Antyayoreva Sutram

$$\therefore S_3=\frac{10-2}{(x+2)(x+10)}=\frac{8}{(x+2)(x+10)}$$

$$= \frac{4}{x+6}\left[\frac{1}{x+2}+\frac{1}{x+10}\right]$$

$$= \frac{4}{x+6}\left[\frac{(2x+12)}{(x+2)(x+10)}\right] \quad \frac{8(x+6)}{(x+6)(x+2)(x+10)}$$

$$= \frac{8}{(x+2)(x+10)}.$$

Eg.(ii) Find S_3 of the given summation of series $\frac{2}{(x+3)(x+5)}+\frac{5}{(x+5)(x+10)}+\frac{3}{(x+10)(x+13)}+\ldots$

Current Method

$$S = \frac{2}{(x+3)(x+5)}+\frac{5}{(x+5)(x+10)}+\frac{3}{(x+10)(x+13)}$$

$$\frac{1}{x+5}\left[\frac{2}{x+3}+\frac{5}{x+10}\right]+\frac{3}{(x+10)(x+13)}$$

$$\frac{1}{x+5}\left[\frac{7x+35}{(x+3)(x+10)}\right]+\frac{3}{(x+10)(x+13)}$$

$$\frac{7(x+5)}{(x+5)(x+3)(x+10)}+\frac{3}{(x+10)(x+13)}$$

$$\frac{7}{(x+3)(x+10)}+\frac{3}{(x+10)(x+13)}$$

$$\frac{1}{x+10}\left[\frac{7}{x+3}+\frac{3}{x+13}\right]=\frac{1}{x+10}\left[\frac{10x+100}{(x+3)(x+13)}\right]$$

Vedic Method

$$S_3 = \frac{2}{(x+3)(x+5)}+\frac{5}{(x+5)(x+10)}+\frac{3}{(x+10)(x+13)}$$

$$= \frac{5-3}{(x+3)(x+5)}+\frac{10-5}{(x+5)(x+10)}+\frac{13-10}{(x+10)(x+13)}$$

It is in the standard form given in (a).

∴ By Antyayoreva Sutram

$$S_3 = \frac{10}{(x+3)(x+13)}$$

$$= \frac{10(x+10)}{(x+10)(x+3)(x+13)} = \frac{10}{(x+3)(x+13)}$$

Eg.(iii) Find S_3 of the given summation of series $\frac{2}{(5x+2)(5x+4)}+\frac{5}{(5x+4)(5x+9)}+\frac{7}{(5x+9)(5x+16)}+\ldots$

Current Method

$$S_3=\frac{2}{(5x+2)(5x+4)}+\frac{5}{(5x+4)(5x+9)}+\frac{7}{(5x+9)(5x+16)}$$

$$=\frac{1}{5x+4}\left[\frac{2}{5x+2}+\frac{5}{5x+9}\right]+\frac{7}{(5x+9)(5x+16)}$$

$$=\frac{35x+28}{(5x+4)(5x+2)(5x+9)}+\frac{7}{(5x+9)(5x+16)}$$

$$=\frac{7(5x+4)}{(5x+4)(5x+2)(5x+9)}+\frac{7}{(5x+9)(5x+16)}$$

$$=\frac{7}{(5x+2)(5x+9)}+\frac{7}{(5x+9)(5x+16)}$$

$$=\frac{7}{5x+9}\left[\frac{1}{5x+2}+\frac{1}{5x+16}\right]=\frac{7}{5x+9}\left[\frac{10x+18}{(5x+2)(5x+16)}\right]$$

$$=\frac{14(5x+9)}{(5x+9)(5x+2)(5x+16)}$$

$$=\frac{14(5x+9)}{(5x+9)(5x+2)(5x+16)}$$

$$=\frac{14}{(5x+2)(5x+16)}$$

Vedic Method

$$S_3=\frac{2}{(5x+2)(5x+4)}+\frac{5}{(5x+4)(5x+9)}+\frac{7}{(5x+9)(5x+16)}$$

$$=\frac{4-2}{(5x+2)(5x+4)}+\frac{9-4}{(5x+4)(5x+9)}+\frac{16-9}{(5x+9)(5x+16)}$$

It is in the standard form given in (b).

$\therefore$ By Antyayoreva Sutram

$$S_3=\frac{16-2}{(5x+2)(5x+16)}$$

$$=\frac{14}{(5x+2)(5x+16)}$$

SECTION – 2

SIMULTANEOUS SIMPLE EQUATIONS

Vedic Method

15) General Method by Paravartya:

The usual simultaneous simple equations in two unknowns can be solved by the method Paravartya Sutram followed by cyclic operations.

$ax + by = c$

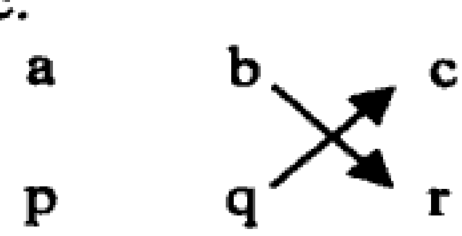
$px + qy = r$

The coefficient of x, coefficient of y and constant term are written for the two equations.

Coefficient of x	Coefficient of y	Constant term
a	b	c
p	q	r

Suppose $x = \frac{N_1}{D_1}, y = \frac{N_2}{D_2}$

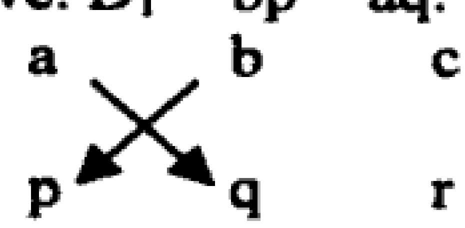
After this set up, to get the value of x, following the cyclic order by starting with y coefficients and the independent terms, cross-multiply forward rightward (i.e., starting from upper row and multiplying across by the lower one) and conversely and the connecting link between the two cross products is always minus. This is the numerator N_1, i.e., $N_1 = br - qc$.



Cross link is –ve

$\therefore N_1 = br - qc$

To get the value of denominator D_1 of x, following cyclic order by starting with y coefficient cross-multiply with x coefficient backward and conversely, the connecting link between them to be negative. $D_1 = bp - aq$.



$$x = \frac{N_1}{D_1} = \frac{br - qc}{bp - aq}$$

To get the value of numerator N_2 of y, following cyclic order by starting with the constant term cross-multiply backward with the coefficient of x and conversely, the connecting link between them to be negative. $N_2 = cp - ra$.

x y constant term
a b c
p q

The Denominator D_2 is same as D_1

$D_2 = bp - aq$

$$\therefore y = \frac{N_2}{D_2} = \frac{cp - ra}{bp - aq}$$

Eg.(i) Solve $x - 4y = 8,$
$3x + 5y = 7$

Current Method

$x - 4y = 8$ --------------- (1)
$3x + 5y = 7$ --------------- (2)
Equlaising the x coefficients:
Equation (2) – 3(1) is
$3x + 5y = 7$
$3x - 12y = 24$
$17y = -17$
$y = -17 / 17 = -1$
Substituting y in eq. (1)
$x - 4(-1) = 8$
$x + 4 = 8$
$x = 4$
$\therefore x = 4, y = -1$ is the solution

Vedic Method

$x - 4y = 8$
$3x + 5y = 7$
Writing down the coefficients of x and y and constant term in the order. x y constant term

Numerator(N_1) of x is
1 –4 8
3 5 7
$N_1 = (-4)(7) - (5)(8)$
$- 28 - 40 = - 68$

Denominator (D_1) of x is
1 –4 8
3 5 7
$D_1 = (-4)(3) - (1)(5)$
$-12 - 5 = - 17$

Therefore, $x = -68 / -17 = 4$

Numerator (N_2) of y is
1 –4 8
3 5 7
$N_2 = (8)(3) - (7)(1)$
$= 24 - 7 = 17$

Denominator (D_2) of y = denominator of x = –17

$\therefore x = 4, y = -1$ is the solution.

Eg. (ii) Solve $2x + y = 10$, $7x + 8y = 53$

Current Method

$2x + y = 10$ ------------ (1)
$7x + 8y = 53$ ------------ (2)
Equalising the coefficient of y
Multiplying eq. (1) with 8, we get,
$16x + 8y = 80$ ----------- (3)
Subtracting eq. (2) from eq. (3), we have

$$\begin{array}{r} 16x + 8y = 80 \\ \underline{7x + 8y = 53} \\ 9x = 27 \end{array}$$

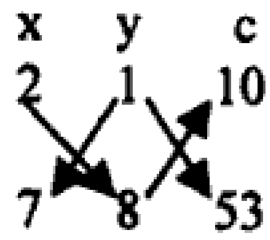
$x = 27 / 9 = 3$

Substituting $x = 3$ in eq. (1)
$2 \times 3 + y = 10$
$6 + y = 10$ or $y = 4$
$\therefore x = 3, y = 4$ is the solution.

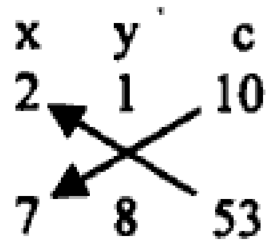
Vedic Method

$2x + y = 10$
$7x + 8y = 53$

Writing down the coefficients of x and y and constant term in order N_1, D_1 and N_2, D_2 are calculated.



$$x = \frac{53-80}{7-16} = \frac{-27}{-9} = 3$$



$$y = \frac{70-106}{7-16} \quad \frac{-36}{-9} = 4$$

Eg.(iii) Solve $\frac{x}{9} + \frac{y}{7} = 10$, $\frac{x}{3} + y = 50$

Current Method

$\frac{x}{9} + \frac{y}{7} = 10$ ---------- (1)

$\frac{x}{3} + y = 50$ ------ (2)
Taking LCM in both equations
$7x + 9y = 630$ ---------- (3)
$x + 3y = 150$ ---------- (4)
Multiplying eq. (4) with 3, we get
$3x + 9y = 450$ ---------- (5)
Subtracting eq. (5) from eq. (3)

$$\begin{array}{l} 7x + 9y = 630 \\ \underline{3x + 9y = 450} \\ 4x \quad = 180 \Rightarrow x = 180 / 4 = 45 \end{array}$$

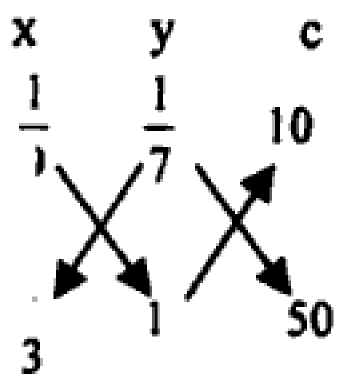
Substituting $x = 45$ in equation (2)
$45/3 + y = 50$
$15 + y = 50$
Therefore, $y = 35$
$x = 45, y = 35$ is the solution.

Vedic Method

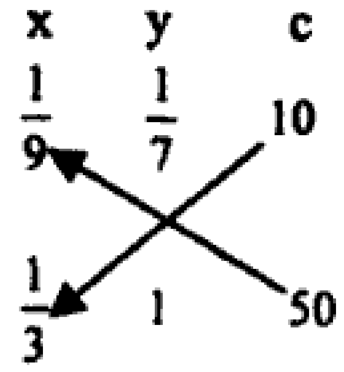
$\frac{x}{9} + \frac{y}{7} = 10$

$\frac{x}{3} + y = 50$

Writing down the coefficients of x and y and constant term in order and N_1, D_1 and N_2, D_2 are calculated.



$$x = \frac{\frac{50}{7} - 10}{\frac{1}{21} - \frac{1}{9}} \quad \frac{-20}{7} \times \frac{63}{-4} = 45$$



$$y = \frac{\frac{10}{3} - \frac{50}{9}}{\frac{-4}{63}} = \frac{-20}{9} \times \frac{63}{-4} = 35$$

Proof:

$ax + by = \ell$ ----- (1)
$cx + dy = m$ ----- (2)

where $b/d = \ell/m$, i.e., $bm = \ell d$
Multiplying eq.(1) by m and eq.(2) by ℓ, we get,

$max + bmy = \ell m$ ----- (3)
$\ell cx + \ell dy = \ell m$ ----- (4)
$\therefore max + bmy = \ell cx + \ell dy$
$(ma - \ell c)x = (\ell d - bm)y$
But $\ell d = bm$
$\therefore (ma - \ell c)x = 0$
$x = 0$

Substituting x in eq.(1)
$by = \ell$
$y = \frac{\ell}{b}$ or $\frac{m}{d}$

There are certain special types of equations, which can be solved by applying different sutras.

16) Special Type (1)
Anurupye, Sunyamanyat

If there are certain relations showing identical ratios, then the sutram Sunyamanyat can be applied. If the coefficients of one variable is in the ratio of the constant terms, then the remaining variable is zero. This is Sunyam anyat by Anurupye. This is applicable to any number of variables as well

$ax + by = \ell$
$cx + dy = m$

If $b/d = \ell/m$ then $x = 0$ and if $\frac{a}{c} = \frac{l}{m}$ then $y = 0$

Eg.(i) Solve $4x + 3y = 9$,
$28x + 5y = 63$

Current Method

$4x + 3y = 9$ ------------- (1)
$28x + 5y = 63$ ------------- (2)

Multiplying eq. (1) with 7, we get
$28x + 21y = 63$ ----------- (3)

Subtracting eq. (2) from eq. (3)
$28x + 21y = 63$
$\underline{28x + 5y = 63}$
$16y = 0$

Vedic Method

$4x + 3y = 9$ ------------- (1)
$28x + 5y = 63$ ------------- (2)
Considering the ratios of coefficients of x and constant terms:
$\frac{4}{28} = \frac{9}{63} = \frac{1}{7}$

$\therefore$ By Anurupye Sunyamanyat Sutram

Therefore, $y = 0$

Substituting y = 0 in eq. (1)

$4x + 0 = 9$

$x = 9 / 4$

$\therefore x = 9 / 4, y = 0$ is the solution. y = 0

$\therefore 4x = 9$

$x = \frac{9}{4}$

Eg.(ii) Solve $673x + 513y = 342,$
$175x + 405y = 270$

Current Method

$673x + 513y = 342$ ---------- (1)
$175x + 405y = 270$ ---------- (2)

Equalising y coefficients:

Multiplying eq. (1) with 15 and eq. (2) with 19, we get

$9895x + 7695y = 5130$ ---------- (3)
$3325x + 7695y = 5130$ ---------- (4)

Subtracting eq. (4) from eq. (3), we get

$6570x = 0$
Therefore, x = 0

$y = 342 / 513 = 2 / 3$

x = 0, y = 2 / 3 is the solution.

Vedic Method

$673x + 513y = 342$
$175x + 405y = 270$

Considering the ratios of coefficients of y and constant term.

$\frac{513}{405} = \frac{19}{15} \qquad \frac{342}{270} = \frac{19}{15}$

$\therefore$ By Anurupye Sunyamanyat Sutram x = 0

$513y = 342 \Rightarrow y = 342 / 513 = 2 / 3$

Therefore, x = 0, y = 2 / 3 is the solution.

Simultaneous Simple Equations

Eg.(iii) Solve

$$ax + cy + (a^3 - b^3)z = cs$$
$$\ell x + my + (m - n)z = ms$$
$$px + ry + (q^2 - 1)z = rs$$

Current Method

$ax + cy + (a^3 - b^3)z = cs$ ------- (1)
$\ell x + my + (m - n)z = ms$ ------ (2)
$px + ry + (q^2 - 1)z = rs$ -------- (3)

Multiplying eq. (1) with ℓ and eq. (2) with a, we get

$\ell ax + \ell cy + \ell(a^3 - b^3)z = \ell cs$ ------- (4)
$\ell ax + amy + a(m - n)z = ams$ ------ (5)

Subtracting eq. (5) from eq. (4), we get,

$(\ell c - am)y + [\ell(a^3 - b^3) - a(m - n)]z = (\ell c - am)s$ ------- (6)

Multiplying eq. (1) with p and eq. (3) with a

$pax + pcy + p(a^3 - b^3)a = pcs$ ----- (7)
$apx + ary + a(q^2 - 1)z = ars$ ------- (8)

Subtracting eq. (8) from eq. (7), we get
$(pc - ar)y + [p(a^3 - b^3) - a(q^2 - 1)]z = (pc - ar)s$ -------(9)

Multiplying eq. (6) with (pc – ar) and eq. (9) with (ℓc – am)
$(pc - ar)(\ell c - am)y + (pc - ar)[\ell(a^3 - b^3) - a(m - n)]z = (pc - ar)(\ell c - am)s$ ------- (10)

Vedic Method

$ax + cy + (a^3 - b^3)z = cs$ ------- (1)
$\ell x + my + (m - n)z = ms$ ------ (2)
$px + ry + (q^2 - 1)z = rs$ -------- (3)

Ratio of 'y' terms and independent terms is equal.
c: m: r = c: m: r.
∴ By Anurupye Sunyamanyat Sutram
$x = 0, z = 0$

$cy = cs$
$y = s$

∴ $x = 0, y = s, z = 0$ is the solution.

As extended to three variables it is clear that variables, x and z are zero. The simplicity of this sutra as applied to this special case is excellent.

One can easily appreciate the elegance of this method in comparison to the existing method.

ℓc – pc – ar)y + (ℓc – m)[p($a^3 - b^3$) – a(q^2– z = (ℓc – pc –ar)

Subtracting eq. from . (10), we get

(pc – ar)[ℓ($a^3 - b^3$) – a(m n)] – (ℓc – am)[p(a^3 - b^3) – a(q z = 0

Therefore, $z = 0$
Substituting $z = 0$ in eq. (1) and (2)

$ax + cy = cs$ ---------- (12)
$\ell x + my = ms$ ---------- (13)

Multiplying eq. (12) with m and equation 3) with c

$max + mcy = mcs$ ---------- (14)
$c\ell x + mcy = cms$ ---------- (15)

Subtracting, we get
$(ma - c\ell)x = 0$
$\therefore x = 0$

Substituting $z = 0, x = 0$ in eq. (1)

$cy = cs$
$\therefore y = s$

$\therefore x = 0$, y s, $z = 0$ is the solution.

Special Type (2)
Sankalana Vyavakalanabhyam VI
(Upasutra)

1. A second speciàl type of equation which can be solved by another sutram Sankalanavyavakanabhyam (i.e., by addition and subtraction).

   This is applicable to solve for x and y when x and y coefficients are noticed interchanged in the given equations.

Eg.(i) Solve $5x + 6y = 17,$
$6x + 5y = 16$

Current Method

$5x + 6y = 17$ -------- (1)
$6x + 5y = 16$ -------- (2)

1) Multiplying eq. (1) with 6 and eq. (2) with 5, we get

$30x + 36y = 102$ ------- (3)
$30x + 25y = 80$ -------- (4)

Subtracting eq. (4) from eq. (3), we get

$11y = 22 \Rightarrow y = 2$
Substituting $y = 2$ in eq. (1)

$5x + 12 = 17$
$5x = 5 \Rightarrow x = 1$
$\therefore x = 1, y = 2$ is the solution.

2) In current method also
By addition and subtraction one can get the solution

$11x + 11y = 33$
$x - y = -1$
$11x - 11y = -11$
$22x = 22$
$x = 1$
$y = 2$

Vedic Method

$5x + 6y = 17$
$6x + 5y = 16$
By addition (Sankalanam), we get

$11x + 11y = 33$
$x + y = 3$ - (1)

By subtraction (Vyavakalanam), we get

$-x + y = 1$ (2)
Solving (1) and (2) we get
$\therefore y = 2, x = 1$

Eg.(ii) Solve $93x + 15y = 123,$
$15x + 93y = 201$

Current Method

$93x + 15y = 123$ ---------- (1)
$15x + 93y = 201$ ---------- (2)

1) $31x + 5y = 41$ ·(3)
$5x + 31y = 67$ ·(4)

Multiplying eq. (3) with 5 and eq. (4) with 31, we get

$155x + 25y = 205$ --- ·(5)
$155x + 961y = 2077$ ·(6)

Subtracting eq. (5) from eq. (6), we get

$936y = 1872 \Rightarrow y = 2$

Substituting $y = 2$ in equation (3)

$31x + 10 = 41$
$31x = 31 \Rightarrow x = 1$

$\therefore x = 1, y = 2$ is the solution.

2) $108x + 108y = 324$ (by addition)
$78x - 78y = -78$ (by subtraction)
$x - y = -1$
$x + y = 3$
$x = 1, y = 2$

Vedic Method

$93x + 15y = 123$ ---------- (1)
$15x + 93y = 201$ ---------- (2)

$31x + 5y = 41$
$5x + 31y = 67$

By addition (Sankalanam), we get

$36x + 36y = 108$

$x + y = 3$ ---------- (3)

By subtraction (Vyavakalanam), we get

$26x - 26y = -26$
$x - y = -1$ ---------- (4)
By solving (3) and (4)
$\therefore x = 1, y = 2$ is the solution.

SECTION – 3

MULTIPLE SIMULTANEOUS EQUATIONS

Solution of Multiple Simultaneous equations of three or more unknowns can be solved by the application of following vedic sutras.

1) Lopana Sthapanabhyam
2) Anurupya sunyamanyat
3) Paravartya

The equations can be generally classified into two types.

FIRST TYPE:

The first type consists of equations wherein only one of the equations has a significant figure on the RHS where as the other equations have zero as RHS.

Example 1: consider three simultaneous equations in three unknowns.

$2x + y - z = 0$ —— A
$x + 3y - 2z = 0$ —— B
$x + y + z = 27$ —— C

Current Method

By successive elimination of z and x
From A + C $3x + 2y = 27$ —— D
From 2C + B $3x + 5y = 54$ —— E
From E – D $3y = 27 \Rightarrow y = 9$
Substituting y in D , $x = 3$
Substituting x , y in A , $z = 15$

Vedic Method

In Vedic Method the above problem can be solved by two methods.

Paravartya Method :

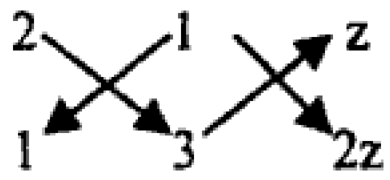
From any set of two homogenous zero equations, new equations of two unknowns in terms of the other unknowns are derived. Applying paravartya at this stage one can find out the unknowns.

From A and B, newly derived two equations are

$2x + y = z$

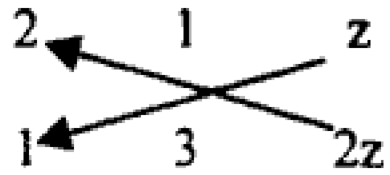
$x + 3y = 2z$

Now the Paravartya sutram is applied to the newly derived equations. For this purpose z can be treated under constant term x y (supposed const)



(for x)

x y . (supposed const)



(for y)

$$x = \frac{1(2z) - 3(z)}{1(1) - 2(3)} = \frac{z}{5} \qquad y = \frac{(z)1 - (2z)2}{1(1) - 2(3)} = \frac{3z}{5}$$

Substituting x , y in C , z = 15

$$x = \frac{z}{5} = 3\ ; \quad y = \frac{3z}{5} = 9 \Rightarrow x = 3, y = 9, z = 15.$$

Lopanasthapanabhyam Method

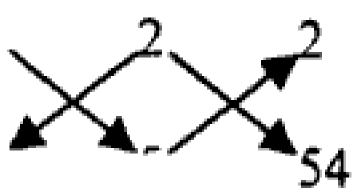
The second method is Lopanasthapanabhyam using judicious additions or subtractions and eliminating one unknown. By this process two simultaneous equations in two unknowns are to be obtained. Then one can apply paravartya method. Details are as follows.

Two Simultaneous equations in two unknowns are to be obtained

From A + C $3x + 2y = 27$

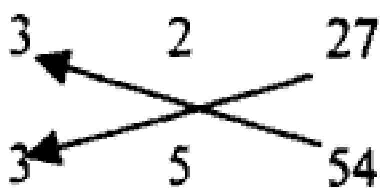
and From 2C + B $3x + 5y = 54$ } are derived

x y Constant term



$$x = \frac{2(54) - 5(27)}{2(3) - 3(5)} = 3$$

x y Constant term



$$y = \frac{(27)3 - (54)3}{(2)3 - 3(5)} = 9$$

Substituting x, y in A, z = 15

$\therefore$ x = 3 , y = 9 , z = 15.

SECOND TYPE:

In the second type of equations all the RHS contain significant figures.

Example 2:

$x - 2y + 3z = 2$ —— A
$2x - 3y + z = 1$ —— B
$3x - y + 2z = 9$ —— C

Current Method

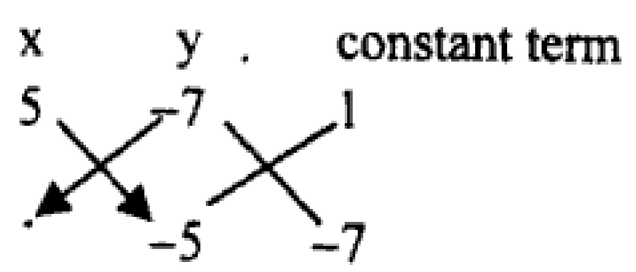
By successive elimination of z and x
From 2B – C $\quad x - 5y = -7$ —— D
From 3B – A $\quad 5x - 7y = 1$ —— E
From E – 5D $\quad 18y = 36 \Rightarrow y = 2$
Substituting y in D, $x = 3$
Substituting x, y in B, $z = 1$

Vedic Method

In this case also different methods can be adopted.

Method of Conversion of Homogenous equations into two Homogenous Zero Equations:

By cross multiplication between any two sets of equations one can derive two equations with RHS as zero. Then the method that is already described under Lopanasthapanabhyam method of first type of equations is applied. That is by eliminating one of the unknowns and getting two simultaneous equations and solve the equations by paravartya method.
From 2B – A $\quad 3x - 4y - z = 0$ —— (1)
From 9B – C $\quad 15x - 26y + 7z = 0$ —— (2) } are obtained
From the above, two simultaneous equations in two unknowns are to be obtained.
From (1) + B; $\quad 5x - 7y = 1$ ——F
From (2) – 7B $\quad x - 5y = -7$ ——G
Applying paravartya to F and G



$$x = \frac{(-7)(-7) - (-5)1}{(-7)1 - 5(-5)} = 3 \qquad y = \frac{1(1) - (-7)5}{(-7)1 - 5(-5)} = 2$$

Substituting x , y in B , $z = 1$

$\therefore x = 3, y = 2, z = 1.$

Paravartya Method (Direct):

Treating this under paravartya method one can solve the three unknowns.
Paravartya can be applied to any two of the given three equations by transposing one of the variables say z.

$$\left.\begin{array}{ll}\text{From A} & x - 2y = 2 - 3z \\ \text{From B} & 2x - 3y = 1 - z\end{array}\right\} \text{ are derived}$$

Applying paravartya to the two equations.

x y constant and z terms are considered as one unit.

$$\begin{array}{ccc} 1 & -2 & 2 - 3z \\ 2 & -3 & 1 - z \end{array}$$

$$x = \frac{(-2)(1-z)-(-3)(2-3z)}{(-2)2-1(-3)} = 7z - 4 \qquad y = \frac{(2-3z)2-(1-z)1}{(-2)2-1(-3)} = 5z - 3$$

Substituting x , y in C, z = 1

$x = 7z - 4 = 3$; $y = 5z - 3 = 2$

$\therefore x = 3, y = 2, z = 1.$

Lopanasthapanabhyam Method

This can also be solved by Lopanasthapanabhyam directly from the given equations i.e., either x or y or z to be eliminated from any two given equations and obtaining two simultaneous equations in two unknowns and solving it by the paravartya method. It is noticed that while current method makes use of successive elimination of two unknowns where as in the Vedic method after the elimination of one unknown, by applying paravartya method the remaining two unknowns are obtained simultaneously. The differences in working out the current and Vedic methods can be clearly seen in the working details.

From 2B – C $x - 5y = -7$ —— (3)
From 3B – A $5x - 7y = 1$ —— (4)

Paravartya applied to (3) (4)

$$\begin{array}{ccc} x & y & \text{constant term} \\ 1 & -5 & -7 \\ 5 & -7 & 1 \end{array}$$

$$x = \frac{(-5)1-(-7)(-7)}{(-5)5-1(-7)} = 3 \qquad y = \frac{(-7)5-1(1)}{(-5)5-1(-7)} = 2$$

Substituting x and y, z = 1

All the above vedic methods can also be applicable to equations with more than three unknowns as well. An example in four unknowns is given below:

Example3:

$2x - y + 3z + 4w = 25$ —— A
$x + 2y - z + 2w = 10$ —— B
$5x - 3y + 3z - 3w = -7$ —— C
$-x + 4y - 4z + w = -1$ —— D

Current Method

By successive elimination of x, z, y
From A + 2D $7y - 5z + 6w = 23$ —— E
From B + D $6y - 5z + 3w = 9$ —— F
From C + 5D $17y - 17z + 2w = -12$ —— G
From E – F $y + 3w = 14$ —— H
From 17F – 5G $17y + 41w = 213$ —— I
From 17H – I $10w = 25 \Rightarrow w = \frac{5}{2}$

Substituting w in H, $y = \frac{13}{2}$

Substituting w, y in F, $z = \frac{15}{2}$

Substituting w, y, z in B, $x = -\frac{1}{2}$

$\therefore\ x = -\frac{1}{2}, y = \frac{13}{2}, z = \frac{15}{2}, w = \frac{5}{2}$

Vedic Method

Method of conversion of given homogenous equations into 3 sets of zero Homogenous equations:

From A + 25D $23x - 99y + 97z - 29w = 0$ —— J
From B + 10D $9x - 42y + 41z - 12w = 0$ —— K
From C – 7D $12x - 31y + 31z - 10w = 0$ —— L
Eliminating x from the above.
From 9J – 23K $15y - 14z + 3w = 0$ —— M
From 3L – 4K $75y - 71z + 18w = 0$ —— N
From K + 9D $6y + 5z - 3w = -9$ —— P
From the above, two simultaneous equations in two unknowns are obtained by eliminating w.
From M + P $y - z = -1$ —— Q
From N + 6P $39y - 41z = -54$ —— R
Paravartya applied to Q and R

y	z	constant term
1	–1	–1
39	–41	–54

$y = \frac{(-1)(-54)-(-41)(-1)}{(-1)39-1(-41)} = \frac{13}{2}$ $z = \frac{(-1)(39)-(-54)(1)}{(-1)(39)-1(-41)} = \frac{15}{2}$

Substituting y, z in H, $w = \frac{5}{2}$

Substituting y, z, w in D, $x = -\frac{1}{2}$

Paravartya Method: (Direct Application) by I
Considering any two of the given equations.

From A $2x - y = 25 - 3z - 4w$ ——— P
From B $x + 2y = 10 + z - 2w$ ——— Q
Applying paravartya to P and Q

x y constant term
2 −1 25 − 3z − 4w
1 2 10 + z − 2w

$x = \frac{(-1)(10+z-2w)-2(25-3z-4w)}{(-1)1-2(2)} = 12 - z - 2w$ ——— R

$y = \frac{(25-3z-4w)1-(10+z-2w)2}{(-1)1-2(2)} = z - 1$ ——— S

Substituting x, y in C $5z + 13w = 70$ ——— T
Substituting x, y in D $z + 3w = 15$ ——— U
Applying paravartya to T and U

z w constant term
5 13 70
1 3 15

$z = \frac{(13)(15)-3(70)}{(13)1-5(3)} = \frac{15}{2}$ $w = \frac{70(1)-5(15)}{13(1)-5(3)} = \frac{5}{2}$

$y = z - 1 = \frac{15}{2} - 1 = \frac{13}{2}$; $x = 12 - z - 2w = -\frac{1}{2}$

$\therefore x = -\frac{1}{2}, y = \frac{13}{2}, z = \frac{15}{2}, w = \frac{5}{2}$

Lopanasthapanabhyam Method: (Successive elimination followed by paravartya.)

From A + 2D $7y - 5z + 6w = 23$ ——— V
From B + D $6y - 5z + 3w = 9$ ——— W
From C + 5D $17y - 17z + 2w = -12$ ——— X
From V − W $y + 3w = 14$ ——— Y
From 17W − 5X $17y + 41w = 213$ ——— Z

Paravartya applied to Y and Z

y w constant term

$$\begin{matrix} 1 & & 3 & & 14 \\ & \times & & \times & \\ 17 & & 41 & & 213 \end{matrix}$$

$$= \frac{3(213) - 41(14)}{3(17) - 1(41)} = \frac{13}{2} \qquad w = \frac{14(17) - 213(1)}{3(17) - 1(41)} = \frac{5}{2}$$

Substituting y, w in W, $z = \frac{15}{2}$

Substituting y, z, w in D, $x = -\frac{1}{2}$

$\therefore x = -\frac{1}{2}, y = \frac{13}{2}, z = \frac{15}{2}, w = \frac{5}{2}$.

Multiple simultaneous equations (Equations containing three or more variables).

These Equations are solved by Lopanasthapanabhyam Sutram, Anurupyena, Paravartya, cross–multiplication etc.

Multiple simultaneous equations of the following two types, one (considered).

(i) Two of the equations have zero on the RHS, whereas one has significant number. In this type one can obtain the value by Lopanasthapanabhyam or by paravartya.

(ii) The second type of equations are that all of them have significant figures on the RHS. The three methods that are applied in the second type to get the solution are clearly explained in the Lecture Notes.

To derive two zero equations using the above by way of combinations i.e., suitable subtraction, addition.

The Paravartya wherein two of the unknowns to be treated on the basis of simultaneous equations and the others are combined with the constant term which is finally as total combination and is as considered as constant.

Lopanasthapanabhyam i.e., elimination of one variable one after another successively to culminate into two simultaneous equations in two unknowns which are subjected to paravartya.

Further extension of these methods could also be clearly worked out based on these above principles and is extendable to any number of unknowns. A few examples with 4 unknowns are also explained.

This method is extendable to linear equations with any number of variables by converting them to a two variable simultaneous equations.

It is felt that there is definitely an ease in solving the equations with the help of the sutras.

1) It is observed that in solving linear equations in three variables if the condition $\Delta \neq 0$, is satisfied; one will arrive at unique solution.
2) But if Δ, Δ_1, Δ_2, Δ_3 are all zero, the set results in infinite number of solutions.
3) If $\Delta = 0$, Δ_1, Δ_2, Δ_3 are not all zero then the set of equations lead to inconsistency and this lead to no solution.
4) If the set of equations are homogeneous i.e. RHS = 0 then a trivial solution exists one can also expect a non trivial solution only, if $\Delta = 0$ $\Delta_1 = \Delta_2 = \Delta_3 = 0$ then one can expect a non trivial and infinite number of solutions.

Based on the above factors it is attempted to study the conversion of a) The set having unique solution to a set having an infinite number of solutions b) To inconsistent set with no solution and so on.

For working out the solutions, the Vedic method of Parvartya principle is applied.

Example 1:

1) Conversion from a set having unique solution to a set having an infinite number of solutions. Starting from a set of equations having a unique solution, one can attempt to change the set suitably so that the modified set of equations will answer infinite number of solutions and vice – versa.

1) Let us consider a set of equations in three variables

$3x + 2y + z = 10$ ———— (1)

$5x + 3y + 2z = 17$ ———— (2) I

$7x + 8y + z = 26$ ———— (3)

Applying Vedic method, the solution of this set is obtained as follows.

3	2	$10 - z$
5	3	$17 - 2z$

$$x = \frac{34 - 4z - 30 + 3z}{1} = 4 - z$$

$$y = \frac{50 - 5z - 51 + 6z}{1} = z - 1$$

It can be also be seen that Δ, Δ_1, Δ_2 and Δ_3 are not equal to zero

Substituting in equation (3), the values of x and y

$28 - 7z + 8z - 8 + z = 26$

$2z + 20 = 26$

$z = 3 \Rightarrow x = 1, \quad y = 2, \quad z = 3$

2) In order to arrive at a set of equations which will give infinite number of solutions one can change suitably either an element or a row or a column such that Δ, Δ_1, Δ_2, Δ_3 are all zero. Here we have considered the change of the first row of I by replacing $3x + 2y + z$ as $ax + by + cz$ and keeping the right hand side values the same. The change should result in the above condition for the Δ's to become zero. If Δ were to be zero then the condition is $-13a + 9b + 19c$ should be zero (by evaluating Δ for the set of equations).

We can give values for a, b and c such that the condition is satisfied. One such combination is that a be given value 2, b be given value '5' when c becomes -1. These are the values for the first row, (These are arbitrary) but satisfying the condition that $-13a + 9b + 19c = 0$ for the Δ to be vanishing.

The modified determinant

$$\Delta = \begin{matrix} 2 & 5 & -1 \\ 5 & 3 & 2 \\ 7 & 8 & 1 \end{matrix} = 0 \quad \Delta_1, \Delta_2, \Delta_3 \text{ are not zero}$$

The second step is to alter the values on the RHS (p,q,r) suitably so that Δ_1, Δ_2, Δ_3 also vanish. For this purpose let p,q,r be the new values for modified equations.

2) $2x + 5y - z = p$
$5x + 3y + 2z = q$ II
$7x + 8y + z = r$

In order to see that $\Delta_1 = \Delta_2 = \Delta_3 = 0$ the condition is that $p + q - r$ should be zero, which can be worked out by evaluating Δ_1, Δ_2 or Δ_3

The original values of p,q,r are 10, 17, 26 (I) will not satisfy the above condition. But a slight change in the values say for example

$p = 10$ $q = 16$ $r = 26$ will satisfy the condition $p + q - r = 0$

Keeping these values for the finally modified set of equations (II) the Vedic Method is applied for evaluation of x,y,z and is as follows

$2x + 5y - z = 10$
$5x + 3y + 2z = 16$ II
$7z + 8y + z = 26$

We can evaluate x, y, z values

$$\begin{matrix} 2 & 5 & 10 + z \\ 5 & 3 & 16 - 2z \end{matrix}$$

$$x = \frac{80 - 10z - 30 - 3z}{19} = \frac{50 - 13z}{19} \qquad y = \frac{50 + 5z - 32 + 4z}{19} = \frac{9z + 18}{19}$$

Substituting the values of x, y in the 3rd equation $7x + 8y + z = 26$.

$7(50 - 13z) + 8(9z + 18) + 19z = 19 \times 26$

$350 - 91z + 72z + 144 + 19z = 494$

z gets cancelled, z can have any value say k.

$x = \frac{50-13k}{19}, \qquad y = \frac{9k+18}{19}, \qquad z = k$

Consistent and infinite number of solutions depending on k (arbitrary).

3) a) To arrive at a set of equations from the previously modified set II for which $\Delta = 0$, so that finally inconsistency results. This is achieved through a relation between p, q and r. In this case of modified equations II for which $\Delta = 0$, the condition for $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_3 = 0$ is that $p + q - r = 0$.

If $p + q - r$ is not zero then in general $\Delta_1\ \Delta_2\ \Delta_3$ not all are zero as such one can also have another set of values for p, q, r which satisfy $p + q \neq r$. This is the condition for inconsistency i.e. $\Delta = 0\ \Delta_1 \neq 0,\ \Delta_2 \neq 0,\ \Delta_3 \neq 0$ so one can arrive at such equations from II when the relation $p + q \neq r$.

One such set is p = 10, q = 16, r = 25 or p = 10 q = 17 r = 26 which are the same values in the set I. From these studies it is clear that one can derive from a set of equations satisfying unique solution, a set giving infinite number of solutions, another set giving inconsistency leading to no solution. These changes can be also brought out by altering an element in a row

For example let us consider set of equations I

Let the element 1z be changed in the first row, let it be cz

In order to represent this set to have infinite number of solutions starting from I.

(1) The condition for Δ to be zero is

$-39 + 18 + 19c = 0$

$-21 + 19c = 0 \Rightarrow c = \frac{21}{19}$

c should be $\frac{21}{19}$ in order to have its determinant vanishing.

Let us study the condition for Δ_1, Δ_2 and Δ_3 to vanish

$$\Delta_1 = \begin{vmatrix} p & 2 & \frac{21}{19} \\ q & 3 & 2 \\ r & 8 & 1 \end{vmatrix} = \frac{p(3-16)}{21} - 2(q - 2r) + \frac{21}{19}(8q - 3r) \qquad \text{II}$$

$$= -13p - 5q + 10r + \frac{168}{19}q - \frac{63}{19}r = 0$$

$$-13p + \frac{130q}{19} + \frac{13}{19}r = 0$$

$$-247p + 130q + 13r = 19p - r - 10q = 0$$

$$\Delta_2 = \begin{vmatrix} 3 & p & \frac{21}{19} \\ 5 & 3 & q \\ 7 & 8 & r \end{vmatrix} = 0$$

$= 3(q - 2r) - p(5 - 14) + \frac{21}{19}(5r - 7q)$

$= 3q - 6r + 9p + \frac{105}{19}r - \frac{147}{19}q = 0$

$9p - \frac{9}{19}r - \frac{90}{19}q = 0$

$17p - 9r - 90q = 0$

$19p - r - 10q = 0$ ———————— (2)

$-247p + 130q + 13r = 0$

$-19p + 10q + r = -19p - 10q - r$ ———————— (1)

$$\Delta_3 = \begin{vmatrix} 3 & 2 & p \\ 5 & 3 & q \\ 7 & 8 & r \end{vmatrix} = 0$$

$3(3r - 8q) - 2(5r - 7q) + p(40 - 21)$

$-r - 10q + 19p = 0$ condition is for $\Delta_1 = \Delta_2 = \Delta_3 = 0$

The values on the RHS of Set I will not satisfy the above condition let us choose a combination of p, q, r such that the condition is satisfied.

$p = 10 \quad q = 17 \quad r = 20$ from I (with changes in z)

Now the set of equations are

$3x + 2y + \frac{21}{9} = 10$

$5x + 3y + 2z = 17$ I

$7x + 8y + 2z = 26$

Solv: g the modified equation by Vedic Method we get

3	2	$10 - \frac{21}{9}z$
5	3	$17 - 2z$

$x = 34 - 4z - 30 + \frac{63}{19}z = \frac{4 \times 19 - 13z}{19}$ $\quad y = 50 - \frac{105}{19}z - 51 + 6z = \frac{-19 + 9z}{19}$

Substituting the values of x and y in the third equation.

$532 - 91z - 152 + 72z + 19z = 380$

z gets cancelled

z can have any value, say k.

$\therefore\ x = \frac{76 - 13k}{19} \qquad y = \frac{9k - 19}{19}$

$\therefore$ Consistent and infinite number of solutions.

4) Inconsistency results for values p, q, r which will not satisfy the relation $19p - 10q - r = 0$ starting from the given set of equation II with C i.e. (which gives zero value for Δ)

Consistent and unique ————→ infinite No. of solution ——→ inconsistent

a) $\Delta \neq 0$, whatever are the values for $\Delta_1, \Delta_2, \Delta_3$ unique
b) $\Delta = 0, \Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0$ for infinite number of solutions
c) $\Delta = 0, \Delta_1, \Delta_2, \Delta_3$ not all are zeros. Inconsistency solutions
d) One can also aim at equation resulting in inconsistency from the given equations giving consistent unique solution

5) The same set of equations (set I or set II) when all are equated to zero, becomes Homogeneous with the trivial solution is $x = 0, y = 0, z = 0$. An attempt is made to find out non – trivial solutions if any is as follows.

We consider the modified equation from II, $\Delta = 0, \Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0$
It leads to infinite number of solutions in addition to the trivial solution.

We consider

$2x + 5y - z = 0$
$5x + 3y + 2z = 0$ $\Delta \neq 0, \Delta_1 = \Delta_2 = \Delta_3 = 0$
$7x + 8y + z = 0$

2	5	z
5	3	– 2z

$x = \frac{-10z - 3z}{19} = \frac{-13z}{19}$

$y = \frac{5z + 4z}{19} = \frac{9z}{19}$ non trivial also exists if $z \neq 0$ leading to infinite number of solutions

$-91z + 72z + 19z = 0$ z can be arbitrary also $z = 0$.

$3x + 2y + z = 0$ ———— (1)
$5x + 3y + 2z = 0$ ———— (2)
$7x + 8y + z = 0$ ———— (3)

Application of Paravartya to the equations (1) & (2).

3	2	– z
5	3	– 2z

$x = \frac{-4z + 3z}{1} = -z$

$y = -5z + 6z = z$

Substituting x and y in (3)

$-7z + 8z + z = 0 \quad \Rightarrow 2z = 0 \Rightarrow z = 0$

$2z = 0 \quad z = 0$

$\therefore \ x = y = z = 0$

There is no non – trivial solution. But there is only trivial solution

But there will non – trivial solutions in addition to trivial solutions in a set of homogenous equations on RHS = 0 when $\Delta = 0$ invariably $\Delta_1, \Delta_2, \Delta_3$ are all zero

For Example

$x + 2y - z = 0$
$3x + 2y - 7z = 0$
$-x + 3y + 6z = 0$

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 2 & -7 \\ -1 & 3 & 6 \end{vmatrix} = 1\,(12+21) - 2(18-7) - 1(9+2) = 33 - 22 - 11 = 0$$

$\Delta_1 = \Delta_2 = \Delta_3 = 0$ Trivial and Non – Trivial solutions

Example 2

I. Let us consider another set of equations

$2x - 4y + z = -7$ ———— (1)

$3x + 2y - 7z = 5$ ———— (2) I

$-x + 3y + 6z = 13$ ———— (3)

0, 0 $\Delta_2 \neq 0$, $\Delta_3 \neq 0$

Hence this is consistent and should result in unique solution.

This can be worked out using Vedic Method.

Applying Paravartya to the equations (1) & (2)

$2x - 4y = -7 - z$

$3x + 2y = 5 + 7z$

2	−4	−7 − z
3	2	5 + 7z

$$x = \frac{-20 - 28z + 14 + 2z}{-12 - 4} \quad \frac{+6 + 26z}{+16} \quad \frac{3 + 13z}{8}$$

$$y = \frac{-21 - 3z - 10 - 14z}{-16} = \frac{+31 + 17z}{+16}$$

Substituting in (3)

$$\left(\frac{6 + 26z}{16}\right) + \frac{3(31 + 17z)}{16} + 6z = 13$$

$-6 - 26z + 93 + 51z + 96z = 208$

$121z = 121$

$z = 1, \;\; x = 2, \;\; y = 3$

II. In order to get the infinite Number of solutions form the set I of equations, The modifications are as follows.

$\Delta = 0, \;\; \Delta_1 = 0, \;\; \Delta_2 = 0, \qquad \Delta_3 = 0$

First step is the condition for $\Delta = 0$ can be obtained by keeping one of the rows say for example the first row as $ax + by + cz = p$

The other two equations are

$3x + 2y - 7z = q$

$-x + 3y + 6z = r$

The condition for Δ to be vanishing is $3a - b + c = 0$

This can be achieved form a number of combinations. A few are given

a	b	c
1	2	– 1
0	2	2
3	2	– 7
– 1	3	-6
---	---	---
---	---	---

Let us consider 1, 2, – 1 for a,b,c respectively. Then the modified equations can be written as

$x + 2y - z = p$ ———————— (1)

$3x + 2y - 7z = q$ ———————— (2)

$-x + 3y + .6z = r$ ———————— (3)

If Δ_1 should be zero then the condition $11p - 5q - 4r = 0$.
Should be satisfied such combination could be 2,2,3

x	y	z		
1	2	– 1	=	2(p)
3	2	– 7	=	2(q)
– 1	3	6	=	3(r)

Applying Paravartya

1	2	2 + z
3	2	2 + 7z

$$x = \frac{4 + 14z - 4 - 2z}{4} = \frac{12z}{4} = 3z$$

$$y = \frac{6 + 14z - 4 - 2z}{4} = \frac{4 - 4z}{4} = 1 - z$$

Substituting the values of x and y in (3).

$-3z + 3 - 3z + 6z = 3$

z gets cancelled and z can have any value k.(k arbitrary)

$\therefore\ x = 3k, \quad y = 1 - k, \quad z = k$

III. To arrive at inconsistency, the modification is as follows

$\Delta = 0, \quad \Delta_1 \neq 0, \quad \Delta_2 \neq 0, \quad \Delta_3 \neq 0$

As such the p, q, r values now take different set say l, m, n.

The condition for Δ_1 to be non zero is $-5m - 4n + 11\ell \neq 0$. One set of values for ℓ, m, n which satisfy the condition $\ell = 1, m = 5, n = 3$

This is same for Δ_2 and Δ_3

Let us take ℓ to be 1, m is 5 and n is 3.

x	y	z		
1	2	– 1	1(l)	—— 1
3	2	– 7	5(m)	—— 2
– 1	3	6	3(n)	—— 3

Apply Paravartya to the equations 1, 2

1	2	1 + z
3	2	5 + 7z

$$x = \frac{10 + 14z - 2 - 2z}{4} = \frac{8 + 12z}{4} = 2 + 3z$$

$$y = \frac{3 + 3z - 5 - 7z}{4} = \frac{-2 - 4z}{4}$$

Substituting the x, y values in (3)

$$\frac{(8 + 12z)}{4} + \frac{3(-2 - 4z)}{4} + 24z = 12$$

$- 8 - 12z - 6 - 12z + 24z = 12$

$- 14 = 12$ in consistent relation

Hence no solution

The conditions required for consistent and unique solution, consistent and infinite number of solutions, inconsistency and no solution and trivial solutions etc can be clearly worked out. Then if one set of equations with a specific conditions are given the other equations satisfying the remaining required conditions can be worked out with suitable modifications. Very interesting results could be obtained. These methods can be applied for any number of variables as well.

CHAPTER II

SECTION – 4

QUADRATIC EQUATIONS

In a quadratic equation of the type $ax^2 + bx + c = 0$.

Current Method

$ax^2 + bx + c = 0$

Transposing, $ax^2 + bx = -c$
Dividing by a,

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Complete the square by adding to each side $\left(\frac{b}{2a}\right)^2$; thus

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

i.e., $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

Extracting the square root,

$$x + \frac{b}{2a} = \frac{\pm\sqrt{(b^2 - 4ac)}}{2a};$$

$$\therefore x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Vedic Method

<u>**General Method**</u>

The differential of each term is obtained by considering its Dhvaja Ghata (the power) as the Anka (coefficient) and reduction of the Dhvaja Ghata by 1.
Ex. Term x^2, First differential is $2x$.

Thus, the first differential is worked out on the basis of above for all terms containing x of the quadratic equations.

In Vedic Method, by equating the first differential to the discriminant, one can directly write down the roots, which is extremely simple. The first Differential $= \pm\sqrt{\text{the discriminant}}$

Using this relation one can solve the two values of x.

First differential is also equal to the sum of the two binomial factors in case the x coefficient is 1. As otherwise this will be satisfied only by making x^2 coefficient1 as follows.

$$ax^2 + bx + c = 0 \qquad \text{———} \qquad (1)$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (2)$$

First differential $D_1 = 2x + \frac{b}{a} = \pm\sqrt{\frac{b^2}{a^2} - \frac{4c}{a}}$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

Factors are $\left| x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \right.$

and $\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right.$

When the two factors are multiplied one gets the equation

Sum of Factors = $2x + \frac{b}{a}$

First differential $D_1 = 2x + \frac{b}{a}$ = Sum of the two factors

$\therefore$ First Differential = Sum of Factors and it is also that the two factors are

1^{st} Differential = $\pm\sqrt{Discriminant}$

Examples:

1. **Solve** $x^2 - 13x + 42 = 0$

Current Method

$x^2 - 13x + 42 = 0$

$x = \frac{13 \pm \sqrt{169 - 168}}{\cdot}$

$\frac{13 \pm \sqrt{1}}{2} = \frac{13 \pm 1}{2}$

$= 7$ or 6

Vedic Method

$ax^2 + bx + c = 0$

First Differential = $\pm\sqrt{\text{The Discriminant}}$

$x^2 - 13x + 42 = 0$

$2x - 13 = \pm\sqrt{169 - 168}$

$2x - 13 = \pm 1$

$x = 7, 6$

Sum of the factors = $(x - 7) + (x - 6) = 2x - 13$ = First differential.

Another method which is called Adyamadyena Antyamantyena followed by Anurupyena is also applicable in solving QE.

The method is to split the x term into 2 units such that the x^2 term by the 1^{st} split term which is called Adyamadyena is equal to the ratio of 2^{nd} split x term to the constant term (Antyamantyena). As these two should be in the same ratio which is called Anurupyena. The two factors of the QE can be obtained as follows. The sum of the numerator and Denominator of the Anurupyena is one of the factors; while the 2^{nd} factor is derived again by Adyamadyena and Antyamantyena. Modus operandi of this sutram to get the 2^{nd} factor is understood as follows. Consider the first factor which is already derived, from this one can write down the 2^{nd} factor as a + b where a is the 1^{st} highest power of x by the first term of the first factor and b is the last constant term of the equation divided by last term of the first factor.

By Adyamadyena Antyamantyena followed by Anurupyena

$x^2 - 7x - 6x + 42 = 0$

$\frac{x^2}{-7x} = \frac{-6x}{42}$ $(x - 7)$ (first factor)

$\frac{x^2}{x}+\frac{42}{-7}=(x-6)$ second factor

∴ Given Quadratic = (x – 7) (x – 6)

2. **Solve** $6x^2 - 25x + 21 = 0$

Current Method

$6x^2 - 25x + 21 = 0$

$x = \frac{25 \pm \sqrt{625-504}}{12}$

$\frac{25 \pm \sqrt{121}}{12} \quad \frac{25 \pm 11}{12}$

= 3 or 7/6

Vedic Method

$6x^2 - 25x + 21 = 0$

First Differential = ± $\sqrt{\text{The Discriminant}}$

$12x - 25 = \pm\sqrt{625-504}$

$12x - 25 = \pm\sqrt{121}$

$12x - 25 = \pm 11$

x = 3 or 7/6

Sum of the factors = $(x - 3) + \left(x + \frac{7}{6}\right) = 2x - \frac{25}{6}$

$x^2 - \frac{25}{6}x + \frac{21}{6} = 0$

$D_1 = 2x - \frac{25}{6}$

First differential (D_1) = Sum of its factors.

By Adyamadyena Antyamantyena followed by Anurupyena

$6x^2 - 25x + 21 = 0$

$6x^2 - 18x - 7x + 21 = 0$

$\begin{matrix} 6x^2 & -7x \\ -18x & 21 \end{matrix} \quad (x-3)$ *is a factor*

$\frac{6x^2}{x}+\frac{21}{-3}=(6x-7)$ is second factor.

$(6x^2 - 25x + 21) = (x - 3)(6x - 7) = 0$

3. **Solve** $x^2 - \frac{7}{6}x - \frac{1}{2} = 0$

Current Method

$x^2 - \frac{7}{6}x - \frac{1}{2} = 0$

$x = \frac{\frac{7}{6} \pm \sqrt{\frac{49}{36}+2}}{2}$

Vedic Method

$x^2 - \frac{7}{6}x - \frac{1}{2} = 0$

First Differential = ± $\sqrt{\text{The Discriminant}}$

$2x - \frac{7}{6} = \pm\sqrt{\frac{49}{36}+2}$

$$x = \frac{\frac{7}{6} \pm \sqrt{\frac{121}{36}}}{2} = \frac{\frac{7}{6} \pm \frac{11}{6}}{2}$$

$$\frac{7}{12} \pm \frac{11}{12}$$

x = 3/2 or −1/3

$$2x - \frac{7}{6} = \pm \sqrt{\frac{1:}{3}}$$

$$2x - \frac{7}{6} = \pm \frac{11}{6}$$

2x = 3 or −2/3

x = 3/2 or −1/3

Sum of the factors $\left(x - \frac{3}{2}\right) + \left(x + \frac{1}{3}\right) = \left(2x - \frac{7}{6}\right)$

= First Differential

$$x^2 - \frac{7}{6}x - \frac{1}{2} = 0$$

$$x^2 - \frac{3}{2}x + \frac{1}{3}x - \frac{1}{2} = 0$$

$$\frac{x^2}{-\frac{3}{2}x} = \frac{\frac{1}{3}x}{-\frac{1}{2}} \Rightarrow \left(x - \frac{3}{2}\right) \text{ is one factor}$$

$$\frac{x^2}{x} = \frac{\frac{1}{2}}{\frac{-3}{2}} \Rightarrow \left(x + \frac{1}{3}\right) \text{ is second factor}$$

$$\therefore \left(x^2 - \frac{7}{6}x - \frac{1}{2}\right) = \left(x - \frac{3}{2}\right)\left(x + \frac{1}{3}\right)$$

1) $E = a^2 + 3a + 2 = 0$

Adyamadyena

$a^2 + 2a + a + 2$

$\frac{a^2}{2a} = \frac{a}{2} \Rightarrow$ (a + 2) is a factor

$\frac{a^2}{a} = \frac{2}{2} \Rightarrow$ (a + 1) is second factor

Differential Relation

$D_1 = 2a + 3 = \pm \sqrt{9 - 8} \Rightarrow 2a = -3 \pm$
1

$\therefore$ (a + 2), (a + 1) are the factors of E

2) $E = a^2 + 7a + 12 = 0$

Adyamadyena

$a^2 + 4a + 3a + 12$

$\frac{a^2}{4a} = \frac{3a}{12} \Rightarrow$ (a + 4) is a factor

$\frac{a^2}{a} + \frac{12}{4} \Rightarrow$ (a + 3) is second factor

Differential Relation

$D_1 = 2a + 7 = \pm \sqrt{49 - 48} \Rightarrow 2a = -7 \pm 1$

$\therefore$ (a + 3), (a + 4) are the factors of E

3) $E = x^2 - 11x + 30 = 0$

Adyamadyena

$x^2 - 6x - 5x + 30$

$\begin{matrix} & -5x \\ -6x & 30 \end{matrix}$ (x – 6) is a factor

$\frac{x^2}{x} + \frac{30}{-6} \Rightarrow$ (x – 5) is second factor

Differential Relation

$D_1 = 2x - 11 = \pm\sqrt{121-120} \Rightarrow 2x = 11 \pm 1$

∴ (x – 5), (x – 6) are the factors of E

4) $E = x^2 - 18x + 45 = 0$

Adyamadyena

$x^2 - 15x - 3x + 45$

$\begin{matrix} & -3x \\ -15x & 45 \end{matrix}$ (x – 15) is a factor

$\frac{x^2}{x} + \frac{45}{-15} \Rightarrow$ (x – 3) is second factor

Differential Relation

$D_1 = 2x - 18 = \pm\sqrt{324-180} \Rightarrow 2x = 18 \pm 12$

∴ (x – 3), (x – 15) are the factors of E

5) $E = a^2 - 24a + 95 = 0$

Adyamadyena

$a^2 - 5a - 19a + 95$

$\begin{matrix} a^2 & -19a \\ -5a & 95 \end{matrix}$ (a – 5) is a factor

$\frac{a^2}{a} + \frac{95}{-5} \Rightarrow$ (a – 19) is second factor

Differential Relation

$D_1 = 2a - 24 = \pm\sqrt{576-380} \Rightarrow 2a = 24 \pm 14$

∴ (a – 5), (a – 19) are the factors of E

6) $E = a^2 + 54a + 729 = 0$

Adyamadyena

$a^2 + 27a + 27a + 729 = 0$

$\frac{a^2}{27a} = \frac{27a}{729} \Rightarrow$ (a + 27) is a factor

$\frac{a^2}{a} + \frac{729}{27}$, (a + 27) is second factor

Differential Relation

$D_1 = 2a + 54 = \pm\sqrt{2916-2916} \Rightarrow 2a = -54$

∴ $(a + 27)^2$ are the factors of E

7) $E = x^2 - 26xy + 169y^2 = 0$

Adyamadyena

$x^2 - 13xy - 13xy + 169y^2$

$\frac{x^2}{-13xy} = \frac{-13xy}{169y^2} \Rightarrow (x - 13y)$ is a factor

$\frac{x^2}{x} + \frac{169y^2}{-13y} \Rightarrow (x-13y)$ is second factor

Differential Relation

$D_1 = 2x - 26y = \pm\sqrt{676y^2 - 676y^2}$

$\Rightarrow 2x = 26y \Rightarrow x = 13y$

$\therefore (x - 13y)^2$ are the factors of E

8) $E = x^4 - 9x^2y + 14y^2 = 0$

Adyamadyena

$x^4 - 7x^2y - 2x^2y + 14y^2$

$\frac{x^4}{-7x^2y} = \frac{-2x^2y}{14y^2}$, $(x^2 - 7y)$ is a factor

$\frac{x^4}{\cdot} + \frac{14y^2}{-7y} \Rightarrow (x^2 - 2y)$ is second factor

$x^2 = 7y$ $x = \pm\sqrt{7y}$

$x^2 = 2y$ $x = \pm\sqrt{2y}$

Differential Relation

Let x^2 be a

$\therefore E = a^2 - 9ay + 14y^2$

$D_1 = 2a - 9y = \pm\sqrt{81y^2 - 56y^2}$

$\Rightarrow 2a = 9y \pm 5y$ $a = 7y, 2y$

$x^2 = 7y, 2y$

$x = \pm\sqrt{7y}, \pm\sqrt{2y}$

9) $E = 20 + 9x + x^2$

Adyamadyena

$x^2 + 5x + 4x + 20$

$\frac{x^2}{5x} = \frac{4x}{20} \Rightarrow (x + 5)$ is a factor

$\frac{x^2}{x} + \frac{20}{5} \Rightarrow (x + 4)$ is second factor

Differential Relation

$D_1 = 2x + 9 = \pm\sqrt{81 - 80} \Rightarrow 2x = -9 \pm 1$

$\therefore (x + 4)(x + 5)$ are the factors of E

10) $E = x^2 + 8x + 7 = 0$

Adyamadyena

$x^2 + 7x + x + 7$

$\frac{x^2}{7x} = \frac{x}{7} \Rightarrow (x + 7)$ is a factor

$\frac{x^2}{x} + \frac{7}{7} \Rightarrow (x + 1)$ is second factor

Differential Relation

$D_1 = 2x + 8 = \pm\sqrt{64 - 28}$

$2x = -8 \pm 6 \Rightarrow x = -7, -1$

$\therefore$ $(x + 1)(x + 7)$ are the factors of E

11) $E = x^2 + 49xy + 600y^2 = 0$

Adyamadyena

$x^2 + 24xy + 25xy + 600y^2$

$\frac{x^2}{24xy} = \frac{25xy}{600y^2} \Rightarrow (x + 24y)$ is a factor

$\frac{x^2}{x} + \frac{600y^2}{24y} \Rightarrow (x+25y)$ is second factor

Differential Relation

$D_1 = 2x + 49y = \pm\sqrt{2401y^2 - 2400y^2}$

$2x = -49 \pm y$

$\therefore (x + 24y)(x + 25y)$ are the factors of E

12) $E = x^2 - 9x - 90 = 0$

Adyamadyena

$x^2 - 15x + 6x - 90$

$\frac{}{-15x} = \frac{6x}{-90} \Rightarrow (x - 15)$ is a factor

$\frac{x^2}{x}\ \frac{-90}{-15}$, $(x + 6)$ is second factor

Differential Relation

$D_1 = 2x - 9 = \pm\sqrt{81+360}$

$2x = 9 \pm 21 \Rightarrow x = 15, -6$

$\therefore$ $(x + 6)(x - 15)$ are the factors of E

13) $E = x^2 - x - 240 = 0$

Adyamadyena

$x^2 - 16x + 15x - 240$

$\frac{x^2}{-16x} = \frac{15x}{-240} \Rightarrow (x - 16)$ is a factor

$\frac{x^2}{x} + \frac{-240}{-16}$, $(x + 15)$ is second factor

Differential Relation

$D_1 = 2x - 1 = \pm\sqrt{1+960} \Rightarrow 2x = 1 \pm 31$

$\therefore$ $(x + 15)(x - 16)$ are the factors of E

14) $E = a^2 + a - 20 = 0$

Adyamadyena

$a^2 + 5a - 4a - 20$

$\frac{a^2}{5a} = \frac{-4a}{-20}$ ⇒ (a + 5) is a factor

$\frac{a^2}{a} + \frac{-20}{5}$: ⇒ (a – 4) is second factor

Differential Relation

$D_1 = 2a + 1 = \pm\sqrt{1+80}$

$\Rightarrow 2a = -1 \pm 9 \Rightarrow a = -5, 4$

$\therefore$ (a – 4), (a + 5) are the factors of E

15) $E = x^2 - 4x - 12 = 0$

Adyamadyena

$x^2 - 6x + 2x - 12$

$\frac{.}{-6x} \quad \frac{2x}{-12}$ (x – 6) is a factor

$\frac{x^2}{x} + \frac{-12}{-6} \Rightarrow$ (x + 2) is second factor

Differential Relation

$D_1 = 2x - 4 = \pm\sqrt{16+48} \Rightarrow 2x = 4 \pm 8$

$\therefore$ (x + 2) (x – 6) are.the factors of E

16) $E = a^2 - 12a - 85 = 0$

Adyamadyena

$a^2 - 17a + 5a - 85$

$\frac{a^2}{-17a} \quad \frac{5a}{-85}$ (a – 17) is a factor

$\frac{a^2}{a} + \frac{-85}{-17}$ ⇒ (a + 5) is second factor

Differential Relation

$D_1 = 2a - 12 = \pm\sqrt{144+340}$

$\Rightarrow 2a = 12 \pm 22$

$\therefore$ (a + 5) (a – 17) are the factors of E

17) $E = x^2 + 7xy - 60y^2 = 0$

Adyamadyena

$x^2 + 12y - 5y - 60y^2$

$\frac{x^2}{12y} = \frac{-5y}{-60y^2}$ (x + 12y) is a factor

$\frac{x^2}{} \quad \frac{60y^2}{12y}$ ⇒ (x – 5y) is second factor

Differential Relation

$D_1 = 2x + 7y = \pm\sqrt{49y^2 + 240y^2}$

$\Rightarrow 2x = -7y \pm 17y \quad x = -12y, 5y$

$\therefore$ (x – 5y) (x + 12y) are the factors of E

18) $E = x^4 - a^2x^2 - 462 = 0$

Adyamadyena

$x^4 - 22a^4x^2 + 21a^2x^2 - 462a^4$

$x^4 \quad 21a^2x^2$

$-22a^2x^2 \quad -462a^4$

$\Rightarrow (x^2 - 22a^2)$ is a factor

$x^4 \quad 462a^4$

$x^2 \quad -22a^2$

$\Rightarrow (x^2 + 21a^2)$ is second factor

Differential Relation

Let x^2 be α

$\therefore\ E = \alpha^2 - a^2\alpha - 46a^4$

$D_1 = 2\alpha - a^2 = \pm\sqrt{a^4 + 1848a^4}$

$2\alpha = a^2 \pm a^2\sqrt{1849}$

$2\alpha = 44a^2, -42a^2$

$\alpha = 22a^2, -21a^2$

$x^2 = 22a^2, -21a^2$

$x = \pm\sqrt{22}a,$

$x = \pm\sqrt{21}ia$

19) $E = 98 - 7x - x^2 = 0$

Adyamadyena

$-x^2 + 7x -$

$-x^2 \quad -14x \qquad (-x + 7)$ is a factor

$7x \quad 98$

$\frac{x^2}{-x} + \frac{98}{7} \Rightarrow (x + 14)$ is second factor

Differential Relation

$D_1 = -2x - 7 = \pm\sqrt{49 + 392}$

$-2x = +7 \pm 21 \Rightarrow x = -14,\ 7$

$(x - 7), (x + 14)$ are the factors of E which will satisfy E.

A few special cases are given below, which can be solved very simply by using Vedic Method

First Special Type 1

(I) Reciprocals: If the equation is in the form of $\frac{ax}{b} + \frac{b}{ax} = \frac{p}{q}$

In this form one has to split the right hand side exactly to have a similar reciprocal form as the left-hand side.

The left-hand side can also be of a binomial or a polynomial having this relation of reciprocity. Even then the right hand side has to be processed as above.

In the problems that are shown below, the ease and elegance with which the problem can be solved using Vedic Method is very clear from the comparison with the existing method wherein very elaborate working details are found necessary for obtaining the result.

Examples:

1. **Solve** $x + \frac{1}{x} = \frac{122}{11}$

Current Method

$$x+\frac{1}{x}=\frac{122}{11}$$

$$x+\frac{1}{x}-\frac{122}{11}=0$$

$$x^2+1-\frac{122}{11}x=0$$

$$x=\frac{122}{11}\quad \frac{14884}{121}-4$$

$$x=\frac{\frac{122}{11}\pm\sqrt{\frac{14400}{11}}}{2}=\frac{\frac{122}{11}\pm\frac{120}{11}}{}$$

$$x=\frac{242}{2\times 11}\text{ or }\frac{2}{2\times 11}$$

$= 11$ or $1/11$

Vedic Method

$$x+\frac{1}{x}=\frac{122}{11}$$

When LHS is of the form $\frac{a}{b}\pm\frac{b}{a}$, we should try to split the RHS into similar reciprocal form $(\frac{c}{d}\pm\frac{d}{c})$

$\therefore$ Split 122/11 as $11+\frac{1}{11}$

$$x+\frac{1}{x}=11+\frac{1}{11}$$

$\therefore x = 11$ or $1/11$

2. **Solve** $\frac{x+5}{x+1}-\frac{x+1}{x+5}=\frac{272}{273}$

Current Method

$$\frac{x+5}{x+1}-\frac{x+1}{x+5}=\frac{272}{273}$$

$$\frac{(x+5)^2-(x+1)^2}{(x+1)(x+5)}=\frac{272}{273}$$

$$\frac{x^2+25+10x-x^2-1-2x}{x^2+6x+5}\quad\frac{272}{273}$$

$273\,(8x + 24) = 272\,(x^2 + 6x + 5)$

$2184x + 6552 = 272x^2 + 1632x + 1360$

$272x^2 - 552x - 5192 = 0$

$34x^2 - 69x - 649 = 0$

$$x=\frac{69\pm\sqrt{4761+88264}}{68}$$

$$x=\frac{69\pm\sqrt{93025}}{68}\quad\frac{69\pm 305}{68}$$

$$x=\frac{374}{68}\text{ or }\frac{-236}{68}$$

$$x=\frac{11}{2}\text{ or }-\frac{59}{17}$$

Vedic Method

$$\frac{x+5}{x+1}-\frac{x+1}{x+5}=\frac{272}{273}$$

$$\frac{x+5}{x+1}-\frac{x+1}{x+5}\quad\frac{21}{13}-\frac{13}{21}$$

$$\frac{x+5}{x+1}=\frac{21}{13}\text{ or }\frac{-13}{21}$$

$$\frac{x+5}{x+1}=\frac{21}{13}$$

$$\therefore x=\frac{65-21}{21-13}\quad\frac{44}{8}\quad\frac{11}{2}$$

$$\frac{x+5}{x+1}\quad\frac{-13}{21}$$

$$x=\frac{105+13}{-13-21}\quad\frac{-118}{34}\quad\frac{-59}{17}$$

$$x=\frac{11}{2},\ \frac{-59}{17}$$

3 Solve $\frac{+a}{+b}\ \frac{x+b}{x+a}=\frac{a^2+b^2+2(1+a+b)}{1+a+b+ab}$

Current Method

$$\frac{x+a}{x+b}+\frac{x+b}{x+a}=\frac{a^2+b^2+2(1+a+b)}{1+a+b+ab}$$

$$\frac{(x+a)^2+(x+b)^2}{(x+b)(x+a)}=\frac{a^2+b^2+2(1+a+b)}{1+a+b+ab}$$

$$\frac{x^2+a^2+2ax+x^2+b^2+2bx}{x^2+ax+bx+ab}=\frac{a^2+b^2+2(1+a+b)}{1+a+b+ab}$$

$$\frac{2x^2+2(a+b)x+a^2+b^2}{x^2+(a+b)x+ab}=\frac{a^2+b^2+2(1+a+b)}{1+a+b+ab}$$

$(2x^2+2(a+b)x+a^2+b^2)(1+a+b+ab)=[a^2+b^2+2(1+a+b)][x^2+(a+b)x+ab]$

$2x^2+2ax+2bx+a^2+b^2+2ax^2+2a^2x+2abx+a^3+ab^2+2bx^2+2abx+2b^2x+ba^2+b^3+2abx^2+2a^2bx+2ab^2x+a^3b+ab^3$
$=a^2x^2+b^2x^2+2x^2+2ax^2+2bx^2+a^3x+ab^2x+2ax+2a^2x+2abx+ba^2x+b^3x+2bx+2abx+2b^2x+a^3b+ab^3+2ab+2a^2b+2ab^2$

$a^2+b^2+a^3+ab^2+ba^2+b^3+2abx^2+2a^2bx+2ab^2x$
$=a^2x^2+b^2x^2+a^3x+ab^2x+ba^2x+b^3x+2ab+2a^2b+2ab^2$

$2abx^2+(2a^2b+2ab^2)x+a^2+b^2+a^3+b^3+ab^2+ba^2$
$=(a^2+b^2)x^2+(a^3+ab^2+ba^2+b^3)x+2ab+2a^2b+2ab^2$

Vedic Method

$$\frac{x+a}{x+b}+\frac{x+b}{x+a}=\frac{a^2+b^2+2(1+a+b)}{1+a+b+ab}$$

$$\frac{x+a}{x+b}+\frac{x+b}{x+a}=\frac{a^2+b^2+2+2a+2b}{1+a+b+ab}$$

$$\frac{x+a}{x+b}+\frac{x+b}{x+a}=\frac{(1+a)^2+(1+b)^2}{(1+a)(1+b)}$$

$$\frac{x+a}{x+b}+\frac{x+b}{x+a}=\frac{1+a}{1+b}+\frac{1+b}{1+a}$$

$$\frac{x+a}{x+b}=\frac{1+a}{1+b}\ \text{or}\ \frac{1+b}{1+a}$$

$$\frac{x+a}{x+b}=\frac{1+a}{1+b}\Rightarrow x=1$$

$$\frac{x+a}{x+b}=\frac{1+b}{1+a}$$

$$\Rightarrow (a^2 + b^2 - 2ab)x^2 + (a^3 + ab^2 + ba^2 + b^3 - 2a^2b - 2ab^2)x + 2ab + 2a^2b + 2ab^2 - a^2 - b^2 - a^3 - b^3 - ab^2 - ba^2 = 0$$

$$\Rightarrow (a - b)^2x^2 + (a^3 + b^3 - ab^2 - a^2b)x + 2ab + a^2b + ab^2 - a^2 - b^2 - a^3 - b^3 = 0$$

$$\Rightarrow (a - b)^2x^2 + [a^2(a - b) + b^2(b - a)]x + a^2(b - a) + b^2(a - b) - (a - b)^2 = 0$$

$$(a - b)^2x^2 + (a - b)(a^2 - b^2)x + (a - b)(b^2 - a^2) - (a - b)^2 = 0$$

$$(a - b)^2x^2 + (a - b)^2(a + b)x - (a - b)^2(a + b) - (a - b)^2 = 0$$

$$x^2 + (a + b)x - a - b - 1 = 0$$

$$x = \frac{-a - b \pm \sqrt{(a+b)^2 + 4a + 4b + 4}}{2}$$

$$= \frac{-a - b \pm \sqrt{a^2 + b^2 + 2ab + 4a + 4b + 4}}{2}$$

$$= \frac{-a - b \pm \sqrt{(a + b + 2)^2}}{2}$$

$$= \frac{-a - b \pm (a + b + 2)}{2}$$

$$= \frac{-a - b + a + b + 2}{2} \text{ or } \frac{-a - b - a - b - 2}{2}$$

$$= 1 \text{ or } -(a + b + 1)$$

$$\Rightarrow x = \frac{a(1+a) - b(1+b)}{1 + b - (1 + a)}$$

$$= \frac{a + a^2 - b - b^2}{1 + b - 1 - a}$$

$$= \frac{a - b + a^2 - b^2}{b - a}$$

$$= \frac{(a - b)(1 + a + b)}{b - a}$$

$$= -(1 + a + b)$$

$$x = 1,\ x = -(1 + a + b)$$

Special Type II

(II) Sunyam Samya Samuccaye Sutra:

Another special type wherein the equation is of the from $\frac{N_1}{D_1}=\frac{N_2}{D_2}$ where N,D's are expression in x then Sunyam Samya Samuccaye Sutram can be applied. Particularly when the x^2 is not getting cancelled on both sides, it leads to a quadratic equation (general form). In such a case the solution can be obtained by verifying the possibility of the application of the Samyam. (Refer to simple equations)

1. $N_1 + D_1 = N_2 + D_2$
2. $N_1 \sim N_2 = D_1 \sim D_2$
3. $N_1 \sim D_1 = N_2 \sim D_2$
4. $N_1 + N_2 = D_1 + D_2$

} for a complete solution this set can be tried

Examples:

1. **Solve** $\frac{x+3}{2x-7} \quad \frac{2x-1}{x-3}$

Current Method

$\frac{x+3}{2x-7} \quad \frac{2x-1}{x-3}$

$(x + 3)(x - 3) = (2x - 1)(2x - 7)$

$x^2 - 9 = 4x^2 - 16x + 7$

$3x^2 - 16x + 16 = 0$

$x = 16 \pm \sqrt{256-192}$

$16 \pm \sqrt{64} \quad 16 \pm 8 \quad 8 \pm 4$

= 4 or 4/3

Vedic Method

$\frac{x+3}{2x-7} \quad \frac{2x-1}{x-3}$

$N_1 + D_1 = 3x - 4$

$N_2 + D_2 = 3x - 4$

$\therefore$ By Sunyam Samya Samuccaye Sutra

$3x - 4 = 0 \Rightarrow x = \frac{4}{3}$ one solution

$N_1 \sim N_2 = x - 4$

$D_1 \sim D_2 = x - 4$

$\therefore$ By Sunyam Samya Samuccaye Sutra

x = 4 another solution

$\therefore x = 4, \frac{4}{3}$

2. **Solve** $\frac{1}{2x-5a}+\frac{5}{2x-a}=\frac{2}{a}$

Current Method

$$\frac{1}{2x-5a}+\frac{5}{2x-a}=\frac{2}{a}$$

$$\frac{1}{2x-5a}=\frac{2}{a}-\frac{5}{2x-a}$$

$$\frac{a}{2x-5a}=\frac{4x-7a}{2x-a}$$

$a(2x-a)=(4x-7a)(2x-5a)$
$2ax-a^2=8x^2-20ax-14ax+35a^2$
$2x^2-9ax+9a^2=0$

$$x=\frac{9a\pm\sqrt{81a^2-72a^2}}{}$$

$9a\pm\sqrt{9a^2}$ $9a\pm 3a$

$=3a$ or $3a/2$

Vedic Method

$$\frac{1}{2x-5a}\cdot\frac{5}{2x-a}=\frac{2}{a}\cdot$$

$$\frac{1}{2x-5a}\quad a\quad 2x-a \text{ (Paravartya)}$$

$\frac{a}{2x-5a}=\frac{4x-7a}{2x-a}$ (LCM and Paravartya to get into standard form)

$N_1\sim D_1=2x-6a=2(x-3a)$
$N_2\sim D_2=2x-6a=2(x-3a)$
$\therefore$ By Sunyam Samya Samuccaye Sutra
$x-3a=0\Rightarrow x=3a$
$N_1+N_2=4x-6a=2(2x-3a)$
$D_1+D_2=4x-6a=2(2x-3a)$
$\therefore$ By Sunyam Samya Samuccaye Sutra
$2x-3a=0\Rightarrow x=3a/2$

$$x=3a,\ \frac{3a}{2}$$

3. **Solve** $\frac{b}{x-a}+\frac{a}{x-b}\cdot=2$

Current Method

$$\frac{b}{x-a}+\frac{a}{x-b}=2$$

$\frac{b}{x-a}=2-\cdot\frac{a}{x-b}$ or $\frac{}{x-a}\quad\frac{2x-2b-a}{x-b}$

$b(x-b)=(x-a)(2x-2b-a)$
$bx-b^2=2x^2-(3a+2b)x+2ab+a^2$
$2x^2-3(a+b)x+(a+b)^2=0$

$$x=\frac{3(a+b)\pm\sqrt{9(a+b)^2-8(a+b)^2}}{4}$$

$$=\frac{3(a+b)\pm\sqrt{(a+b)^2}}{4}=\frac{3(a+b)\pm(a+b)}{4}$$

$=a+b$ or $(a+b)/2$

Vedic Method

$$x-a\cdot\frac{a}{x-b}=2$$

$$\frac{b}{x-a}\cdot=2-\cdot\frac{a}{x-b}$$

$$\frac{b}{x-a}\quad\frac{2x-2b-a}{x-b}\cdot$$

$N_1\sim D_1=x-b-a$
$N_2\sim D_2=x-b-a$
$\therefore$ By Sunyam Samya Samuccaye Sutra
$x-a-b=0\Rightarrow x=a+b$
$N_1+N_2=2x-b-a$
$D_1+D_2=2x-b-a$
$\therefore$ By Sunyam Samya Samuccaye Sutra
$2x-a-b=0\Rightarrow x=\frac{a+b}{2}$

$x=(a+b),\ (a+b)$

Special Type III

(II) Sunyam Samya Samuccaye and Sunyamanyat Sutras:

Another special type of equation is of the form $\frac{p}{ax+b}+\frac{q}{cx+d}=\frac{r}{ex+f}+\frac{s}{gx+h}$, where p, q, r, s, a, b, c, d, e, f, g and h are numerical values.

Then two tests are applied before the Samyam or Sunyamanyat are considered.

1. $\frac{p}{a}+\frac{q}{c}=\frac{r}{e}+\frac{s}{g}$
2. $\frac{p}{b}+\frac{q}{d}=\frac{r}{f}+\frac{s}{h}$

If the test (2) is satisfied then Sunyamanyat is applicable, i.e., x = 0. (One solution has value, Sunyamanyat means the other solution in the sense, which is valueless).

If the first test is satisfied then one has to try several possibilities.

a) To verify if $D_1 + D_2 = D_3 + D_4$ in which case that relation is Samyam and hence is equal to zero. In case it is not satisfied, even then one has to apply Paravartya to obtain a modified equation (1), which is subject to the test $D_1' + D_2' = D_3' + D_4'$.

b) Even at this stage, if the relation is not satisfied then one can try if $N_1D_2 + N_2D_1 = N_3D_4 + N_4D_3$ in which case this relation acts as Samyam and hence it is zero. This relation can be first tested for the given equation and if it is not satisfied then one can try this relation after performing Paravarthya division to the given equation. If it is satisfied then this relation acts as Samyam and is equal to zero.

c) If the equation is in the form of $\frac{a}{x+a}+\frac{b}{x+b}=\frac{c}{x+c}+\frac{d}{x+d}$ and a + b = c + d then the Sunyam Samya Samuccaya Sutram is applied to $D_1 + D_2 = D_3 + D_4$

Proof:

$\frac{a}{x+a}+\frac{b}{x+b}=\frac{c}{x+c}+\frac{d}{x+d}$ and a + b = c + d

$\frac{a}{x+a}-\frac{d}{x+d}=\frac{c}{x+c}-\frac{b}{x+b}$

$\frac{ax+ad-dx-ad}{(x+a)(x+d)} \quad \frac{cx+bc-bx-bc}{(x+c)(x+b)}$

$\frac{(a-d)x}{(x+a)(x+d)}=\frac{(c-b)x}{(x+c)(x+b)}$

Since a – d = c – b

$\therefore x = 0$

$(x + a)(x + d) = (x + c)(x + b)$

$x^2 + (a+d)x + ad = x^2 + (c+b)x+bc$

$(a + d - c - b)x = bc - ad$

$2(d - b)x = b(a + b - d) - ad = ab + b^2 - bd - ad = a(b - d) + b(b - d) -2x = a + b$

$x = -(a + b)/2$

Examples:

1. $\frac{5}{x+5}+\frac{7}{x+7}=\frac{1}{x+1}+\frac{11}{x+11}$

Current Method

$$\frac{5}{x+5}+\frac{7}{x+7}=\frac{1}{x+1}+\frac{11}{x+11}$$

$$\frac{5}{x+5}-\frac{1}{x+1}=\frac{11}{x+11}-\frac{7}{x+7}$$

$$\frac{5x+5-x-5}{(x+5)(x+1)}=\frac{11x+77-7x-77}{(x+11)(x+7)}$$

$$\frac{4x}{(x+5)(x+1)}=\frac{4x}{(x+11)(x+7)}$$

$$4x[\frac{1}{(x+5)(x+1)}-\frac{1}{(x+11)(x+7)}]=0$$

$x = 0$ or $(x + 5)(x + 1) = (x + 11)(x + 7)$
$x = 0$ or $x^2 + 6x + 5 = x^2 + 18x + 77$
$x = 0$ or $12x = -72$
$\therefore\ x = 0$ or -6

Vedic Method

$$\frac{5}{x+5}+\frac{7}{x+7}=\frac{1}{x+1}+\frac{11}{x+11}$$

$\frac{5}{5}+\frac{7}{7}=\frac{1}{1}+\frac{11}{11}$ (Yes)

$\therefore$ By Sunyamanyat, one root is zero.

$\frac{5}{1}+\frac{7}{1}=\frac{1}{1}+\frac{11}{1}$ (Yes) by Sunyam Samya Samuccaye Sutra,

$D_1 + D_2 = D_3 + D_4 = 2x + 12$
$2x + 12 = 0$
$x = -6$

$\therefore\ x = 0, -6$

2. Solve $\frac{1}{2x+1}+\frac{3}{5x+3}=\frac{1}{5x+1}+\frac{9}{10x+9}$

Current Method

$$\frac{1}{2x+1}+\frac{3}{5x+3}=\frac{1}{5x+1}+\frac{9}{10x+9}$$

$$\frac{1}{2x+1}-\frac{9}{10x+9}=\frac{1}{5x+1}-\frac{3}{5x+3}$$

$$\frac{10x+9-18x-9}{(2x+1)(10x+9)}=\frac{5x+3-15x-3}{(5x+1)(5x+3)}$$

$$\frac{-8x}{(2x+1)(10x+9)}=\frac{-10x}{(5x+1)(5x+3)}$$

$$x[\frac{4}{(2x+1)(10x+9)}-\frac{5}{(5x+1)(5x+3)}]=0$$

$x = 0$ or $4(5x + 1)(5x + 3)$
$= 5(2x + 1)(10x + 9)$
$4(25x^2 + 20x + 3) = 5(20x^2 + 28x + 9)$
$100x^2 + 80x + 12 = 100x^2 + 140x + 45$
$-33 = 60x$
$x = -33/60 = -11/20$

$x = 0$ or $\frac{-11}{20}$

Vedic Method

$$\frac{1}{2x+1}+\frac{3}{5x+3}=\frac{1}{5x+1}+\frac{9}{10x+9}$$

$\frac{1}{1}+\frac{3}{3}=\frac{1}{1}+\frac{9}{9}$ (Yes)

$\therefore$ By Sunyamanyat $x = 0$

$\frac{1}{2}+\frac{3}{5}=\frac{1}{5}+\frac{9}{10}$ (Yes)

By Paravartya Division

$$1-\frac{2x}{1+2x}+1-\frac{5x}{5x+3}=1-\frac{5x}{1+5x}+1-\frac{10x}{10x+9}$$

$$\frac{2x}{1+2x}+\frac{5x}{5x+3}=\frac{5x}{1+5x}+\frac{10x}{10x+9}$$

$N_1D_2 + N_2D_1 = 10x + 6 + 5 + 10x = 20x + 11$
$N_3D_4 + N_4D_3 = 50x + 45 + 10 + 50x$
$= 100x + 55 = 5(20x + 11)$

By Sunyam Samya Samuccaye Sutra

$20x + 11 = 0 \Rightarrow x = -11/20$

$\therefore\ x = 0, \frac{-11}{20}$

3. Solve $\frac{a+b}{x+a+b}-\frac{a+b-c}{x+a+b-c}=\frac{b+c-a}{x+b+c-a}-\frac{b-a}{x+b-a}$

Current Method

$$\frac{a+b}{x+a+b}-\frac{a+b-c}{x+a+b-c}=\frac{b+c-a}{x+b+c-a}-\frac{b-a}{x+b-a}$$

$$\frac{(a+b)(x+a+b-c)-(x+a+b)(a+b-c)}{(x+a+b)(x+a+b-c)}=\frac{(b+c-a)(x+b-a)-(b-a)(x+b+c-a)}{(x+b+c-a)(x+b-a)}$$

$$\frac{ax+bx+a^2+ab+ab+b^2-ac-bc-ax-a^2-ab-bx-ab-b^2+cx+ac+bc}{(x+a+b)(x+a+b-c)}$$

$$=\frac{bx+cx-ax+b^2+bc-ab-ab-ac+a^2-bx+ax-b^2+ab-bc+ac+ab-a^2}{(x+b+c-a)(x+b-a)}$$

$$\frac{cx}{(x+a+b)(x+a+b-c)}=\frac{cx}{(x+b+c-a)(x+b-a)}$$

$x=0$ or $(x+a+b)(x+a+b-c)=(x+b+c-a)(x+b-a)$

$x^2+ax+bx+ax+a^2+ab+bx+ab+b^2-cx-ac-bc$

$=x^2+bx+cx-ax+bx+b^2+bc-ab-ax-ab-ac+a^2$

$2ax+2ab-cx-bc=cx-2ax-2ab+bc$

$(2a-c)x+2ab-bc=(c-2a)x+bc-2ab$

$(2a-c-c+2a)x=bc-2ab-2ab+bc$

$2(2a-c)x=2b(c-2a)$

$x=\frac{b(c-2a)}{(2a-c)}=-b$

$\therefore x=-b, 0$

Vedic Method

$$\frac{a+b}{x+a+b}-\frac{a+b-c}{x+a+b-c}=\frac{b+c-a}{x+b+c-a}-\frac{b-a}{x+b-a}$$

by paravartya

$$\frac{a+b}{x+a+b}+\frac{b-a}{x+b-a}=\frac{b+c-a}{x+b+c-a}+\frac{a+b-c}{x+a+b-c}$$

$$\frac{a+b}{a+b}+\frac{b-a}{b-a}=\frac{b+c-a}{b+c-a}+\frac{a+b-c}{a+b-c}\text{ (Yes)}$$

$\therefore$ By Sunyamanyat, One Root is Zero

$$\frac{a+b}{1}+\frac{b-a}{1}=\frac{b+c-a}{1}+\frac{a+b-c}{1}$$

$2b=2b$ (Yes)

By Sunyam Samya Samuccaye Sutra

$D_1+D_2=D_3+D_4=2x+2b$

$x+a+b+x+b-a=0$

$2x+2b=0$

$x=-b$

$\therefore x=-b, 0$

Special Type IV

(III) Sunyamanyat and Paravartya (Merger) Sutra:

This makes use of Sunyamanyat and Paravartya (merger).
This can be divided into two forms:

$bx + c \quad ex + f = \frac{g}{hx + j}$, where a, b, c, d, e, f, g, h, and j are numbers.

Merger is possible if the right hand side contains one expression and left-hand side may contain any number of expressions. The first test for the application of merger is summation of numerators on the LHS = numerator on the RHS.

Application of merger Paravartya can be considered when $N_1 + N_2$ is equal to N_3, in which case merger is processed and the solution is obtained. (Refer Merger in simple equations).

In case $N_1 + N_2 \neq N_3$ then test the sum of the ratios of numbers in $\frac{N}{D}$ on LHS = RHS, i.e., $\frac{a}{c} + \frac{d}{f} = \frac{g}{j}$.

This is the condition for Sunyamanyat, i.e., x = 0.

For the second solution one has to apply Paravartya Division and test for merger feasibility, i.e., $N_1 + N_2$ (LHS) = N_3 (RHS). After performing the merger and equalizing the numerators one can get second solution by Sunyam Samya Samuccaye.

Examples:

1. Solve $\frac{1}{x+1} + \frac{9}{x+3} = \frac{16}{x+4}$

Current Method

$\frac{1}{x+1} \quad x+3 \quad \frac{16}{x+4}$

$\frac{9}{x+3} = \frac{16}{x+4} \quad \frac{1}{x+1}$

$\frac{9}{x+3} \quad \frac{16x+16-x-4}{(x+4)(x+1)}$

$\frac{9}{x+3} \quad \frac{15x+12}{(x+4)(x+1)}$

$\frac{3}{x+3} = \frac{5x+4}{(x+4)(x+1)}$

$3(x + 4)(x + 1) = (x + 3)(5x + 4)$

$3(x^2 + 5x + 4) = 5x^2 + 19x + 12$

$3x^2 + 15x + 12 = 5x^2 + 19x + 12$

$x(3x + 15) = x(5x + 19)$

$\therefore x = 0$

Vedic Method

$\frac{1}{x+1} + \frac{9}{x+3} \quad \frac{16}{x+4}$

$\frac{1}{1} + \frac{9}{3} = \frac{16}{4}$ (Yes)

By Sunyamanyat x = 0
For another solution

Paravartya Division

$1 - \frac{x}{x+1} + 3 - \frac{3x}{x+3} = 4 - \frac{4x}{x+4}$

$3x + 15 = 5x + 19$
$2x + 4 = 0$
$x = -2$
$\therefore x = -2, 0$

$\frac{x}{x+1} + \frac{3x}{x+3} \quad \frac{4x}{x+4}$

$\frac{1}{x+1} + \frac{3}{x+3} \quad x+4$

1 + 3 = 4. Hence we can apply Merger.

By Merger method

$\frac{-3}{x+1} + \frac{-3}{x+3} = 0$

Numerators are equal

Therefore, $D_1 + D_2 = 0$

$2x + 4 = 0$
$x = -2$
$\therefore x = -2, 0$

2. **Solve** $\frac{8}{3x+2} + \frac{5}{12x+1} \quad \frac{27}{8x+3}$

Current Method

$\frac{8}{3x+2} + \frac{5}{12x+1} \quad \frac{27}{8x+3}$

$3x+2 \quad \frac{27}{8x+3} \quad 12x+1$

$\frac{8}{3x+2} \quad \frac{324x+27-40x-15}{(8x+3)(12x+1)}$

$\frac{8}{3x+2} \quad \frac{284x+12}{(8x+3)(12x+1)}$

$3x+2 \quad \frac{71x+3}{(8x+3)(12x+1)}$

$2(8x + 3)(12x + 1) = (71x + 3)(3x + 2)$
$2(96x^2 + 44x + 3) = 213x^2 + 151x + 6$
$192x^2 + 88x + 6 = 213x^2 + 151x + 6$
$x(192x + 88) = x(213x + 151)$
$\therefore x = 0$
$192x + 88 = 213x + 151$
$21x = -63$
$x = -3$
$\therefore x = -3, 0$

Vedic Method

$\frac{8}{3x+2} + \frac{5}{12x+1} \quad \frac{27}{8x+3}$

$\frac{8}{2} + \frac{5}{1} = \frac{27}{3}$ (Yes)

By Sunyamanyat, $x = 0$

$\therefore$ By Paravartya Division

$4 - \frac{12x}{3x+2} + 5 - \frac{60x}{12x+1} = 9 - \frac{72x}{8x+3}$

$\frac{12}{3x+2} + \frac{60}{12x+1} \quad \frac{72}{8x+3}$

$\frac{1}{3x+2} + \frac{5}{12x+1} \quad 8x+3$

By LCM, to make the coefficients in the denominator equal.

$\frac{8}{24x+16} \quad \frac{10}{24x+2} \quad \frac{18}{24x+9}$

$$\frac{4}{24x+16}+\frac{5}{24x+2}=\frac{9}{24x+9}$$

As $N_1+N_2 = N_3$ on the (RHS)

By Merger Method

$$\frac{28}{24x+16}-\frac{35}{24x+2}=0$$

$$\frac{1}{3x+2}-\frac{5}{12x+1}=0$$

Making Numerators Equal

$$\frac{5}{15x+10}-\frac{5}{12x+1}=0$$

$D_1 + D_2 = 0$

by Sunyam Samyam Samucayah

$15x + 10 - 12x - 1 = 0$

$3x + 9 = 0$

$x = -3$

$\therefore x = -3, 0$

3. Solve $\frac{a+b-c}{x+a+b-c}=\frac{2(b-a-c)}{x+(b-a-c)}-\frac{b+c-a}{x+b+c-a}$

Current Method

$$\frac{a+b-c}{x+a+b-c}=\frac{2(b-a-c)}{x+(b-a-c)}-\frac{b+c-a}{x+b+c-a}$$

$$=\frac{2(b-a-c)(x+b+c-a)-(b+c-a)(x+b-a-c)}{(x+b-a-c)(x+b+c-a)}$$

$$=\frac{(2b-2a-2c)(x+b+c-a)-(b+c-a)(x+b-a-c)}{(x+b-a-c)(x+b+c-a)}$$

$$(2bx+2b^2+2bc-2ab-2ax-2ab-2ac+2a^2-2cx$$

$$-2bc-2c^2+2ac-bx-b^2+ab+bc-cx-bc+ac$$

$$\frac{a+b-c}{x+a+b-c}=\frac{+c^2+ax+ab-a^2-ac)}{(x^2+bx+cx-ax+bx+b^2+bc-ab-ax-ab \quad -ac+a^2-cx-bc-c^2+ac)}$$

$$\frac{a+b-c}{x+a+b-c}=\frac{(b-a-3c)x+a^2+b^2-c^2-2ab}{x^2+(2b-2a)x+a^2+b^2-c^2-2ab}$$

$$(b-a-3c)x^2+(a^2+b^2-c^2-2ab)x+(a+b-c)(b-a-3c)x+(a^2+b^2-c^2-2ab)$$
$$(a+b-c)$$

Vedic Method

$$\frac{a+b-c}{x+a+b-c}=\frac{2(b-a-c)}{x+(b-a-c)}-\frac{b+c-a}{x+b+c-a}$$

$$\frac{a+b-c}{x+a+b-c}+\frac{b+c-a}{x+b+c-a}=\frac{2(b-a-c)}{x+b-a-c}$$

$$\frac{a+b-c}{a+b-c}+\frac{b+c-a}{b+c-a}=\frac{2(b-a-c)}{b-a-c}\text{ (Yes)}$$

∴ By Sunyamanyat $x = 0$

By Paravartya Division

$$1-\frac{x}{x+a+b-c}+1-\frac{x}{x+b+c-a}=2-\frac{2x}{x+b-a-c}$$

$$\frac{1}{x+a+b-c}+\frac{1}{x+b+c-a}=\frac{2}{x+b-a-c}$$

$1+1=2$ (Numerators)

By Merger Method

$$\frac{2a}{x+a+b-c}+\frac{2c}{x+b+c-a}=0$$

Making Numerators Equal

$= (a+b-c)x^2 + (a+b-c)(2b-2a)x + (a+b-c)(a^2+b^2-c^2-2ab)$

$x[(b-a-3c)x + a^2+b^2-c^2-2ab + (a+b-c)(b-a-3c)]$

$= x[(a+b-c)x + (a+b-c)(2b-2a)]$

$\therefore x = 0$

$(b-a-3c-a-b+c)x = (a+b-c)(2b-2a-b+a+3c) - a^2 - b^2 + c^2 + 2ab$

$(-2a-2c)x = (a+b-c)(b-a+3c) - a^2 - b^2 + c^2 + 2ab$

$= ab + b^2 - bc - a^2 - ab + ac + 3ac + 3bc - 3c^2 - a^2 - b^2 + c^2 + 2ab$

$= -2a^2 - 2c^2 + 2bc + 4ac + 2ab$

$(a+c)x = a^2 + c^2 - bc - 2ac - ab$

$= (a-c)^2 - b(a+c)$

$$x = \frac{(a-c)^2}{a+c} - b$$

$$\therefore x = 0, \frac{(a-c)^2}{a+c} - b$$

$$\frac{ac}{cx+ac+bc-c^2} + \frac{ac}{ax+ab+ac-a^2} = 0$$

$D_1 + D_2 = 0$ Sunyam Samya Samuccaye

$ax + ab + ac - a^2 + cx + ac + bc - c^2 = 0$

$(a+c)x = a^2 + c^2 - 2ac - bc - ab$

$= (a-c)^2 - b(a+c)$

$$x = \frac{(a-c)^2}{a+c} - b$$

$$\therefore x = 0, \frac{(a-c)^2}{a+c} - b$$

2) $\frac{ax+b}{cx+d} + \frac{ex+f}{gx+h} = \frac{px+q}{rx+s}$ where a, b, c, d, e, f, g, h, p, q, r and s are numbers. Even here one can test for the validity of the relation that $\frac{b}{d} + \frac{f}{h} = \frac{q}{s}$. If so, one solution is got by Sunyamanyat, i.e., $x = 0$.

The second solution is obtained by mere division by Paravartya combined with merger at a suitable stage. The stage of merger is chosen after one or more Paravartya division operations.

Examples:

Eg.(i) Solve $\frac{3x+2}{3x+1}+\frac{2x+3}{2x+1}=\frac{8x+5}{4x+1}$

Current Method

$$\frac{3x+2}{3x+1}+\frac{2x+3}{2x+1}=\frac{8x+5}{4x+1}$$

$$\frac{(3x+2)(2x+1)+(2x+3)(3x+1)}{(3x+1)(2x+1)}=\frac{8x+5}{4x+1}$$

$$\frac{6x^2+7x+2+6x^2+11x+3}{6x^2+5x+1}=\frac{8x+5}{4x+1}$$

$$\frac{12x^2+18x+5}{6x^2+5x+1}=\frac{8x+5}{4x+1}$$

$48x^3+72x^2+20x+12x^2+18x+5$
$=48x^3+40x^2+8x+30x^2+25x+5$
$8\ x^2+38x+5=70x^2+33x+5$
$14x^2+5x=0$
$x(14x+5)=0$
$\therefore x=0$ and $x=-5/14$

Vedic Method

$$\frac{3x+2}{3x+1}+\frac{2x+3}{2x+1}=\frac{8x+5}{4x+1}$$

$\frac{2}{1}+\frac{3}{1}=\frac{5}{1}$ (Yes)

$\therefore$By Sunyamanyat one root is 0, i.e., $x=0$.

By Paravartya Division,

$$\frac{1}{3x+1}+\frac{2}{2x+1}=\frac{3}{4x+1}$$

$\frac{1}{1}+\frac{2}{1}=\frac{3}{1}$

Again by Paravartya division

$$\frac{3}{3x+1}+\frac{4}{2x+1}=\frac{12}{4x+1}$$

Equalizing x coefficients in the denominators

$$\frac{12}{12x+4}+\frac{24}{12x+6}=\frac{36}{12x+3}$$

$$\frac{1}{12x+4}+\frac{2}{12x+6}=\frac{3}{12x+3}$$

$1+2=3$

$\therefore$ We can apply merger method.

$$\frac{1}{12x+4}+\frac{6}{12x+6}=0$$

$$\frac{1}{12x+4}+\frac{1}{2x+1}=0$$

$\therefore D_1+D_2=0$ By Sunyam Samya Samuccaye

$14x+5=0$

$x=-5/14.$

$\therefore x=0, \frac{-5}{14}$

SECTION – 5

SIMULTANEOUS QUADRATIC EQUATIONS (IN TWO UNKNOWNS)

Simultaneous Quadratic equations have been solved wherein several methods have been applied.

These are shown by way of examples so that similar procedures can be adopted for problems coming under different categories.

The Sutras that are applied are:

1) Vilokanam
2) Methods adopted for Simple Quadratic Equations
3) Substitution methods
4) Methods converting into standard equation form
5) Factorization
6) Sunyamanyat
7) Paravartya and the like

Solving these equations can be carried out in different ways, mostly it is confined to one method for each problem. These equations can be classified in combinations of homogenous and non – homogenous equations.

1)

Current Method

In current method simultaneous quadratic equations are solved in the following way (generally).

From the two given equations an attempt is made to obtain two simultaneous simple equations with the help of standard formulae.

For example the given equations are in the form of $x \pm y = a$, $xy = b$.
Given $x \pm y$, one can obtain $x \mp y$ using the given data and the standard formula for square. After solving for one variable, it is substituted in one of the given equations to obtain other variable or the simultaneous equations x+y, x−y can be solved.

Vedic Method

Modus operandi:
After obtaining one set of values by Vilokanam then the other set can be simply written down by reversal operation:

In this case where $x \pm y$ and xy are given, then the values of x and y can be obtained by Vilokanam which tantamounts to working out standard forms to obtain the counter part $x - y$ or $x + y$, making use of xy. After obtaining one set of values for x and y, the second set is obtained by simple reversal of the first. In the Vedic method this symmetry is conspicuously elicited.

Note :

i) It is interesting to note that among the two equations if the value of xy has got different set of factors, then select one set of factors which will satisfy the given equations by Vilokanam.

ii) In the given simple equation, if x and y can be read by Vilokanam. Then those values satisfy also the Quadratic Expression. (Eg 6)

So for all such problems it is indicated by " * ".

*** Example 1(a):** $x + y = 28$,
$xy = 187$
$(x - y)^2 = (x + y)^2 - 4xy = 784 - 748 = 36$
$x - y = \pm 6$

Current Method

1st set
$x + y = 28$
$x - y = 6$ by elimination

$2x = 34$
$\therefore x = 17, y = 11$

2nd set
$x + y = 28$
$x - y = -6$

$2x = 22$
$\therefore x = 11, y = 17$

$x = 17, 11$; $y = 11, 17$ is the solution.

Vedic Method

$x + y = 28$
$x - y = 6$

Lopanasthapanabhyam
$2x = 34 \Rightarrow x = 17, \therefore y = 11$
Other set is simply written down by reversal operation as $x = 11, y = 17$
* $xy = 187$

$\therefore$ The set of factors of 187 are (1, 187) (11, 17). Out of these if we select (11, 17) it will satisfy the given simple equation by mere Vilokanam. Hence one set of values of x and y are 11 and 17

Other set can be simply written down by reversal operation as $x = 17$ and $y = 11$

Example 1 (b) Solve $x + y = 11$

$xy = 30$

Current Method

$x + y = 11$

$(x + y)^2 = 121$

$4xy = 120$

$(x - y)^2 = (x + y)^2 - 4xy$

$= 121 - 120$

$= 1$

$x - y = \pm 1$

$x + y = 11$ ---------- (1)

$x - y = 1$ ---------- (2)

Adding eq: (1) and eq. (2), we get

$2x = 12 \Rightarrow x = 6$

Substituting x, we get $y = 5$

Vedic Method

$x + y = 11$

$xy = 30$

$(x + y)^2 = 121$

$4xy = 120$

$(x - y)^2 = (x + y)^2 - 4xy$

$= 121 - 120 = 1$

$x - y = \pm 1$

$x + y = 11$ ------ (1)

$x - y = 1$ ------ (2)

By addition, we get

$2x = 12 \Rightarrow x = 6 \therefore y = 5$

After getting one set, i.e., $x = 6$ and $y = 5$ we can directly write another set as follows.

$x + y = 11$ ---------- (1)

$x - y = -1$ ---------- (3)

Adding eq. (1) and eq. (3), we get

$2x = 10 \Rightarrow x = 5$

Substituting x, we get $y = 6$

Other set is simply written down by reversal operation

$\therefore x = 5, y = 6$

$\therefore$ The set of factors of 30 are (1, 30) (2, 15), (3, 10) (5, 6). Out of these if we select (5,6) it will satisfy the given simple equation by mere Vilokanam. Hence one set of values of x and y are 5 and 6. Other se can be simply written down by reversal operations $x = 6$ and $y = 5$

2) If x – y and xy are given, then the second set has the reversal with a minus sign.

*** Example2:** $x - y = 5$
$xy = 126$
$(x + y)^2 = (x - y)^2 + 4xy = 25 + 504 = 529$
$x + y = \pm 23$

Current Method

$x - y = 5$
$\underline{x + y = 23}$
$2x = 28 \Rightarrow x = 14, y = 9$

$x - y = 5$
$\underline{x + y = -23}$
$2x = -18 \Rightarrow x = -9, y = -14$

Vedic Method

$x - y = 5$
$\underline{x + y = 23}$
$2x = 28 \Rightarrow x = 14, \quad \therefore y = 9$
* $xy = 126$

$\therefore$ The set of factors of 126 are (1, 126) (2, 63) (3, 42) (6, 21) (7, 18) (9, 14) out of these sets of factors if x = 14 and y = 9 then the given simple equation will be satisfied by Vilokanam.

Now the second set can simply written down by reversal operation with minus sign as $x = -9$, $y = -14$.

(3) These can also be extendable to the most general form, as given below.

Current Method

If the equations are of the form $ax \pm by = d$, $c\,xy = e$. By using the given data we can obtain $ax \mp by$ and can solve x and y.

Vedic Method

If the two equations are in the form $ax \pm by = p$, $cxy = q$, even then the counter part of $(ax \mp by)$ is first obtained by using the standard form and later, from one set, the second set can be simply written down by reversal operation. The formula for the internal relation between the two sets of the values can be worked out.
If one set of values of x and y are m and n respectively, then the other set is obtained by using the following formula.

$$x = n \left| \frac{b}{a} \right. \qquad y = m \left| \frac{a}{b} \right.$$

*** Example3 :** $3x + 2y = 16$, $xy = 10$
$(3x - 2y)^2 = (3x + 2y)^2 - 24xy = 256 - 240 = 16$
$3x - 2y = \pm 4$

Current Method

$3x + 2y = 16$
$\underline{3x - 2y = 4}$
$6x = 20$
$x = \frac{10}{3}, y = 3$

$3x + 2y = 16$
$\underline{3x - 2y = -4}$
$6x = 12$
$x = 2, y = 5$

Vedic Method

$3x + 2y = 16$
$3x - 2y = 4$
$6x = 20 \Rightarrow x = \frac{10}{3}, y = 3$

$m = \frac{10}{3}\ n = 3$

Other set is $x = 3\left|\frac{2}{3}\right| = 2;\ y = \left(\frac{3}{2}\right) =$

* xy = 10 The sets of factors of 10 are (1, 10), (2, 5) by mere Vilokanam if x = 2 and y = 5 will satisfy the given simple equation.

∴ one set of values are x = 2 & y = 5 and the other set is simply written down by the general formula.

m = 2 n = 5

$\therefore\ x = 5\left(\frac{2}{3}\right) = \frac{10}{3},\qquad y = \left(\frac{3}{2}\right) = 3$

*** Example 4:** $x - 3y = 1$

$xy = 4$

Current Method

$(x + 3y)^2 = (x - 3y)^2 + 12xy$

$= 1 + 48 = 49$

$x + 3y = \pm 7$

1st Set	2nd Set
$x - 3y = 1$	$x - 3y = 1$
$x + 3y = 7$	$x + 3y = -7$
$2x = 8$	$2x = -6$
$x = 4$	$x = -3$
$\therefore\ y = 1$	$\therefore\ y = -\frac{4}{3}$

$\therefore\ x = 4$ or -3 } is the solution

$\therefore\ y = 1$ or $-\frac{4}{3}$

Vedic Method

By mere Vilokanam one can solve for values of x any y. As xy = 4 it has the factors (4, 1) and (2, 2). But x = 4 , y = 1 will satisfy the given simple equation.

m = 4 ; n = 1

The other set is

$x = n\left(\frac{b}{a}\right) = 1\left(\frac{-3}{1}\right) = -3$

$y = m\left(\frac{a}{b}\right) = 4\left(\frac{1}{-3}\right) = -\frac{4}{3}$

4) When the equations are of the form $x^2 + y^2 = a$, $xy = b$.

It can be brought into standard forms $(x \pm y)^2$ leading to two simultaneous equations $x + y$ and $x - y$ as given below.

*** Example5 :** $x^2 + y^2 = 170$, $xy = 13$

Current Method

$(x+y)^2 = x^2 + y^2 + 2xy = 170 + 26 = 196$

$\therefore \quad x + y = \pm 14$

$(x-y)^2 = x^2 + y^2 - 2xy = 170 - 26 = 144$

$\therefore x - y = \pm 12$

From x + y and x − y, using the procedure given in (1)

x, y can be solved.

$x = 13, 1, -1, -13 \quad y = 1, 13, -13, -1$

$x = 13$ or $y = 1$; $x = 1$ or $y = 13$; $x = -1$ or $y = -13$; $x = -13$ or $y = -1$

Vedic Method

$(x+y)^2 = x^2 + y^2 + 2xy = 170 + 26 = 196$

$\therefore \quad x + y = \pm 14$

$(x-y)^2 = x^2 + y^2 - 2xy = 170 - 26 = 144$

$\therefore \quad x - y = \pm 12$

a	b
$x + y = 14$	$x + y = -14$
$x - y = 12$	$x - y = 12$
$2x = 26$	$2x = -2$
$x = 13, y = 1$	$x = -1, y = -13$

a) Other set is simply written down by reversal operation as $x = 1, y = 13$

Other set is simply written down by reversal operation as $x = -13, y = -1$

5) If the given equations are of the form $x \pm y = a$, $x^2 + y^2 = b$, then one should aim at the value of xy and solve for x and y.

*** Example 6:** $x + y = 13, x^2 + y^2 = 97$

Current Method

$$xy = \frac{(x+y)^2 - (x^2+y^2)}{2} = 36$$

$(x-y)^2 = (x+y)^2 - 4xy = 25$

$x - y = \pm 5$

From the set of x + y and x − y, the values of x and y can be obtained by using the Procedure given in (1) as

$x = 9, 4 \quad ; \quad y = 4, 9$

Vedic Method

$$xy = \frac{(x+y)^2 - (x^2+y^2)}{2} = 36$$

$(x-y)^2 = (x+y)^2 - 4xy = 25$

$x - y = \pm 5$

$x + y = 13$

$x - y = 5$

$2x = 18 \quad x = 9 \quad y = 4$

Other set is simply written down by reversal operation as

$x = 4$, and $y = 9$.

6) If the given equations are of the form $x^3 \pm y^3 = p$, $x \pm y = q$ one can try different methods.

i) Standard form of one equation in terms of the other equation. From this a second degree equation together with the first degree equation are solved.

*** Example: 7** $x^3 - y^3 = 218$

$x - y = 2$

$(x-y)(x^2 + xy + y^2) = 218$

$x^2 + xy + y^2 = 109$

$x^2 - 2xy + y^2 = 4$

$3xy = 105 \quad xy = 35$

Current Method

With the value (x – y) and xy one can obtain (x + y) as
$(x + y)^2 = (x - y)^2 + 4xy = 4 + 4(35) = 144$
$x + y = \pm 12$

1st Set	2nd set
$x - y = 2$	$x - y = 2$
$x + y = 12$	$x + y = -12$
$2x = 14$	$2x = -10$
$x = 7$	$x = -5$
$y = 5$	$y = -7$

Vedic Method

By mere Vilokanam one can solve for values of x and y as x = 7 and y = 5 as xy = 35 has the factors 7, 5 which will satisfy the given data.

The second set can simply be written down by reversal operation with minus sign as
x = – 5 and y = – 7

ii) A direct substitution of one variable in terms of other variable will be useful in writing down the higher degree equation in a single variable in Quadratic form. Then by solving Quadratic equation, values of x and y are obtained.

Example : $x^3 - y^3 = 218$
$x - y = 2$
$x = 2 + y$

Substituting this value in the first equation
$(2 + y)^3 - y^3 = 218$
$y^3 + 6y^2 + 12y + 8 - y^3 = 218$
$6y^2 + 12y - 210 = 0$

Solving the simple quadratic equation by differentiation method
$12y + 12 = \pm\sqrt{144 + 5040} = \pm\sqrt{5184}$
$12y + 12 = \pm 72 \quad y = 5, -7 \quad x = 7, -5$

Example 8 : $x^3 + y^3 = 3473$
$x + y = 23$
$(x^3 + y^3) = (x + y)(x^2 - xy + y^2) = 3473$
$x^2 - xy + y^2 = 151$
$x^2 + 2xy + y^2 = 529$
$3xy = 378 \quad xy = 126$

Current Method

$(x - y)^2 = (x + y)^2 - 4xy = 25$
$x - y = \pm 5$

1st Set	2nd Set
$x + y = 23$	$x + y = 23$
$x - y = 5$	$x - y = -5$
$2x = 28$	$2x = 18$
$x = 14$	$x = 9$
$y = 9$	$y = 14$

Vedic Method

Following similar steps as above in Eg 7 (i) the first set of solution give the values as 14 and 9 for x and y respectively and the second set is simply written down by reversal operation as x = 9 and y = 14

The above methods can also be applicable for reciprocal equations.

7) When reciprocal equations are of the form $\frac{1}{x} \pm \frac{1}{y} = k_1$ and $xy = k_2$. It can be brought into standard form $(x \pm y)^2$ leading to two simultaneous equations x + y and x – y.

*** Example 9 :** $\frac{1}{x} + \frac{1}{y} = \frac{9}{20}$

$xy = 20$

Current Method

$$\left(\frac{1}{x} - \frac{1}{y}\right)^2 = \left(\frac{1}{x} + \frac{1}{y}\right)^2 - \frac{4}{xy}$$

$$= \left(\frac{9}{20}\right) \quad 20 \quad \frac{1}{400}$$

$$\therefore \quad \frac{1}{x} - \frac{1}{y} = \pm\frac{1}{20}$$

Solving

1st set	2nd set
$\frac{1}{x} + \frac{1}{y} = \frac{9}{20}$	$\frac{1}{x} + \frac{1}{y} = \frac{9}{20}$
$\frac{1}{x} - \frac{1}{y} \quad \frac{1}{20}$	$\frac{1}{x} - \frac{1}{y} \quad \cdot\frac{1}{20}$
$\frac{2}{x} = \frac{10}{20}$	$\frac{2}{x} = \frac{8}{20}$
$\therefore x = 4$	$\therefore x = 5$
$y = 5$	$y = 4$

Vedic Method

$\frac{x+y}{xy} \quad \frac{9}{20}$

$\Rightarrow x + y = 9 \qquad xy = 20$

$(x - y)^2 = (x + y)^2 - 4xy = 81 - 80 = 1$

$x - y = \pm 1$

$x + y = 9$

$\underline{x - y = 1}$

$2x = 10$

$x = 5 \quad \therefore \; y = 4$

$xy = 20$ The set of factors for 20are (5, 4), (10, 2), (20, 1). Out of these if we select(5, 4)

It will satisfy the given simple equation by mere Vilokanam. Hence one set of values of x and y are 5 and 4.

Other set is simply written down by reversal operation as x = 4 and y = 5

x = 4, 5 ; y = 5,4 is the solution.

8) A reciprocal form of $\frac{1}{x} + \frac{1}{y} = a$, $\frac{1}{x^2} + \frac{1}{y^2} = b$ can also be fit into the above working with the help of suitable standard forms.

Example 10: $\frac{1}{x^2} + \frac{1}{y^2} = \frac{481}{576}$, ·(1)

$\frac{1}{x} + \frac{1}{y} = \frac{29}{24}$ ·(2)

Current Method

From (2) by squaring $\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=\frac{841}{576}$

By subtraction $\frac{2}{xy}=\frac{360}{576}=\frac{5}{8}$

$\left[\frac{1}{x}-\frac{1}{y}\right]^2=\frac{1}{x^2}+\frac{1}{y^2}-\frac{2}{xy}=\frac{481}{576}-\frac{360}{576}=\frac{121}{576}$

$\therefore \frac{1}{x}-\frac{1}{y}=\pm\frac{11}{24}$

1[st] Set	2[nd] Set
$\frac{1}{x}+\frac{1}{y}=\frac{29}{24}$	$\frac{1}{x}+\frac{1}{y}=\frac{29}{24}$
$\frac{1}{x}-\frac{1}{y}=\frac{11}{24}$	$\frac{1}{x}-\frac{1}{y}=\frac{-11}{24}$
$\frac{2}{x}=\frac{40}{24}$	$\frac{2}{x}=\frac{18}{24}$
$x=\frac{6}{5}; y=\frac{8}{3}$	$x=\frac{8}{3}; y=\frac{6}{5}$

Vedic Method

Converting the reciprocals to other variables,

Put $\frac{1}{x}=p, \frac{1}{y}=q$

$\therefore$ Given equations are $p+q=\frac{29}{24}$,

$p^2+q^2 = \frac{481}{576}$

$pq=\frac{(p+q)^2-(p^2+q^2)}{2}=\frac{5}{16}$

$(p-q)^2=p^2+q^2-2pq=\frac{481}{576}-\frac{5}{8}=\frac{121}{576}$;

$\therefore p-q = \pm\frac{11}{24}$,

$p+q=\frac{29}{24}$

$p-q=\frac{11}{24}$

$2p = \frac{5}{3} \Rightarrow p=\frac{5}{6}$,

$\therefore x=\frac{6}{5}, y=\frac{8}{3}$

Other set is simply written down by reversal operation as $x = \frac{8}{3}, y=\frac{6}{5}$

9) If in the given equations one is a second degree general equation of the form $ax^2+2hxy+by^2+2gx+2fy+c=0$ and the other is first degree equation of the form $ax \pm by = c$, then one can solve the equations in several methods..

*** Example 11:** $x + 3y = 10$ ——— (1)

$x^2 + 3xy + 2x = 48$ ——— (2)

By Vedic Method Vilokanam helps us in solving the problem

If x = 4 and y = 2 the second equation can be explained.

(Or)

Taking out common factor in the 2[nd] degree equation.

$x(x+3y)+2x=48$

$x(10)+2x=48 \quad x=4, \quad y=2$

(Or)

It's equivalent to writing down the Second degree equation in terms of the 1[st] equation and see if the resulting equation formed solves the problem.

On substituting $x = 10 - 3y$ in the 2[nd] equation one gets $36y = 72$; $y = 2$, $x = y$

*** Example 12:** $2x + y = 3$

$2x^2 + 3xy + y^2 + 3x + 2y = 9$

One can write down the second degree equation and see if two Simultaneous equations can be formed. Solve the two Simultaneous equations by Paravartya or Sunyamanyat.

$(2x + y)(x + y) + 3x + 2y = 9$

$3x + 3y + 3x + 2y = 9$

$6x + 5y = 9$

Current Method

$2x + y = 3$
$6x + 5y = 9$

$6x + 3y = 9$
$6x + 5y = 9$

$-2y = 0$

$\Rightarrow y = 0$

$x = \frac{3}{2}$

Vedic Method

$2x + y = 3$
$6x + 5y = 9$

Here it is seen that

$$\frac{2x}{6x} = \frac{3}{9}$$

Hence by Sunyamanyat $y = 0$

$x = \frac{3}{2}$

iii) An attempt is made to convert the higher degree equation to the same degree equation in terms of the given lower degree equation. This is achieved by division of the given second degree equation by the given lower degree homogenous expression. This is followed by proper substitution of the terms to arrive at quadratic equation in single variable which can be solved for y and hence x also or the 2nd degree part can be factorised in terms of the lower degree equation and the difference counted. Finally aim at single variable equation.

Example 13 : $x + 2y = 5$

$x^2 + 3xy - 3y^2 + 6x + 3y = 0$

$(x + 2y)(x + y) - 5y^2 + 6x + 3y = 0$

$5x + 5y - 5y^2 + 6x + 3y = 0$

$11x - 5y^2 + 8y = 0$ But $x = 5 - 2y$

$11(5 - 2y) - 5y^2 + 8y = 0$

$55 - 14y - 5y^2 = 0$

$5y^2 + 14y - 55 = 0$

Vedic Method

First differential = $\pm\sqrt{Discriminant}$

$10y + 14 = \pm\sqrt{196 + 1100}$

$10y + 14 = \pm 36$

$y = -5, \frac{11}{5}, \quad x = 15, \frac{3}{5}$

Example 14 : $x + y = 3$

$4x^2 + y^2 = 26$

$(x + y)(4x - 4y) + 5y^2 = 26$ But $x + y = 3$

$12x - 12y + 5y^2 = 26$

$12(3 - y) - 12y + 5y^2 = 26$

$36 - 24y + 5y^2 = 26$

$5y^2 - 24y + 10 = 0$

Vedic Method

$10y - 24 = \pm\sqrt{576 - 200} = \pm\sqrt{376}$

$= \pm 2\sqrt{94}$

$10y = 24 \pm 2\sqrt{94}$

$y = \dfrac{12 \pm \sqrt{94}}{5}$

and

$x = \dfrac{3 \mp \sqrt{94}}{5}$

i) Substitution of one variable in terms of the other variable is followed, thus converting the second degree equation into one variable as quadratic. The quadratic equation is solved by factorization methods or differentiation method and the values are substituted to evaluate the values of the other variable.

Example 15 : $x + 2y = 9$, $3y^2 - 5x^2 = 43$

$x = 9 - 2y$

$3y^2 - 5(9 - 2y)^2 = 43$

$3y^2 - 5(81 + 4y^2 - 36y) = 43$

$17y^2 - 180y + 448 = 0$

Vedic Method

Factorization by Argumentation considering the factors of 448 which will explain the y terms

$(17y - 112)(y - 4) = 0$

$y = 4$ or $y = \dfrac{112}{17}$

$x = 1$ or $x = -\dfrac{71}{17}$

10) If both the equations are of the form $ax^2 + 2hxy + by^2 = k$ then one can solve it by several methods.

Vedic Method

All the following methods are suggested by Swamiji in his book on Vedic Mathematics.

i) If the given two expressions are homogenous with the same degree and have significant numbers on the RHS, a direct cross multiplication of the two given equations is suggested to derive one homogenous equation in the same degree. After this, one can attempt factorization and solve for x and y. (If it doesn't yield to factorization one can try direct substitution of one variable in terms of other and try to derive equation in a single variable. When this is also not possible one has to resort to a simplification by subtraction or addition which may result in factorization. On substitution for x and y, we can solve the values of x and y.)

Example 16: $x^2 + y^2 = 5$ ______ (1)

$2xy - y^2 = 3$ ______ (2)

By cross multiplication,

$3x^2 + 3y^2 = 10xy - 5y^2$

$3x^2 - 10xy + 8y^2 = 0$

$3x^2 - 6xy - 4xy + 8y^2 = 0.$ Adyamadyena Anurupyena.

$(3x - 4y)(x - 2y) = 0$

$\therefore y = \frac{x}{2}$ or $y = \frac{3x}{4}$

Substituting $y = \frac{x}{2}$ in eq.(1) $x^2 = 4 \Rightarrow x = \pm 2, \therefore y = \pm 1$

Substituting $y = \frac{3x}{4}$ in eq.(1) $x^2 = \frac{16}{5} \Rightarrow x = \pm\frac{4}{\sqrt{5}},\quad y = \pm\frac{3}{\sqrt{5}}$

Example 17 : $7x^2 + y^2 - 2xy = 7$ ______ (1)

$14x^2 + xy = 10$ ______ (2)

By cross multiplication

$70x^2 + 10y^2 - 20xy = 98x^2 + 7xy$

$28x^2 + 27xy - 10y^2 = 0$

$(7x - 2y)(4x + 5y) = 0$

$y = \frac{7}{2}x$ or $y = -\frac{4}{5}x$

Substituting $y = \frac{7}{2}x$ in equation 2

$14x^2 + x\left(\frac{7}{2}\right)x = 10$

$x = \pm\frac{2}{\sqrt{7}}$

$y = \pm\sqrt{7}$

Substituting $y = \frac{-4}{5}x$ in equation 2.

$14x^2 + x\left(-\frac{4}{5}\right)x = 10$

$x = \pm\frac{5}{\sqrt{33}}$

$y = \mp\frac{4}{\sqrt{33}}$

ii) In case one of the equation is a zero homogenous equation, cross multiplication results in solving only that equation. If it is factorizable then the solution of x and y can be obtained.

Example 18 : $2x^2 - 9xy + 4y^2 = 0$ ______ (1)

$x^2 + 3xy + y^2 = 11$ ______ (2)

By cross multiplication

$2x^2 - 9xy + 4y^2 = 0$

$(2x - y)(x - 4y) = 0$

$y = 2x$ or $y = \frac{x}{}$

By Substituting $y = 2x$ in equation 2

$x^2 + 3x \cdot 2x + 4x^2 = 11$

$x = \pm 1$

$y = \pm 2$

Substituting $y = \frac{x}{4}$ in equation 2.

$x^2 + 3x\,\frac{x}{4} + \frac{x^2}{16} = 11$

$x = \pm\, 4\sqrt{\frac{11}{29}}$

$y = \pm\sqrt{\frac{11}{29}}$

Example 19 : $6x^2 + xy + 2y^2 = 4$ (1)

$y^2 - 4x^2 = 0$ (2)

By cross multiplication

$y^2 - 4x^2 = 0$

$y = \pm 2x$

Substituting $y = 2x$ in equation 1

$6x^2 + x(2x) + 8x^2 = 4$

$x = \pm\frac{1}{2}$

$y = \pm 1$

Substituting $y = -2x$ in equation 1.

$6x^2 + x(-2x) + 8x^2 = 4$

$x = \pm\frac{1}{\sqrt{3}}$

$y = \mp\frac{2}{\sqrt{3}}$

Example 20 : $2x^2 + 5xy = 12y^2$ (i)

$+ y^2 = 13$ (ii)

$2x^2 + 5xy - 12y^2 = 0$

$x^2 + y^2 = 13$

By cross multiplication

$2x^2 + 5xy - 12y^2 = 0$

$(2x - 3y)(x + 4y) = 0.$

$x = \frac{3}{}.$ or $x = -4y$

Substituting $x = \frac{3}{2}y$ in equation (ii)

$\frac{9}{4}y^2 + y^2 \;\; 13$

$y = \pm 2$

$x = \pm 3$

Substituting $x = -4y$ in equation (ii)

$16y^2 + y^2 = 13$

$y = \pm\sqrt{\frac{13}{17}}$

$x = \mp 4\sqrt{13}$

i) By subtraction one may expect an easy solution of second degree equation. Then substitution of the result in the equation which has been used in the subtraction, gives the value with ease.

Example 21 : $7x^2 + 12xy + 5y^2 = 16$

$y^2 - 2x^2 = -9$

By subtraction

$9x^2 + 12xy + 4y^2 = 25$

$(3x + 2y)^2 = 25$

$3x + 2y = \pm 5$

$y = \dfrac{\pm 5 - 3x}{2}$

By substituting the value of y in second equation $y^2 - 2x^2 = -9$

$x^2 \mp 30x + 61 = 0$

$x^2 - 30x + 61 = 0$	$x^2 + 30x + 61 = 0$
$2x - 30 = \pm\sqrt{900 - 244}$	$2x + 30 = \pm\sqrt{900 - 244}$
$2x - 30 = \pm\sqrt{656}$	$2x + 30 = \pm\sqrt{656}$
$x = 15 \pm 2\sqrt{41}$	$x = -15 \pm 2\sqrt{41}$
$y = -20 \mp 3\sqrt{41}$	$y = 20 \mp 3\sqrt{41}$

iv) When the equations are of the same degree and homogenous having significant numbers in the RHS, then a substitution for one of the variables say y = mx form is attempted to derive new equations in m i.e., in single variable . By taking the ratio of the two such derived equations followed by cross multiplication one gets a quadratic equation in m, then the value of m is solved through which x and y can be worked out.

Example: $x^2 + y^2 = 5$ ——— (1)

$2xy - y^2 = 3$ ——— (2)

Put y = mx in both the given equations

Eq.(1) becomes $x^2(1 + m^2) = 5$ ——— (3)

Eq.(2) becomes $x^2(2m - m^2) = 3$ ——— (4)

By dividing, $\dfrac{1+m^2}{2m-m^2} = \dfrac{5}{3}$

By cross multiplication, $8m^2 - 10m + 3 = 0$

By factorization $(4m - 3)(2m - 1) = 0$

$m = \dfrac{1}{2}$ or $m = \dfrac{3}{4}$

$y = \dfrac{x}{2}$ or $y = \dfrac{3x}{4}$

Substituting $y = \dfrac{x}{2}$ in eq.(1) $x^2 = 4 \Rightarrow x = \pm 2, \therefore y = \pm 1$

Substituting $y = \dfrac{3x}{4}$ in eq.(1) $x^2 = \dfrac{16}{5} \Rightarrow x = \pm\dfrac{4}{\sqrt{5}}$, $y = \pm\dfrac{3}{\sqrt{5}}$

v) To eliminate xy and to write down one of the variable x^2 or y^2 in terms of square of the other variable. In such a case one has to take the square root of x^2 or y^2 in order to substitute for xy which results in a complicated working but still it will give the result.

Example22: $2x^2 - 8xy + 3y^2 = 0$ —— A
$x^2 + xy + y^2 = 13$ —— B

Eliminating xy term, from A + 8B

$$10x^2 + 11y^2 = 104 \Rightarrow x^2 = \frac{104 - 11y^2}{10}$$

Substituting for x^2 in B,

$$\frac{104-11y^2}{10} + y\left(\sqrt{\frac{104-11y^2}{10}}\right) + y^2 = 13$$

$$\frac{-y^2-26}{10} = y\left(\sqrt{\frac{104-11y^2}{10}}\right)$$

$$\frac{y^4+676+52y^2}{100} = y^2\left(\frac{104-11y^2}{10}\right)$$

$$y^4 + 676 + 52y^2 = 1040y^2 - 110y^4$$

$$y^2\left(\frac{104-11y^2}{10}\right) = \frac{104y^2-11y^4}{10}$$

$$111y^4 - 988y^2 + 676 = 0$$

Put $y^2 = p$,

$$111p^2 - 988p + 676 = 0$$

First differential = $\pm\sqrt{\text{discriminant}}$

$$222p - 988 = \pm\sqrt{976144-300144} = \pm\sqrt{676000} = \pm 260\sqrt{10}$$

$$p = \frac{494 \pm 130\sqrt{10}}{111}$$

$$y = \pm\sqrt{\frac{494 \pm 130\sqrt{10}}{111}}$$

$$x^2 = \frac{104-11y^2}{10} = \frac{104 - 11\left(\frac{494 \pm 130\sqrt{10}}{111}\right)}{10} = \frac{611 \mp 143\sqrt{10}}{111}$$

$$x = \pm\sqrt{\frac{611 \pm 143\sqrt{10}}{111}}$$

11) If both are non–homogenous equations, they are classified in different ways.
(a) One variable higher degree with no xy term or with xy term in one equation.
(b) One variable higher degree with xy term in both the equations.
(c) Two variables higher degree with no xy term.
(d) Two variable higher degree with xy term (General non–homogenous second degree equation).

In all the following non–homogenous equations, the classification details can be considered as below. This is based on square terms and product terms occurring in the equations.

(a) In the following case two methods can be tried
 (i) To express the variable of the equation which has no higher degree in terms of other variable. This is substituted in the second equation thus a quadratic equation in other variable is obtained. Solution of quadratic equation gives x and y.

Example 23: $x^2 + 3x - 2y = -2$ —— A

$2x^2 - 5x + 3y = 6$ —— B

from A, $y = \frac{x^2+3x+2}{2}$

substituting y in B,

$$2x^2 - 5x + 3\left(\frac{x^2+3x+2}{2}\right) = 6$$

$$4x^2 - 10x + 3x^2 + 9x + 6 = 12$$

$$7x^2 - x - 6 = 0$$

$$(x-1)(7x+6) = 0$$

$$\therefore x = 1, \quad \frac{-6}{7} \qquad \therefore y = 3, \quad \frac{4}{49}$$

ii) To express x and y as a linear combination by eliminating the second degree term in the given equations by Lopanasthapanabhyam. Substituting this value in any of the equations one can get quadratic equation in one variable gives x and y.

Eliminating x^2, from 2A – B

$$11x - 7y = -10 \Rightarrow y = \frac{11x+10}{7}$$

substituting for y in B,

$$2x^2 - 5x + 3\left(\frac{11x+10}{7}\right) = 6$$

$$14x^2 - 35x + 33x + 30 = 42$$

$$7x^2 - x - 6 = 0$$

$$(x-1)(7x+6) = 0$$

$$\therefore x = 1, \quad \frac{-6}{7} \qquad \therefore y = 3, \quad \frac{4}{49}$$

(b) The other conditions being same, xy term being present in both the equations, then two methods can be tried.

(i) Lopanasthapanabhyam i.e., elimination of xy term followed by the procedure described in (a).

Example 24: $x^2 + 3x - 2y + xy = 1$ —— A

$2x^2 - 5x + 3y + 3xy = 15$ —— B

Eliminating xy term, from 3A – B

$$x^2 + 14x - 9y = -12$$

$$y = \frac{x^2+14x+12}{9}$$

Substituting for y in equation A,

$$x^2 + 3x + (x-2)\left(\frac{x^2+14x+12}{9}\right) = 1$$

$$9x^2 + 27x + x^3 + 12x^2 - 16x - 24 = 9$$

$$x^3 + 21x^2 + 11x - 33 = 0$$

This cubic equation is solved for x, consequent on which y can be derived Ref section on cubic equation.

(ii) The method is same as given in (a) simplifying xy terms also in the working.

$$x^2 + 3x - 2y + xy = 1$$
$$2x^2 - 5x + 3y + 3xy = 15$$

From A, $y = \dfrac{x^2 + 3x - 1}{2 - x}$

Substituting y in B,

$$2x^2 - 5x + 3(1 + x)\left(\frac{x^2 + 3x - 1}{2 - x}\right) = 15$$
$$9x^2 - 2x^3 - 10x + 3x^3 + 12x^2 + 6x - 3 = 30 - 15x$$
$$x^3 + 21x^2 + 11x - 33 = 0$$

Ref. To section on cubic equation.

$$x = 1,\ 11 + 2\sqrt{22},\ 11 - 2\sqrt{22}$$
$$y = 3,\ -25.8704,\ 17.0133$$

c) i) When the two variables have second degree but with no xy term, anyone of the higher degree terms is to be eliminated thus converting one variable in terms of other variable which on proper substitution will give the values x and y.

Example 25: $x^2 + 3x - 2y + y^2 = 7$ ——— (A)

$2x^2 - 5x + 3y + 3y^2 = 33$ ——— (B)

Eliminating y^2, from 3A − B

$$x^2 + 14x - 9y = -12 \Rightarrow y = \frac{x^2 + 14x + 12}{9}$$

Substituting for y in A,

$$x^2 + 3x - 2\left(\frac{x^2 + 14x + 12}{9}\right) + \left(\frac{x^2 + 14x + 12}{9}\right)^2 = 7$$
$$81x^2 + 243x - 18x^2 - 252x - 216 + x^4 + 196x^2 + 144 +$$
$$24x^2 + 336x + 28x^3 = 567$$
$$x^4 + 28x^3 + 283x^2 + 327x - 639 = 0$$

Refer higher order equation

The fourth degree equation is to be solved for x

$$x^4 + 28x^3 + 283x^2 + 327x - 639 = 0$$
$$(x - 1)(x^3 + 29x^2 + 312x + 639) = 0$$
$$\Rightarrow x^3 + 29x^2 + 312x + 639 = 0$$

One of the roots is in between − 2 and − 3 and the other two are complex.

Details for the solution of this problem is given in the Lecture Notes III (b).

d) In this case the following three possibilities can be tried

i. One can try a proper combination of subtraction or addition of the two equations which may result in simpler form of the equations which can be solved.
ii. Try to eliminate one of the higher degree terms to see the solution can be arrived.
iii. Finally one can also look at the ratios of variables in relation to the ratio of constant term. If for any variable it is equal, then the other variable is zero by the application of Sunyamanyat.

Example 26 : $9x^2 - 6xy + y^2 = 15x - 5y + 6$ ——— (1)
$3x^2 + 2xy + y^2 = 5x - y + 2$ ——— (2)

i) By Sunyamanyat y = 0 as $\frac{9x^2}{3x^2} = \frac{15x}{5x} = \frac{6}{2}$

Let $3x - y = a$
$\therefore a^2 - 5a - 6 = 0$
$\therefore a = 6, -1$
$\therefore 3x - y = 6$ or -1
$\therefore y = 3x - 6$ or $y = 3x + 1$

Substituting $y = 3x - 6$ in equation (2)
$3x^2 + 2x(3x - 6) + (3x - 6)^2$
$= 5x - (3x - 6) + 2$
$18x^2 - 50x + 28 = 0$
$9x^2 - 25x + 14 = 0$
$D_1 = \pm\sqrt{Discriminant}$
$18x - 25 = \pm\sqrt{25^2 - 4\times14\times9}$
$18x - 25 = \pm 11$
$\therefore x = 2, \frac{7}{9}$
$\therefore y = 0, -\frac{11}{3}$

Substituting $y = 3x + 1$ in equation (2)
$3x^2 + 2x(3x + 1) + (3x + 1)^2$
$= 5x - (3x + 1) + 2$
$18x^2 + 6x = 0$
$6x(3x + 1) = 0$
$\therefore x = 0; \quad x = -\frac{1}{3}$
$\therefore y = 1; \quad y = 0$

ii) By subtracting
Eq (1) – 3x eq (2)
$-12xy - 2y^2 = -2y$
$2y(6x + y - 1) = 0$
$y = 0$ or $6x + y = 1$

Substituting $y = 0$ in equation (2)
$3x^2 = 5x + 2$
$3x^2 - 5x - 2 = 0$
$(3x + 1)(x - 2) = 0$
$\therefore x = -\frac{1}{3}$ or $x = 2$

Substituting $y = 1 - 6x$ in equation (2)
$3x^2 + 2x(1 - 6x) + (1 - 6x)^2$
$= 5x - (1 - 6x) + 2$
$27x^2 - 21x = 0$
$3x(9x - 7) = 0$
$\therefore x = 0; \quad \frac{7}{9}$
$\therefore y = 1; \quad y = -\frac{11}{3}$

The solutions of (x, y) are $\left(-\frac{1}{3}, 0\right)$, (2, 0), (0, 1) and $\left(\frac{7}{9}, -\frac{11}{3}\right)$

12) In certain equations both will lead after a little readjustment to the same equation, in which case any value of one variable is equally good a solution, depending on which the value of other variable will be.

Example 27 : $8x^2 + 10xy - 3y^2 = 0$ ——— 1
$3y^2 + 2x^2 = 10x(y + x)$ ——— 2
Equation 2 $\Rightarrow 3y^2 + 2x^2 = 10xy + 10x^2$
$-8x^2 - 10xy + 3y^2 = 0$

13) If in the given equation one is non – homogeneous equation and the other is a second degree homogenous with significant number in RHS, one can solve by the following methods.

i) To express higher degree non–homogenous equation in terms of one variable depending on the second degree equation.

Example28: $x^2y^2 + 5xy = 84$ ——— (1)

$xy + y^2 = 8$ ——— (2)

From (2) $x = \dfrac{8-y^2}{y}$

Substituting x in equation (1),

$$\left(\frac{8-y^2}{y}\right)^2 y^2 + 5\left(\frac{8-y^2}{y}\right)y = 84$$

$$64 + y^4 - 16y^2 + 40 - 5y^2 = 84$$

$$y^4 - 21y^2 + 20 = 0$$

$$(y^2 - 1)(y^2 - 20) = 0$$

$$\therefore y^2 = 1 \quad \text{or} \quad 20$$

$$\therefore y = \pm 1 \quad \text{or} \quad \pm 2\sqrt{5}$$

$$\therefore x = \pm 7 \quad \text{or} \quad \mp\frac{6}{\sqrt{5}}$$

ii) One can try the possibility of expressing the non–homogenous equation in terms of x and y as a quadratic form of the type

$$a(x \pm my)^2 + b(x \pm my) + c = 0$$

Example: $x^2y^2 + 5xy = 84$ ——— (1)

$xy + y^2 = 8$ ——— (2)

$x^2y^2 + 5xy - 84 = 0$

This is quadratic equation in xy.

$x^2y^2 + 12xy - 7xy - 84 = 0$

$xy(xy + 12) - 7(xy + 12) = 0$

$xy = 7$ or $xy = -12$

(a) $xy = 7$

substituting $xy = 7$ in equation(2),

$7 + y^2 = 8 \Rightarrow y^2 = 1$

$\therefore y = \pm 1, x = \pm 7$

(b) $xy = -12$

substituting $xy = -12$ in equation(2),

$-12 + y^2 = 8 \Rightarrow y^2 = 16$

$\therefore y = \pm 2\sqrt{5}, x = \mp \dfrac{6}{\sqrt{5}}$

CHAPTER III

FACTORISATION OF SIMPLE QUADRATICS, HARDER QUADRATICS, CUBICS AND HIGHER DEGREE EQUATIONS

Factorization of Simple Quadratics, harder Quadratics and Simple Cubics can be carried out using the Sutras and Upa – Sutras

1) Vilokanaṁ
2) Adyamadyena Antya mantyena
3) Anurupyena
4) Argumentation
5) Gunita Samuccayah Samuccaya Gunitah
6) Paravartya
7) Lopana Sthapanabhyam
8) Purana Apuranabhyam
9) Differential Relations.
10) Successive Differentiation

The modus operandi is explained along with the working details.

Simple Quadratic Equations :

The central term is split into two parts so that the ratio of the first term in the Quadratic Expression to the first part of the split terms is equal to the ratio of the second part to the constant term. This method is called Adyamadyena Antyamantyena and Anurupyena.

From the ratio (which is the Anurupyena) one can write down the first factor as the numerator + denominator.

To obtain the second factor (a + b) of the given Quadratic expression, again by applying Adyamadyena by

a) Dividing the first term of the Quadratic expression by the first term of the derived factor and

b) Dividing the last term (constant) of the Quadratic expression by the second term of the derived factor. The sum of a and b is the second factor.
This can be seen in the following example
$x^2 + 7x + 12 = x^2 + 4x + 3x + 12$

Applying Adyamadyena $\frac{x^2}{4x} = \frac{3x}{12} = \frac{x}{4}$

By Anurupyena (x + 4) is one factor.

To get the second factor, $\frac{x^2}{x}+\frac{12}{4}=(x+3)$

∴ Given Quadratic expression is factorised as (x + 4) (x + 3)

It is also seen that an interesting relation is prominently observed by applying "Gunita Samuccayah, Samuccaya Gunitah Sutram" which means "the product of the sum of the coefficients in the factors is equal to the sum of the coefficients in the product".

As applied to the above problem $x^2 + 7x + 12 = (x + 4)(x + 3)$
$1 + 7 + 12 = (1 + 4)(1 + 3) = 20$

Example 1: $E = x^2 + 13x + 42 = (x + 7)(x + 6)$ Applying Gunita Samuccayah Sutram, Sum of the coefficients of all the terms = Product of the sum of the coefficients of each factor i.e. $1 + 13 + 42 = (1 + 7)(1 + 6)$
$56 = 8 \times 7 = 56$

The application of the above sutram is useful

1) In factorising the expression of any degree whatsoever.
2) In working out the unknown term if some of the other terms are known.
3) In verifying the factorisation or it helps us in filling the gaps if some of the factors are known in an equation.

Coming to the harder quadratics such as homogeneous expressions in Second degree of any number of variables, it is difficult to solve by the current method if the variables are too many. But they can be solved using the Vedic Method wherein the sutram i.e. "Lopana Sthapanabhyam" is easily applied. By this process one can eliminate all the other variables excepting two at a time, step wise; so that quadratic expression in those two variables can be obtained. This can be solved for its factors, in the manner described under Adyamadyena and Anurupyena / differential relation (Section 4).

The process is continued with the elimination of other variables so that Quadratic in two variables is obtained. The process is still continued till all the combinations of variables are eliminated, successively.

Example 2 : $E = 2x^2 + 3y^2 + 3z^2 + 2w^2 + 7xy + 10yz - 5zw - 4wx + 5xz - 7yw$

Eliminating z and w	Eliminating y and w	Eliminating z and y
$2x^2 + 7xy + 3y^2$	$2x^2 + 5xz + 3z^2$	$2x^2 - 4xw + 2w^2$
$2x^2 + 6xy + xy + 3y^2$	$2x^2 + 2xz + 3xz + 3z^2$	$2x^2 - 2xw - 2xw + 2w^2$
$(x + 3y)(2x + y)$	$(x + z)(2x + 3z)$	$(x - w)(2x - 2w)$

By Lopana Sthapanabhyam the given expression can be factorized as $(x + 3y + z - w)(2x + y + 3z - 2w)$

Application of Lopana Sthapanabhyam :

The operation is first to consider all the factors in sets so obtained i.e. $(x + 3y)(2x + y)$; $(x + z)(2x + 3z)$; $(x - w)(2x - 2w)$. In each factor two variables are missing (Lopana). They are to be established (sthapana) in each factor as follows. Consider $(x + 3y)$ from 1st set and establish the two missing terms one taken each from the other two sets. $(x + 3y + z - w)$ is a factor. The other factor is written similarly using the second term of the 1st set as $(2x + y + 3z - 2w)$. The same result is arrived starting with any set of factors and also retaining any other variable throughout.

This procedure can be extendable to any number of variables.

The method of Lopana Sthapanabhyam which is alternate elimination and retention, is useful in solid Geometry, co-ordinate Geometry, of Straight line, hyperbola, conjugate hyperbola, Asymptotes, HCF and the like.

Cubic Equation :

Coming to the Factorisation of Cubics which are in the general form of $ax^3 + bx^2 + cx + d$, one can apply the Adyamadyena Sutram in the following way.

One has to arrive at atleast one factor by Vilokanam by testing if $(x + 1)$ or $(x - 1)$ is a factor. If it is not satisfied then a trial is carried out for a + ve or – ve values (integers) in succession to see if it satisfies the equation. If one factor is established. apply, the Adyamadyena Sutram as follows.

The first term is divided by the first term of the factor. This is added to the ratio of constant of the equation to the constant of the derived factor. The two together will form the second factor, only when the x term is also added.

Let the x term be αx where α is the coefficient of x.

Applying the Gunita Samuccayah Sutram, one can evaluate the coefficient of x i.e α value. We can also get the value of α by Argumentation i.e. (by comparing the like terms on both sides).

Example 3 : $E = x^3 + 8x^2 + x - 42 = 0$

By Vilokanam 1 and – 1 are not roots of the equation as $S_0 \neq S_e$, $S_c \neq 0$

S_o = sum of coefficients of odd powers

S_e = sum of coefficients of even powers.

S_c = Sum of the coefficients

But $x = 2$ satisfies the equation

$(x - 2)$ is a factor

By Paravartya Division of E by (x – 2), we get the quadratic expression as a quotient, remainder being 0

$\frac{x-2}{2}$	$x^3 + 8x^2 + x$	-42
	$2x^2 + 20x$	42
	$x^2 + 10x + 21$	0

$Q = x^2 + 10x + 21$
$R = 0$

More simply, the same result can be obtained as follows

∴ let us apply Adyamadyena Sutram i.e. $\frac{x^3}{x} = x^2, \frac{-42}{-2} = 21$

$\therefore (x^2 + \alpha x + 21)$ is another factor of the given equation.

∴ Applying Gunita Samuccayah Sutram, one can evaluate α

$S_c = -32 = (1-2)(1+21+\alpha)$

$\therefore \alpha = 10$

α can be obtained by Argumentation as follows

$E = (x-2)(x^2 + \alpha x + 21)$

$21 - 2\alpha = 1$

$\therefore \alpha = 10$

$1 = (x-2)(x^2 + 10x + 21)$

This is verified by Gunita Samuccayah Sutram as $E = -1(1 + 10 + 21) = -32 = S_c$

∴ The equation is factorised as

$(x-2)(x^2 + 10x + 21) = 0$

Now the Quadratic equation can be further factorised by applying Adyamadyena as. $x^2 + 7x + 3x + 21$ and the factors are (x + 7) and (x + 3)

∴ The given cubic equation = (x – 2) (x + 7) (x + 3)

We describe here a more general form of the Cubic Equation and its factors (x + a), (x + b) and (x + c) as follows (– a, – b and – c are the three roots of the equation).

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc = 0$$

1) If the Sum of the coefficients $S_c = 0$ then (x – 1) is a factor.

2) If the sum of the co–efficients of odd powers S_0 = Sum of the co–efficients of even powers S_e then (x + 1) is a factor.

One can try this in the beginning to get one of the factors if any. The other factors can be worked out from the remaining Expression.

3) It is also seen that one can obtain the three factors by the process of Argumentation. This is achieved from a comparison of the coefficients of various powers in the given Cubic equation with the supposed factors. (or)

one can deduce the factors from such a comparison using the general expansion of Cubic Expression.

4) Consider the possible three factors of the constant term of the given Cubic equation and then see if any one of the combinations can explain the x^2 and x coefficient accordingly.

5) In case the constant term in the given equation has a number of combinations of factors then one has to select three factors from them so that, finally they can explain the coefficient of x^2 as well as x term, in addition to the constant term. One can consider also negative values if it is necessary.

6) From such possible factors of the constant term one has to test each factor to see whether it is an allowed factor of the equation or not. In order to test it, x + a, the factor of the constant term = (x + a) satisfies the rule of the divisibility of the sum of the coefficients of S_c i.e. in other words (x + a) (coeff of x + a) should be a factor of S_c. This should be the case with all the factors of the constant term.

Such combination of factors should finally explain the coefficients of x^2 and x.

7) In case either the S_c or the constant term happens to be – ve, then both negative and positive factors should be considered.

Example 4 : $E = x^3 - 4x^2 - 11x + 30$

$S_c \neq 0$ $S_0 \neq S_c$ (x – 1) and (x + 1) are not factors.

Argumentation :

The constant value 30 have the following factors 1, 2, 3, 5, 6, 10, 15, 30, (– ve also should be considered when needed)

The factors of $S_c = 16$ are 1, 2, 4, 8, 16 (– ve also needs to be considered if necessary)

* The sum of the coefficients in each factor must be a factor of S_c

Out of the factors of the constant term, one has to select only three possibilities which should explain the coefficients of x^2 term and x term and at the same time each factor should be a factor of S_c as well
The possibilities of factors of the given equation are (x – 2) (x + 3) (x – 3) and (x – 5)

- Rule

The detailed procedure for selection of the possibilities is as follows.

Considerate of factors 1, 2, 3, 5, 10, 15 and 30 (each one should be subject to test)
(x + 1) is not a factor as already stated.
(x + 2) is not a factor as (1 + 2) cannot be a factor of S_c
But (x – 2) can be a factor of S_c as (1 – 2) is a factor of S_c.
(x + 3), (x – 3) may be factors as (1 + 3), (1 – 3) are factors of S_c.
(x + 5) cannot be a factor as (1 + 5) is not a factor of S_c
(x – 5) may be a factor as (1 – 5) is a factor of S_c

Similarly
(x + 6), (x – 6), (x + 10) (x – 10) (x + 15) (x – 15) are not factors of S_c
(x – 2), (x + 3), (x – 3), and (x – 5) are possible factors of the equation.

The possible factors of the given equation are (x – 2), (x + 3) (x – 3) (x – 5). Out of this series, (x – 2), (x + 3) and (x – 5) explain the coefficients of x^2 and x of the given equation, whereas (x – 3) is not.
Coefficient of $x^2 = -2 + 3 - 5 = -4$
Coefficient of $x = -6 + 10 - 15 = -11$
The given equation is factorised as (x – 2) (x + 3) (x – 5)

Purana Method

The Cubic equation can also be factorized by Purana Apuranabhyam Method.

Example 5 : $E = x^3 + 3x^2 - 10x - 24 = 0$ ——— (1)

By Purana Apuranabhyam Method

Re writing the equation as $x^3 + 3x^2 = 10x + 24$ ——— (2)

Consider the perfect cube in which the first two terms of the given equation $x^3 + 3x^2$ occur as they are; $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$ — (3) this is by Purana method. Substituting for $(x^3 + 3x^2)$ in the standard equation (3), its value as (10x + 24) from the given equation (2), ie
$(x + 1)^3 = 10x + 24 + 3x + 1$ $(x + 1)^3 = 13x + 25$
Let (x + 1) = y
$y^3 = 13(y - 1) + 25;$ $y^3 - 13y - 12 = 0$ ——— (3)
To solve the y^3 equation, (y + 1) is a factor as $S_0 = S_e$
By Adyamadyena we can get the second factor

$$\frac{y^3}{y} = y^2, \frac{-12}{1} = -12$$

The second factor is $(y^2 + \alpha y - 12)$
Where α is to be determined by Gunita Samuccayah Sutram
$S_c = -24 = (1 + 1)(1 + \alpha - 12)$

$-24 = 2(-11 + \alpha)$; $\alpha = -1$

The second factor is $(y^2 - y - 12)$

The equation (3) is factorized as $(y + 1)(y^2 - y - 12)$

The Quadratic expression can be further factorized as $(y - 4)(y + 3)$

$(y^3 - 13y - 12) = (y + 1)(y + 3)(y - 4)$ But, $y = (x + 1)$

$y = -1 \Rightarrow x = -2 \Rightarrow (x + 2)$ is a factor

$y = -3 \Rightarrow x = -4 \Rightarrow (x + 4)$ is a factor

$y = 4 \Rightarrow x = 3 \Rightarrow (x - 3)$ is a factor

The given equation can be factorized as $(x + 2)(x + 4)(x - 3)$

This method can be extended to a more general Cubic form (where the factors may exhibit decimals) and also any higher degree equation which are dealt with in the Lecture Notes III(b) more elaborately.

SECTION– 6

VILOKANAM

1) Solving of Cubic equations can be attempted in many ways using different Sutras. The first step is to test if the sum of the coefficients S_c of the given equation is zero. If $S_c = 0$, $(x - 1)$ is a factor of the given equation.

2) Similarly if the sum of the coefficients of even powers (S_e) = The sum of the coefficients of odd powers (S_0). Then $(x + 1)$ is a factor

3) If any one of the above is satisfied then one can obtain the remaining expression "A" by dividing the given equation by that factor using paravartya Method (Refer Lecture notes II for division)

or

4) By applying Argumentation followed by Gunita Samuccayah Samuccaya Gunitah Sutram, the remaining expression can be obtained. The second expression will be a Quadratic expression. That Quadratic expression can be solved by Adyamadyena followed by Anurupyena (or) Differential relation. Thus the three factors can be obtained. There is still another method for obtaining the factors

1) By rewriting the given equation with the coefficient of x^3 as unity, if it is not so.

Consider cubic expression in the form $(x + a)(x + b)(x + c) = x^3 + x^2(a + b + c) + x(ab + bc + ca) + abc$; where $-a, -b -c$ are the roots, $(x + a)(x + b)$ and $(x + c)$ are the factors of the Cubic equation.

Then consider the constant term of the equation. The three possible factors a, b, c of the (Ref. Factorisation section) constant term which can explain the coefficient of x^2 term as $(a + b + c)$ and the coefficient of x term as $(ab + bc + ca)$. Then the three expressions $(x + a)$, $(x + b)$, $(x + c)$ are the factors of the given equation.

2) The Vilokanam Method, by trial in Succession, is extended to other integers also as 2,3,.......... etc and $-2, -3$, ... etc. Through this process, one can arrive at exact solutions or find out the ranges of its roots which lie between any two successive integers. If one can find out the roots to lie between two successive integers; r_1 and r_2, then one can try as a finer step with the values as r_1 or $r_2 \pm 0.25$, r_1 or $r_2 \pm 0.5$, r_1 or $r_2 \pm 0.75$ as the case may be. Even after this trial, if it doesn't satisfy exactly the equation then it has to be considered that such roots may be irrational or complex.

Such problems are dealt with separately in the Lecture Notes IIIb.

Even if (x + 1) (or) (x – 1) are not the factors, if it shows by trial that any other definite factor can satisfy the equation then it can be used as the starting point for solving the equation. A few examples of the above are given below

Examples :

1) $E = 27x^3 - 39x^2 - 52x + 64 = 0$
   $S_c = 0$ ∴ (x – 1) is a factor
   $S_0 \neq S_e$ ∴ (x + 1) is not a factor

2) $E = x^3 - 2x^2 - 7x - 4 = 0$
   $S_c \neq 0$ ∴ (x – 1) is not a factor
   $S_0 = S_e$ ∴ (x + 1) is a factor

3) $E = 4x^3 - 24x^2 + 23x + 18 = 0$
   $S_c \neq 0$
   $S_0 \neq S_c$ ∴ (x – 1) and (x + 1) are not factors
   On Reorientation.
   Let $f(x) = 4x^3 - 24x^2 + 23x = -18$
   But x = 2 satisfies the equation ∴ (x – 2) is one factor

4) $E = x^5 + 2x^4 - 42x^3 - 8x^2 + 258x - 400 = 0$
   $S_c \neq 0$ $S_0 \neq S_c$ ∴ (x – 1) and (x + 1) are not factors.
   Let $f(x) = x^5 + 2x^4 - 42x^3 - 8x^2 + 258x = 400$

Trial in Succession for x	LHS f (x) value	RHS 400	Diff (RHS – LHS)	
x = 1	211		189	
x = 2	212		188	
x = 3	– 27		427	
x = 4	– 248		648	
x = 5	215		185	Region in
x = 6	2556		– 2156	which one (
x = – 1	– 223		623	locate a roo
x = – 2	– 212		612	
x = – 3	207		193	”
x = – 4	1016		– 616	
x = – 5	1885		– 1485	
x = – 6	2052		– 1652	”
x = – 7	203		197	
x = – 8	– 5648		6048	

In this trial one can expect that the roots lie between 5 and 6; – 3 and – 4 and – 6 and – 7. For the exact values of all the roots; details are given in the Lecture Notes III b.

5) $E = 32x^3 + 80x^2 - 122x - 35 = 0$

$S_c \neq 0$ (x – 1) is not a factor $S_0 \neq S_e$ (x + 1) is not a factor

Let $f(x) = 32x^3 + 80x^2 - 122x = 35$

	LHS f(x)	RHS	RHS – LHS
x = 1	– 10	35	45
x = – 1	170		– 135

The roots lie between 1 and – 1

One can try with x = 0.75, 0.5, 0.25, 0, – 0.25, –0.5, – 0.75 for locating the factors.

	LHS	RHS	RHS – LHS
If x = 1	– 10	35	45
If x = – 0.5	77		– 42
If x = – 0.25	35		0
If x = 0.75	– 33		68

x = – 0.25 is a solution $\left(x + \frac{1}{4}\right)$ is one factor Ref.

6) $E = x^3 + 6x^2 + 11x + 6$

$S_c \neq 0$ (x – 1) is not a factor $S_0 \neq S_e$ (x + 1) is not a factor

Let $f(x) = x^3 + 6x^2 + 11x = -6$

It is possible to have an approximate idea of the number of + roots and –ve roots form a study of the number of changes in the sign in the given equation, (when x = 1 is substituted in the equation the maximum number of + ve roots, when x = – 1, then maximum number of – ve roots can be determined). As this equation has only all – ve roots, only – ve value for x are tried under Vilokanam

		RHS	Diff (R.H.S – L.H.S)
x = – 1	– 6	– 6	0
x = – 2	– 6		0
x = – 3	– 6		0

As three solutions are formed there is no necessity for further trials.

x = – 1 , – 2, – 3 are three solutions of the given equation and (x + 1) (x + 2) (x + 3) are the three factors

Example 7 : $E = x^4 - 4x^3 - 3x + 23 = 0$

By Vilokanam

$S_c \neq 0$ $\therefore$ $(x - 1)$ is not a factor $S_0 \neq S_e$ $\therefore$ $(x + 1)$ is not a factor

Let $f(x) = x^4 - 4x^3 - 3x = -23$

	LHS f(x)	RHS −23	RHS – LHS	
x = 1	−6	−23	−17	
x = 2	−22	−23	−1	Region of
x = 3	−36	−23	13	occurrence of root
x = 4	−12	−23	−11	"
x = 5	110	−23	−133	
x = 6	414	−23	−437	
x = −1	8	−23	−31	
x = −2	54	−23	−77	
x = −3	198	−23	−221	
x = −4	524	−23	−547	
x = −5	1140	−23	−1163	
x = −6	2178	−23	−2201	

From this trial one can expect that the roots lie between 2 and 3; 3 and 4. for the exactness of the roots, details are given in Lecture Notes.III b

Example 8 : $E = x^5 + 2x^4 - 42x^3 - 8x^2 + 257x - 210 = 0$

By Vilokanam as

$S_c = 0$ $\therefore$ $(x - 1)$ is a factor

Let $f(x) = x^5 + 2x^4 - 42x^3 - 8x^2 + 257x = 210$

	LHS f(x)	RHS 210	RHS – LHS
x = 1	210	210	0
x = 2	210	210	0
x = 3	−30	210	240
x = 4	−252	210	462
x = 5	210	210	0
x = 6	2550	210	−2340
x = 7	8610	210	−8400
x = −1	−222	210	432
x = −2	−210	210	420
x = −3	210	210	0
x = −4	1020	210	−810
x = −5	1890	210	−1680
x = −6	2058	210	−1848
x = −7	210	210	0
x = −8	−5640	210	−5850

$x = 1, 2, 5, -3, -7$ are the solutions of the given equations the given Fifth degree equation can be factorized as $(x-1)(x-2)(x-5)(x+3)(x+7)$

Example 9 : $E = x^3 + 6x^2 - 37x + 30 = 0$
By Vilokanam as
$S_c = 0$ (x – 1) is a factor
Let $f(x) = x^3 + 6x^2 - 37x = -30$

	LHS f(x)	RHS 210	RHS – LHS
x = 1	– 30	– 30	0
x = 2	– 42		12
x = 3	– 30		0
x = 4	12		– 42
x = 5	90		– 120
x = 6	210		– 240
x = – 1	42		– 72
x = – 2	90		– 120
x = – 3	138		– 168
x = – 4	180		– 210
x = – 5	210		– 240
x = – 6	222		– 252
x = – 7	210		– 240
x = – 8	168		– 198
x = – 9	90		– 120
x = – 10	– 30		0
x = – 11	– 198		+ 168

The roots of the given equation are 1, 3, – 10

The given equation can be factorized as $(x-1)(x-3)(x+10)$

It may be noted that the Vilokanam method of locating the roots or intervals will not give information on repeated roots. It cannot also give any information on the various values if any in the same interval, unless one tries decimal – wise value, in a given interval. But from a different method called Successive Differentiation of the given equation, one can get the repeated factors.

SECTION – 7

GUNITA SAMUCCAYAH, SAMUCCAYA GUNITAH

This Sutram when used in conjunction with the factors of the given expression means.

"The Sum of the coefficients of all the terms in the given expression is equal to the product of the sum of the coefficients of each factor".

For Eg: $x^2 + 7x + 12 = (x + 3)(x + 4)$

On the LHS, the Sum of the coefficients S_c = the product of the sum of the coefficients in each factor.(R.H.S)

$S_c = (1 + 7 + 12) = 20 = (1 + 3)(1+4) = 4 \times 5 = 20$

This is a very elegant sutram in the sense that this can be used for finding out the factors of a given expression. More generally this sutram when coupled with the argumentation or Vice Versa will solve all the factors of a given expression

$x^2 + (a + b)x + ab = (x + a)(x + b)$

$1 + a + b + ab = (1 + a)(1 + b)$

$= 1 + a + b + ab$

This is applicable in solving any degree equation . The roots may be real (rational, irrational) or complex. One has to consider the following steps either in combination or individually as the case may be, in factorizing the given equations of any degree.

1) **By Vilokanam :** One can have an idea of + 1, – 1 as the solutions in case $S_c = 0$ or $S_0 = S_e$ respectively.
   (Where S_c = Sum of the coefficients, S_o = sum of the coefficients of odd powers, S_e = sum of the coefficients of even powers). If the above condition is not satisfied then one can try 2,3 etc and –2, –3.......... in succession to see if any one of these can be a solution. Thus by way of inspection or trial (Vilokanam)if one can locate one solution of the given equation , then one can solve the remaining solutions, through the process of argumentation together with the application of an elegant sutram namely "Gunita Samuccayah – Samuccaya Gunitah" This method is considered to be a general one in solving the equations. Even if one cannot get even one solution, one can still proceed by writing down the equation into a product of two lower degree expressions. For example a fifth degree equation can be written as a product of quadratic and cubic equations or as a product of a 1st degree and 4th degree expressions. The two different equations can be further solved separately.

In case of a quadratic equation $ax^2 + bx + c = 0$ a very important relationship is established by Swamiji between the first differential D_1 and the $\sqrt{\text{discriminant}}$. As $D_1 = \pm\sqrt{Discriminant} = \pm\sqrt{b^2 - 4ac}$

From which one can directly read the two values of x.

Based on the above principles the following examples are worked out. Examples of solving a few equations are given below.

1) $E = x^3 - 2x^2 - 7x - 4 = 0$
S_c = Sum of the coefficients $\neq 0$ (x – 1) is not a factor
But sum of the coefficients of even powers $S_e = -2 \ -4 = -6$
Sum of the coefficients of odd powers $S_0 = \ -7 = -6$
$S_e = S_0$ (x + 1) is a factor

Given equation E = (x + 1) (A) where A is a quadratic expression. To find out the expression A, let us apply Argumentation process by comparing the coefficients of like terms on both sides. A should have an x^2, x and constant terms. The coeff., of x^3 should be 1 on the R.H.S also. Hence in the expression A, one has to write x^2. Regarding the constant value, in the L.H.S it is – 4 and on the R.H.S the already derived factor (x + 1) has 1 as the constant term and hence in the expression A, the constant is – 4 (by Adyamadyena).

Now it remains us to determine the coefficient of x in the expression A. Let this be α and one has to resort to the Sutram "Gunita Samucayah Samuccaya Gunitah" to get the value α. When applied to this problem.,
$x^3 - 2x^2 - 7x - 4 = (x + 1)(x^2 + \alpha x - 4)$
$1 - 2 - 7 - 4 = 2(-3 + \alpha)$
$\alpha = -3$, the coefficient of x in A is – 3.

Hence the two factors of the given cubic equation are (x + 1) & $(x^2 - 3x - 4)$

The second factor can be further factorized using differentiation concept; or applying Adyamadyena.

The quadratic equation can be solved by the differentiation method.
D_1 the first differential = $\pm\sqrt{Discriminant}$
$2x - 3 = \pm\sqrt{9 + 16} = \pm 5$
$x = 4, \ -1$

The second factor A, the quadratic expression can be also factorized using Adyamadyena Antyamantyena as $x^2 - 4x + x - 4$

$\frac{x^2}{-4x} = \frac{x}{-4}$ which is Anurupyena (x – 4) is a factor.

The second factor is $\frac{x^2}{x} + \frac{-4}{-4}$, (x + 1) is another factor.
Hence (x + 1) (x + 1) (x – 4) are the factors for the given Cubic equation.
It is interesting to note that one of the factors (x + 1) is repeated.

In order to arrive at the repeated factors working details are shown separately (under successive differentiation – factorisation)

$D_1 \quad = 3x^2 - 4x - 7$
$\quad = 3x^2 - 7x + 3x - 7 = (3x - 7)(x + 1)$

Now either $(x + 1)$ or $(3x - 7)$ can be considered for repetition. If $(x + 1)$ is repeated then the given equation E should be explained with $(x + 1)^2$ as the factor.

By Argumentation $E = (x^2 + 2x + 1)A$. A should have x and constant term
By Adyamadyena $A = (x - 4)$

It is evident that $3x - 7$ is not a factor of E.
Let us consider fourth degree equation

2) $E = 4x^4 - 4x^3 - 39x^2 + 36x + 27 = 0$
$S_c \neq 0$, $S_0 \neq S_e$ $\quad \therefore (x - 1)$ & $(x + 1)$ are not factors
Let $f(x) = 4x^4 - 4x^3 - 39x^2 + 36x = -27$
By Trial in Succession
Let x be 2, then L.H.S $= -52, \neq$ RHS
$\therefore (x - 2)$ is not a factor.
Let x be 3
Then L.H.S $= 324 - 108 - 351 + 108 = -27$
$\therefore (x - 3)$ is a factor.
Let $x = -2$ LHS $= -132$ $\therefore (x + 2)$ is not a factor
Let $x = -3$
L.H.S $= 324 + 108 - 351 - 108 = -27$
$\therefore (x + 3)$ is a factor.
The two factors by applying Vilokanam are $(x - 3)$ & $(x + 3)$
Now the given equation can be written as $(x - 3)(x + 3)(A)$
To find out A, one can resort to Argumentation followed by Gunita Samuccayah
$4x^4 - 4x^3 - 39x^2 + 36x + 27 = (x^2 - 9)(A)$. A should contain x^2, x and constant terms.

Considering coefficient of the first term x^4 on L.H.S, the coefficient of x^2 term in A should be 4. Considering the constant term 27 on the L.H.S the constant term in A should be -3. To work out the coefficients of the middle term x. Let it be αx in A. By applying Gunita Samuccayah Sutram, one can write
$24 = -8(1 + \alpha)$
$\therefore \alpha = -4$
Hence the third factor A is $(4x^2 - 4x - 3)$
The third factor A can be further factorized using the differential concept.
$D_1 = \pm\sqrt{Discriminant}$

$8x - 4 = \pm\sqrt{16 + 48} \qquad \therefore x = \frac{3}{2}, -\frac{1}{2}$

The given equation = $(x + 3)(x - 3)\left(x-\frac{3}{2}\right)\left(x+\frac{1}{2}\right)$

$\frac{1}{4} S_c = 6 = (4)(-2)\left(-\frac{1}{2}\right)\left(\frac{3}{2}\right) = 6$

The above second degree equation can also be factorized by Adyamadyena Antyamantyena Sutram

$4x^2 - 4x - 3 = 4x^2 - 6x + 2x - 3$

$\frac{4x^2}{-6x} = \frac{2x}{-3}$ (Which is Anurupyena) $(2x - 3)$ is a factor

$\frac{4x^2}{2x} + \frac{-3}{-3} =$ $(2x + 1)$ is the second factor.

$E = (x + 3)(x - 3)(2x - 3)(2x + 1) = 0$

$S_c = 24 = (4)(-2)(-1)(3) = 24$

3) **Consider Solving the Equation** :

$84x^4 + 724x^3 + 2297x^2 + 3160x + 1575 = 0$ This equation has all the 4 roots – ve.
Let f(x) be $84x^4 + 724x^3 + 2297x^2 + 3160x = -1575$
By Vilokanam trial, for the roots:
$S_c \neq 0$ & $S_0 \neq S_c$ $(x - 1)$ & $(x + 1)$ are not factors.
By Trial in Succession.
Let $x = -2$ the difference between R.H.S & L.H.S = 5
Let $x = -3$ the difference between R.H.S & L.H.S = – 24
Let $x = -4$ the difference between R.H.S & L.H.S = – 855
One root is extended to lie between $x = -2$ and $x = -3$ one can try
Let $x = -2.5$; the difference between R.H.S & L.H.S = 0 $(x + 2.5)$ is a factor.

In all cases where there is likelihood of the values for x lying between two specific values, for example, here, between – 2 and – 3 one can also try a specific value for example such as – 2 .25, – 2.5, – 2.75

Trial in this direction gave an exact value – 2.5 for x which satisfies the equation.

To obtain the second factor, one can try by argumentation method and apply Gunita Samuccayah Sutram.

Given equation can be written as $(x + 2.5)A$ where $A = (ax^3 + bx^2 + cx + d)$
$E = 84x^4 + 724x^3 + 2297x^2 + 3160x + 1575 = 0$

Step 1 : By Argumentation and comparison of coefficient of x^4 with the coefficients of x^4 on L.H.S, of the given equation one can write down a = 84 in A.

Step 2 : Comparing the constant term of the given equation with the product of 2.5 and d. One can obtain 2.5d $= 1575 \Rightarrow d = 630$

Step 3: The coefficient of x in the given equation is 3160. Comparison of coeff. of x of with that of x in the result of the product of two factors (on the R.H.S) gives the following identity $2.5C + 630 = 3160$; $C = 1012$

Step 4 : Let us compare the x^2 coefficient 2297 in the given equation, with the coefficient of x^2 by way of multiplying the two factors. One can write down that $1012 + 2.5 \times b = 2297$; $b = 514$

Thus the expression $A = 84x^3 + 514x^2 + 1012x + 630$

Given equation is $= (x + 2.5)(84x^3 + 514x^2 + 1012x + 630)$

Verification of this factorization by Gunita Samuccayah Sutram is as follows

$S_c = 84 + 724 + 2297 + 3160 + 1575 = 7840$ (L.H.S)

$(1 + 2.5)(84 + 514 + 1012 + 630) = 7840$ (R.H.S)

One of the factors of the given 4^{th} degree equation is Cubic expression. This can be solved in different ways.

As we have already got one factor, one can also try if that factor repeats by applying Successive Differentiation followed by verification by using Gunita Samuccayah Sutram

Application of Successive Differentiation:

Differentiating the 4^{th} degree equation.

$D_1 = 336x^3 + 2172x^2 + 4594x + 3160 = 0$

It can be factorized by using Gunita Samuccayah. Taking into consideration that there is a repetition of $(x + 2.5)$ then $D_1 = (x + 2.5)(168x^2 + 666x + 632)$

Now we should explain the factorisation of 4^{th} degree equation E with $(x + 2.5)$ as the two factors (repeated)

Given equation $E = (x + 2.5)^2B$

$E = 84x^4 + 724x^3 + 2297x^2 + 3160x + 1575 = (x^2 + 6.25 + 5x)$ B. B should contain x^2, x and constant terms.

We can find out B by Argumentation. Coefficient. of x^2 in B is 84. The constant term in B is $\frac{1575}{6.25} = 252$. The coefficient of x is obtained as $252 \times 5 + 6.25\alpha = 3160$ (by constant of x terms)

Coefficient of $x = 304$

Given 4^{th} degree equation is factorised as $(x + 2.5)^2(84x^2 + 304x + 252)$

The Quadratic equation can be solved by Differential method

$D_1 = 168x + 304 = \pm\sqrt{7744}$ $x = -\frac{7}{3}, -\frac{9}{7}$

Hence the factorisation of E is finally as $(2x + 5)^2 (3x + 7)(7x + 9)$

If (2x + 5) is not repeatable in E then D_1 cannot have (2x + 5) as the factor. One can test for it by reduction and absurdum also i.e. Considering (2x + 5) as a factor of D_1, find out the other factor and then confirm the factorisation of D_1 by applying Gunita Samuccayah Sutram. After confirming, this factor can be carried out as once repeated factor in E to get the other two factors of the 4^{th} degree equation E.

Now we can take that $(x + 2.5)^2$ are the two factors of the given equation E. We have to obtain the remaining two factors.

We can also verify if $(x + 2.5)^2$ are the once repeated factors of the given equation using the Gunita Samuccayah.

(84 + 724 + 2297 + 3160 + 1575) = (1 + 6.25 + 5) (84 + 304 + 252) = 7840

Successive Differentiation Method for locating the repeated factors is dealt with separately in a section under Differentiation as a method for factorisation of equations of any degree.

It is interesting to note that if the coefficient of highest power is not unity, one may follow any one of the following procedures for obtaining the factors of the equation which would finally satisfy Gunita Samuccayah, Samuccaya Gunitah of the original equation.

1) To divide the given expression E by the coefficient of highest power of the equation to enable, it to have unity

2) Then to proceed to solving such modified equation E and then apply Gunita Samuccaya for final verification with the first equation E.

For Example : $8x^3 + 2x^2 - 37x - 18 = 0$

On dividing $\left(x^3 + \frac{1}{4}x^2 - \frac{37x}{8} - \frac{9}{4}\right) = 0$

Now the given equation is

$E_1 = x^3 + \frac{1}{4}x^2 - \frac{37}{8}x - \frac{9}{4} = 0$

By Trial (x + 1) and (x – 1) are not factors.

But x = – 2 Satisfies the equation

∴ (x + 2) is one factor.

$E_1 = (x + 2) A$. A should have x^2, x and constant term. By Adyamadyena the first and last terms of A are $x^2, -\frac{9}{8}$.

$\therefore E_1 = (x + 2)(x^2 + \alpha x - \frac{9}{8}) = 0$

By comparing the x – coefficient on both sides.

$-\frac{9}{8} + 2\alpha = \frac{-37}{8} \quad \therefore 2\alpha = \frac{-28}{8} \qquad \alpha = -\frac{7}{4}$

$\therefore E = (x + 2)(x^2 - \frac{7}{4}x - \frac{9}{8}) = 0$

The Quadratic expression A, $(x^2 - \frac{7}{4}x - \frac{9}{8})$ is solved by using first differential

$D_1 = \pm\sqrt{Discriminant}$

$2x - \frac{7}{4} = \pm\sqrt{\frac{49}{16} + \frac{9}{2}} = \pm\sqrt{\frac{121}{16}} = \frac{11}{4}$

$2x - \frac{7}{4} = \pm \frac{11}{4} \quad \therefore x = \frac{9}{4}, -\frac{1}{2}$

$\therefore \left(x - \frac{9}{4}\right)\left(x + \frac{1}{2}\right)$ are the factors of A.

$\therefore E_1 = (x + 2)\left(x - \frac{9}{4}\right)\left(x + \frac{1}{2}\right) = 0$

By Gunita Samuccaya this is verified

$S_c \text{ of } E_1 = 1 + \frac{1}{4} - \frac{37}{8} - \frac{9}{4} = (1 + 2)\left(1 - \frac{9}{4}\right)\left(1 + \frac{1}{2}\right) = -\frac{45}{8}$

If the S_c of the given equation is to be verified then $S_c(8E_1) = S_c(E)$

$S_c(E) = -45; S_c(8E_1) = -\frac{45}{8} \times 8 = -45$

2) Some times it may be difficult to solve a Cubic Equation with fractions which is obtained by dividing the coefficient of highest power of x to get the equation with unity for it.

In such a case one can proceed without dividing as follows.

$E = 8x^3 + 2x^2 - 37x - 18 = 0$

By Vilokanam $(x - 1)$ & $(x + 1)$ are not factors

But a trial in Succession shows

$x = -2$ Satisfies the equation.

$E = (x + 2) A$, A should have x^2, x and constant terms

By Adyamadyena, the first and last terms of A are $8x^2, -9$.

$\therefore E = (x + 2)(8x^2 + \alpha x - 9) = 0$

By comparing the x coefficient on both sides.

$-9 + 2\alpha = -37$ $\therefore\ \alpha = -14$

$\therefore\ E = (x + 2)(8x^2 - 14x - 9) = 0$

Even if we take out 8 from the entire Quadratic expression the roots will be the same as for the original Quadratic expression i.e. roots of $8x^2 - 14x - 9 = 0$ are same as $x^2 - \frac{7}{4}x - \frac{9}{8} = 0$. But unless we multiply it by 8, we cannot get back the original expression. Keeping this in view, taking out 8 as common factor,

The Quadratic Expression will be $\left(x^2 - \frac{7}{4}x - \frac{9}{8}\right)$. On solving this $x = \frac{9}{4},\ x = \frac{-1}{2}$

The factors of the Quadratic Expression can be written as $\left(x - \frac{9}{4}\right)\left(x + \frac{1}{2}\right) = 0$

But the factors of $(8x^2 - 14x - 9)$ will be $8\left(x - \frac{9}{4}\right)\left(x + \frac{1}{2}\right)$

$\therefore$ The given Cubic Equation is factorised as $8(x + 2)\left(x - \frac{9}{4}\right)\left(x + \frac{1}{2}\right)$

Now applying Gunita Samuccayah

$$S_c = -45 = 8(3)\left(-\frac{5}{4}\right)\left(\frac{3}{2}\right) = -45$$

One can also simplify the factors as $(x + 2)(4x - 9)(2x + 1)$

$$8(x + 2)\left(\frac{4x - 9}{4}\right)\left(\frac{2x + 1}{2}\right) = (x + 2)(4x - 9)(2x + 1) = -45 = S_c$$

Note : One can also make the highest power as unity at any stage of working for example, in this method we have divided by 8 at the stage of solving the Quadratic equation.

c) The above procedure can be extended in making the coefficient of highest power as 1 at any stage of working of any degree. But the cognizance of such feature has to be carried out at the respective stages when Gunita Samuccaya is to be applied.

In another problem
$E = 2x^5 + x^4 - 12x^3 - 12x^2 + x + 2 = 0$ (Refer working details Section)
In solving this problem
We come across $A = (x + 0.5)(2x^2 - 6x + 2) = 0$

We can take out 2 out of the quadratic expression to make the coefficient of x^2 as unity i.e. division of Quadratic Expression by 2. In order to preserve the value of A we have to multiply by 2 which can be written as (2x + 1). Thus the division by 2 is preserved at the end. The roots of $x^2 - 3x + 1$ are $\frac{3}{2} \pm \frac{\sqrt{5}}{2}$. We can apply Gunita Samuccayah for the equation written in the form.

$$S_c = (x+1)(x+2)(2x+1)\left(x - \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\left(x - \frac{3}{2} + \frac{\sqrt{5}}{2}\right)$$

$$-18 = (2)(3)(3)\left(-\frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right) = 18\left(\frac{1}{4} - \frac{5}{4}\right) = -18$$

SECTION – 8

FACTORIZATION USING DIFFERENTIAL CALCULUS (CALANA – KALANA)

(Gunaka Samuccayah)

The sutras that are used are given as

1) Gunaka Samuccayah.

2) Method of Argumentation, Adyamadyena and Anurupyena.

Let $y = \ell m$ (ℓ, m are functions of x and factors of y)

Then $\frac{dy}{dx} = l\frac{dm}{dx} + m\frac{dl}{dx}$ is defined as Gunaka Samuccayah. The method of finding out the factors using this differential calculus is as follows. Let us consider the equation $E = x^2 + 7x + 12 = 0$. This can be factorized into (x + 3) and (x + 4) using Adyamadyena followed by Anurupyena. The relations between the factors and co-efficients of various powers of x in the given expression E and the successive differentials are exemplified below. Given the expression

$$E = x^2 + 7x + 12 = \overset{\ell}{(x+3)}\,\overset{m}{(x+4)}$$

Let the first factor (x + 3) be 'ℓ' and the 2nd factor (x + 4) be 'm'.

The first differential D_1 of the given expression E is 2x + 7. This is equal to ℓ + m i.e. (x + 3) + (x + 4) which is simply $\sum l$, the sum of the factors.

Let E be $x^2 + 7x + 12 = (x + 3)(x + 4)$, By Calana Kalana

$$D_1 = 2x + 7 = (x + 3)1 + (x + 4)1 = l\,\frac{dm}{dx} + m\,\frac{dl}{dx}$$

$$= 2x + 7 = \sum l$$

Consider a cubic equation

General Cubic equation

$$(x + a)(x + b)(x + c) = x^3 + x^2(a + b + c) + x(ab + ac + bc) + abc$$

Then (x + a), (x + b), (x + c) are the factors (ℓ, m, n) of the Cubic equation and – a , – b, – c are the three roots of the equation. By Calana Kalana

$$D_1 = 3x^2 + 2x(a + b + c) + (ab + ac + bc)$$

$$= \sum(x+a)(x+b) = \sum lm$$

$$= (x + a)(x + b) + (x + a)(x + c) + (x + b)(x + c)$$

$$= 3x^2 + 2x(a + b + c) + (ab + ac + bc)$$

$D_2 = 6x + 2(a + b + c) = 2! \sum l$

$2! \sum (x + a) = (x + a) + (x + b) + (x + c)$

$= 2! [3x + (a + b + c)] = 6x + 2(a + b + c)$

Example 1 :

$y = x^3 + 16x^2 + 71x + 56 = 0$. Find the factors.

Let (x + a), (x + b) and (x + c) i.e. (ℓ, m and n) be its factors

$D_1 = 3x^2 + 32x + 71 = \sum lm$

$D_2 = 6x + 32 = 2! \sum l$

By Argumentation the factors ℓ, m, n can be written down. Consider the equation $x^3 + 16x^2 + 71x + 56 = 0$.

The procedure is to find out the three possible factors a, b, c of the constant term 56 and to select from those various combinations, the one which satisfies the above expressions for the differentials. Out of all possible factors of 56 (integer) (4, 14, 1), (28, 2, 1), (56, 1, 1), (7, 8, 1), (2,4,7) and (2,2,14), (– ve values also if necessary may be considered) the combination (7,8,1) alone satisfies the following.

1) The condition of the divisibility of S_c (sum of the coefficients of the given Cubic equation E) by the sum of the coefficients of each factor of the combination, should be satisfied.

2) The coefficients of x^2 term and coefficient of x term derivable from the given combination should explain the corresponding terms of the given cubic equation E.

3) The 1st, and 2nd differentials should be explainable as per their General relations with the factors. The combination of factors will satisfy the derivative expressions.

When the factors are (x + 8), (x + 7) and (x + 1) as ℓ, m and n, one may note that any one of these factors can be taken as ℓ and any of the remaining as m and n.

Let us apply them to the above case. The factors of 56 as 7, 8, 1 alone satisfies, the above three conditions. When 7, 8, 1 are factors of the constant terms 56 one can write down (x + 7) (x + 8) (x + 1) as the factors of E and then verify the above three conditions.

1) The factors of the equation considered are (x +7), (x + 8) and (x + 1) as ℓ, m, n. Then the sum of the coefficients of each factor are 8, 9 and 2 respectively. These are factors of $S_c = 1 + 16 + 71 + 56 = 144 = 8 \times 9 \times 2$. But all the other combinations for the constant term 56 do not satisfy this rule.

2) x^2 coefficient of the given equation E is 16.

This is equal to Sum of the factors of the constant term 56. $\sum a = (a + b + c) = 7 + 8 + 1 = 16$

Similarly x coefficient (71) of the given equation E = Sum of the products of two factors (of constant term 56) taken two at a time. $\sum ab = (7 \times 8) + (8 \times 1) + (7 \times 1) = 71$

3) $D_1 = 3x^2 + 32x + 71 = \sum lm = \sum (x + a)(x + b) = (x + 7)(x + 8) + (x + 7)(x + 1) + (x + 1)(x + 8) = 3x^2 + 32x + 71$

$D_2 = 6x + 32 = 2! \sum l = 2! (l + m + n) = 2(x + 7 + x + 8 + x + 1) = 6x + 32$

$E = (x + 7)(x + 8)(x + 1)$

Example 2 : Consider $24x^3 + 66x^2 - 81x - 189 = (2x + 3)(3x + 9)(4x - 7) = 0$

Verifying the Differential relation.

The factors of this equation are $(2x + 3)$ as ℓ ; $(3x + 9)$ as m; $(4x - 7)$ as n.

$D_1 = 72x^2 + 132x - 81$

$D_1 = \sum lm = (2x + 3)(3x + 9)4 + (4x - 7)(3x + 9)2 + (2x + 3)(4x - 7)3$

$= (6x^2 + 27x + 27)4 + (12x^2 + 15x - 63)2 + (8x^2 - 2x - 21)3 = 72x^2 + 132x - 81$

As x^3 coefficient is not unity, one may also divide the given equation by coefficient of x^3 to get the required equation.

Dividing by 24 to make the coeff of x^3 as 1

The equation $= x^3 + \frac{66x^2}{24} \quad \frac{81x}{24} \quad \frac{189}{24}$

$= x^3 + \frac{11x^2}{4} - \frac{27x}{8} - \frac{63}{8}$

$$\left(x^3 + \frac{11x^2}{4} - \frac{27x}{8} - \frac{63}{8}\right) = \underset{\ell}{\left(x + \frac{3}{2}\right)} \underset{m}{(x + 3)} \underset{n}{\left(x - \frac{7}{4}\right)}$$

$D_1 = 3x^2 + \frac{22}{4}x - \frac{27}{8} = \sum lm = \ell m + \ell n + m n = \left(x^2 + \frac{9}{2}x + \frac{9}{2}\right) + \left(x^2 - \frac{1}{4}x - \frac{21}{8}\right) + \left(x^2 + \frac{5}{4}x - \frac{21}{4}\right) = 3x^2 + \frac{22}{4}x - \frac{27}{8}$

$D_2 = 6x + \frac{22}{4} = 2! \sum l = 2! (\ell + m + n) = 2\left[\left(x + \frac{3}{2}\right) + (x + 3) + \left(x - \frac{7}{4}\right)\right] = 6x + \frac{22}{4}$

Example 3 :

Let us consider another Cubic Equation $x^3 + 12x^2 + 47x + 60 = 0$. The factors of 60 are many out of which 3, 4, 5 will satisfy x coeff. and x^2 coeff. also. This is factorised into three factors ℓ, m, and n as $(x + 3) = \ell$, $(x + 4) = m$, and $(x + 5) = n$. In this case the first differential will be equal to $\sum lm$ i.e. , the sum of the product of the factors taken two at a time and the summation is extended over all such products. The 2nd differential D_2 is equal to $2! \sum l$.

This method can be extended to solve equations of any degree wherein the differentials D_1 D_2 D_3 …….. will be equal to $\sum lmn$, $2! \sum lm$, $3! \sum l$, for a fourth degree equation having 4 factors ℓ, m, n, p and $\sum lmnp$, $2!\sum lmn$, $3!\sum lm$, $4!\sum l$ for a 5th degree equation having 5 factors as ℓ, m, n, p and q and so on.

<u>4th Degree Equation</u> :

$(x + a)(x + b)(x + c)$ and $(x + d)$ the product of the four factors $(\ell, m, n, p) =$
$(x + a)(x + b)(x + c)(x + d) = x^4 + x^3(a + b + c + d) + x^2(ab + ac + ad + bc + bd + cd) +$
$x(abc + abd + acd + bcd) + abcd$
First differential $D_1 = 4x^3 + 3x^2(a + b + c + d) + 2x(ab + ac + ad + bc + bd + cd) +$
$(abc + abd + acd + bcd) = \sum lmn$

$= \sum (x + a)(x + b)(x + c) = (x + a)(x + b)(x + c) + (x + a)(x + b)(x + d)$
$(x + a)(x + c)(x + d) + (x + b)(x + c)(x + d)$
$= 4x^3 + 3x^2(a + b + c + d) + 2x(ab + ac + ad + bc + bd + cd) + (abc + abd + acd + bcd)$
Second differential $D_2 = 12x^2 + 6x(a + b + c + d) + 2(ab + ac + ad + bc + bd + cd) = 2!\sum lm$

$= 2!\sum (x + a)(x + b) = 2![(x + a)(x + b) + (x + a)(x + c) +$
$(x + a)(x + d) + (x + b)(x + c) + (x + b)(x + d) + (x + c)(x + d)]$
$= 12x^2 + 6x(a + b + c + d) + 2(ab + ac + ad + bc + bd + cd)$
Third differential $D_3 = 24x + 6(a + b + c + d)$
$= 3!\sum (x + a) = 3![(x + a) + (x + b) + (x + c) + (x + d)] = 3!\sum l$
$= 24x + 6(a + b + c + d)$

<u>5th Degree Equation</u> :

Similarly one can work out 5th degree equation which has
$(x + a)(x + b)(x + c)(x + d)(x + e)$ as factors.
ℓ m n p q

$D_1 = \sum (x + a)(x + b)(x + c)(x + d) = \sum lmnp$

$D_2 = 2!\sum (x + a)(x + b)(x + c) = 2!\sum lmn$

$D_3 = 3! \sum (x + a)(x + b) = 3! \sum lm$

$D_4 = 4! \sum (x + a) = 4! \sum l$

General r^{th} degree equation

Let us consider an r^{th} degree equation having r factors as $(x + a)(x + b)(x + c) \ldots (x + r)$
This is expanded as $= x^r + x^{r-1}(a + b + c + \ldots\ldots + (r - 1) + r)$

$+ x^{r-2}(ab + bc + ad \ldots\ldots ar + bc + bd + b(r - 1) + br + \ldots\ldots (r - 1)r)$

$+ x^{r-3}[(abc + abd + abe + acd + \ldots\ldots (r - 2)(r - 1)(r)]$

$+ x(abcde.. (r - 1) + bcde\ldots r) + a\,b\,c\,d\,e \ldots (r - 1)r.$

$D_1 = \sum (x + a)(x + b) \ldots\ldots [x + (r - 1)] = \sum lmn\ldots(r - 1)$

$D_2 = 2! \sum (x + a)(x + b) \ldots\ldots (x + (r - 2)) = 2! \sum lmn\ldots (r - 2)$

$D_3 = 3! \sum (x + a)(x + b) \ldots\ldots (x + (r - 3)) = 3! \sum lmn\ldots (r - 3)$

$D_{r-3} = (r - 3)! \sum (x + a)(x + b)(x + c) = (r - 3)! \sum lmn$

$D_{r-2} = (r - 2)! \sum (x + a)(x + b) = (r - 2)! \sum lm$

$D_{r-1} = (r - 1)! \sum (x + a) = (r - 1)! \sum l$

Note :

The formula that are derived by Swamiji under Gunaka Samuccayah, the relation between Differentiation and factorization are in reference to

1) Factors with unit coefficient of x and expression and the product of the factors having unit coefficient for the highest degree.

If the given expression, which needs to be factorized is having coefficient other than 1 for the highest degree term, then one can divide the expression by the coefficient so as to apply directly the formula derived by Swamiji.
i.e.

Example : $E = x^3 + 4x^2 + x - 6 = (x + 2)(x + 3)(x - 1)$

$(x + 2) = \ell \quad (x + 3) = m \quad (x - 1) = n$

The first differential

$D_1 = \sum lm = (x + 2)(x + 3) + (x + 2)(x - 1) + (x + 3)(x - 1)$

$= 3x^2 + 8x + 1$

Second differential

$D_2 = 2! \sum l = 2![(x + 2) + (x + 3) + (x - 1)] = 6x + 8$

This formulae for D_1, D_2 etc are concerned with unit coefficient for the highest degree.

If the given expression has a coefficient not equal to 1 for the highest degree, then one can solve in two ways.

I. By simply dividing the equation by the coefficient of highest degree, one sets it as an equation with the coefficient of highest degree as 1. Then it can be worked out exactly similar to that explained earlier

i.e. $D_1 = \sum lm$

$D_2 = 2! \sum l$

1) $E = 4x^3 - 12x^2 - 15x - 4$

$E = 4\left(x^3 - 3x^2 - \frac{15}{4}x - 1\right) = 0$

$\therefore\ E = x^3 - 3x^2 - \frac{15}{4}x - 1 = 0$

Let the factors be $\left(x + \frac{1}{2}\right)\left(x + \frac{1}{2}\right)(x - 4)$

ℓ m n

$D_1 = 3x^2 - 6x - \frac{15}{4}$

$D_1 = \sum lm = \left(x^2 + x + \frac{1}{4}\right) + \left(x^2 - \frac{7}{2}x - 2\right) + \left(x^2 - \frac{7}{2}x - 2\right) = 3x^2 - 6x - \frac{15}{4}$

$D_2 = 6x - 6$

$D_2 = 2! \sum l = 2\left(x + \frac{1}{2}\right) + \left(x + \frac{1}{2}\right) + (x - 4) \quad 2(3x - 3) = 6x - 6$

II. The General method of solving by the application of Gunaka Samuccayah for the cubic equation.

$D_1 = \sum lm\frac{dn}{dx} = lm\frac{dn}{dx} + \ln\frac{dm}{dx} + mn\frac{dl}{dx}$

$D_2 = \sum \frac{dn}{dx}\left(l\frac{dm}{dx} + m\frac{dl}{dx}\right) + \sum \frac{d^2l}{dx^2}(mn)$

$= \frac{dn}{dx}\left(l\frac{dm}{dx} + m\frac{dl}{dx}\right) + \frac{dl}{dx}\left(m\frac{dn}{dx} + n\frac{dm}{dx}\right) + \frac{dm}{dx}\left(n\frac{dl}{dx} + l\frac{dn}{dx}\right) + \frac{d^2l}{dx^2}(mn) + \frac{d^2m}{dx^2}(nl) + \frac{d}{d}$

$E = 4x^3 - 12x^2 - 15x - 4 = 0$

$(2x + 1)(2x + 1)(x - 4)$

ℓ m n

$D_1 = 12x^2 - 24x - 15$

$D_1 =$

1) $(4x^2 + 4x + 1) \left| \frac{dn}{dx} \right| = 4x^2 + 4x + 1, \ \frac{dn}{dx} = 1$

2) $(2x^2 - 7x - 4)\left(\frac{dl}{dx}\right) = 4x^2 - 14x - 8 \quad \frac{dl}{dx} = 2$

3) $(2x^2 - 7x - 4)\left(\frac{dm}{dx}\right) = \underline{4x^2 - 14x - 8} \quad \frac{dm}{dx} = 2$

$D_1 = \underline{12x^2 - 24x - 15}$

$D_2 = 24x - 24$

$$D_2 = \sum \frac{d}{dx}\left(lm \frac{dn}{dx}\right)$$

$$= \frac{d^2n}{dx^2}(lm) + \frac{dn}{dx}\left(l\frac{dm}{dx} + m\frac{dl}{dx}\right) + \frac{d^2l}{dx^2}(mn) + \frac{dl}{dx}\left(m\frac{dn}{dx} + n\frac{dm}{dx}\right) + \frac{d^2m}{dx^2}(nl) + \frac{dm}{dx}\left(n\frac{dl}{dx} + l\frac{dn}{dx}\right)$$

But $\frac{d^2n}{dx^2} = \frac{d^2l}{dx^2} = \frac{d^2m}{dx^2} = 0$

Hence $D_2 =$ $1[(2x + 1)2 + (2x + 1)2] = 8x + 4$

$+ 2[(2x + 1)1 + (x - 4)2] = 8x - 14$

$+ 2[(x - 4)2 + (2x + 1)1] = 8x - 14$

$D_2 = 24x - 24$

SECTION – 9

SUCCESSIVE DIFFERENTIATION – FACTORISATION (REPEATED FACTORS)

There are certain significant internal relationships between factors and successive differentials of polynomials as such Swamiji has given very elegant methods showing these relations for the equations of any degree which has factors ℓ, m, n ...... r (r factors)

In general, this is shown in the general r^{th} degree equation.

If " t " is one of the factors of the first differential D_1 of the given equation E. Then one can find out if this is a factor also of the E. If so it should appear once repeated factor as t^2. The factorisation of E should be in terms of t^2 which can be verified by Gunita Samuccayah Sutram.

This is extended to all the factors of D_1. Through this process the equation E can be obtained. If t occurs as repetition for more than once say 3, 4. D_1 should be factorisable in terms t^2, t^3 etc. At every stage the verification is to be carried out using Gunita Samuccayah. If D_1 is factorizable in terms of t^2 , t^3 etc then D_2 should be factorizable in terms of t, t^2 etc respectively in which case t^3 should be a factor of E. This is shown in the following general table

E	D_1	D_2	D_3	D_4	D_4	–	–	D_{n-1}
t^2	t	–	–	–	–	–	–	–
t^3	t^2	t	–	–	–	–	–	–
t^4	t^3	t^2	t	–	–	–	–	–
t^n	t^{n-1}	t^{n-2}	t^{n-3}	t^{n-4}	t^{n-5}	–	–	t

A few examples are given for each degree equation, where the degree is 2,3,4,5& 6.

Swamiji has further extended the concept of Successive differentiation to locate the repeated factors, of the given equation E.

Working out for the location of the repetition of the factors is exemplified in the following problems.

Example 4 : $E = x^3 + 5x^2 + 7x + 3 = 0$

The first differential $D_1 = 3x^2 + 10x + 7$

It can be factorized by Adyamadyena as $(x + 1)(3x + 7)$

If there is a possibility of any one of these as the factors in the given cubic equation, then the given equation E should be factorizable as the square of that factor.

First let us consider the possibility of the factor (x + 1) of D_1 to be a factor of the equation E, then it should occur as $(x+1)^2$ in E. As such the given cubic equation should be written as $(x+1)^2$ A. A is to be determined by Adyamadyena and Argumentation followed by Gunita Samuccayah Sutram as follows $E = x^3 + 5x^2 + 7x + 3 = (x^2 + 2x + 1)A$ [(A will have only x term and a constant term] Determination of A by

Adyamadyena $\frac{x^3}{x^2} = x$, $\frac{3}{1} = 3$ $\quad A = (x+3)$

The given cubic equation E is factorized as $(x+1)^2(x+3)$,

This factorisation is to be also verified by applying Gunita Samuccayah Sutram

$S_c = 16 = (1+2+1)(1+3) = 4 \times 4 = 16$

$(x+1)^2(x+3)$ are the factors of the given cubic equation.

It is obvious that (3x + 7) cannot be a factor of the given E.

(x + 1) alone repeats in the Cubic equation with the roots – 1, – 1 and – 3.

A few more examples are given below under successive differentiation.

Example 5 : $x^3 + 7x^2 - 5x - 75 = 0$

The first differential $D_1 = 3x^2 + 14x - 5$

Applying Adyamadyena followed by Anurupyena = $3x^2 + 15x - x - 5$

$\frac{3x^2}{15x} = \frac{-x}{-5} \Rightarrow (x+5)$ is a factor of D_1. The other factor is $\frac{3x^2}{x} - \frac{5}{5} = (3x-1)$

D_1 is factorized as $(x+5)(3x-1)$

If (x + 5) is a factor of cubic equation, then the given cubic equation should be factorizable in terms of $(x+5)^2$ i.e.

$E = (x+5)^2 A$ and A is to be determined

$x^3 + 7x^2 - 5x - 75 = (x^2 + 10x + 25)A$, (A will have only x term & constant term)

Applying Adyamadyena $\frac{x^3}{x^2} = \frac{-75}{25} = (x-3) = A$. Applying Gunita Samuccayah for verification of factorization.

$S_c = -72 = (1+10+25)(-2) = -72$

Given equation $E = (x+5)^2(x-3)$

It can be shown that (3x – 1) of D_1 is not a factor of E.

Example 6 : $x^4 - 13x^3 + 42x^2 + 32x - 224 = 0$

The first Differential $D_1 = 4x^3 - 39x^2 + 84x + 32$

$D_2 = 12x^2 - 78x + 84 = 6(2x^2 - 13x + 14)$ This has no rational factors.

Applying Vilokanam to D_1 to find out its factors.

Let $f(x) = 4x^3 - 39x^2 + 84x = -32$

	LHS	RHS	RHS – LHS
x = 1	49		– 81
x = 2	44		– 76
x = 3	9		– 41
x = 4	– 32	·32	0

x = 4 is a solution

(x – 4) is a factor of D_1

$D_1 = 4x^3 - 39x^2 + 84x + 32 = (x - 4)$ A. A should have x^2, x, constant term

By Argumentation,

$D_1 = (x - 4)(4x^2 - 23x - 8)$. This is verified by Gunita Samuccayah.

$S_c = 81 = (1 - 4)(4 - 23 - 8) = 81$.

In order to find out if (x– 4) is also a factor of the original Equation E, then the latter should be factorizable in terms of $(x - 4)^2 = x^2 - 8x + 16$

$E = (x^2 - 8x + 16)$ (B) B will have x^2, x and constant terms.

B can be determined by Adyamadyena & Argumentation.

By Adyamadyena the first and last terms of B will be x^2 and – 14

$E = (x^2 - 8x + 16)(x^2 + \alpha x - 14)$, where α is to be determined. By comparing the coefficients of x with that of E, one gets $16\alpha + 112 = 32$; $\alpha = -5$

Verifying the above factorisation by applying Gunita Samuccayah Sutram .

$S_c = -162 = (1 - 8 + 16)(1 - 5 - 14) = -162$

$(x^2 - 5x - 14)$ is another factor of E. This quadratic expression can be further factorized by applying Adyamadyena as (x + 2) (x – 7).

The given equation E can be factorized as $(x - 4)^2 (x + 2)(x - 7)$

x = 4, 4, – 2, + 7 are the solutions of E.

Example 7 : $x^5 - 7x^4 - 15x^3 + 243x^2 - 702x + 648 = 0$

Let $E = x^5 - 7x^4 - 15x^3 + 243x^2 - 702x + 648$ (given equation)

First differential $D_1 = 5x^4 - 28x^3 - 45x^2 + 486x - 702$

$D_2 = 20x^3 - 84x^2 - 90x + 486$

$D_3 = 60x^2 - 168x - 90$

$D_4 = 120x - 168$

One can also start working with higher differentials

Applying Vilokanam to $D_2 = 20x^3 - 84x^2 - 90x + 486 = 0$

$S_c \neq 0$ (x – 1) is not a factor

$S_0 \neq S_e$ (x + 1) is not a factor

Let $f(x) = 20x^3 - 84x^2 - 90x = -486$

	LHS f(x)	RHS – 486	Diff RHS – LHS
x = 1	– 154		– 332
x = 2	– 356		– 130
x = 3	– 486	– 486	0
x = 4	– 424		– 62

$(x - 3)$ is a factor of D_2 . The remaining expression can be obtained by applying Adyamadyena followed by Argumentation

$D_2 = 20x^3 - 84x^2 - 90x + 486 = (x - 3)(20x^2 - 24x - 162)$. This is verified by applying Gunita Samuccayah Sutram.

$332 = (1 - 3)(20 - 24 - 162)$

$= -2(-166) = 332$

Hence $(x - 3)$ is a factor of D_2. In order to find out if $(x - 3)$ is also a factor of E then D_1 should be factorizable in terms of $(x - 3)^2$.

$D_1 = 5x^4 - 28x^3 - 45x^2 + 486x - 702 = (x^2 - 6x + 9)(A)$

A is to be determined by applying Adyamadyena followed by Argumentation. A should have x^2, x and constant term. The x^2 term and the constant can be determined by Adyamadyena as $5x^2$, and -78; The x term is determined by comparison of x terms on both sides.

$D_1 = (x^2 - 6x + 9)(5x^2 + 2x - 78)$

It can be verified by applying Gunita Samuccayah Sutram as

$S_c = -284 = (1 - 6 + 9)(5 + 2 - 78) = -284$

$(x - 3)$ repeats twice in the D_1.

If $(x - 3)$ is a factor of E than it should be factorizable in terms of $(x - 3)^3 = (x^3 - 9x^2 + 27x - 27)$

This is worked out by applying Adyamadyena and Argumentation as follows.

$E = x^5 - 7x^4 - 15x^3 + 243x^2 - 702x + 648 = (x^3 - 9x^2 + 27x - 27)B$

B should have x^2, x and constant term. The first and the last term of B are obtained as x^2 and -24 by Adyamadyena. The x term is to be determined by a comparison of x on both sides

$B = (x^2 + \alpha x - 24)$

$E = (x^3 - 9x^2 + 27x - 27)(x^2 + \alpha x - 24)$

$(-24 \times 27)x - 27\alpha x = -702x$

$\alpha = 2$

This is verified by applying Gunita Samuccayah Sutram as.

$S_c = 168 = (1 - 9 + 27 - 27)(1 + 2 - 24)$

$= -8(-21) = 168$

E has $(x - 3)^3$ as its three factors (repeated)

The remaining Quadratic expression of $E = (x^2 + 2x - 24)$ can be further factorized by using Adyamadyena. (followed by Anurupyena)

$x^2 + 2x - 24 = x^2 + 6x - 4x - 24 = (x + 6)(x - 4)$

The given equation E is factorized as $(x - 3)^3 (x + 6)(x - 4)$ with 3, 3, 3 and -6 and 4 as its roots.

Example 8: $E = x^6 + 12x^5 + 57x^4 + 136x^3 + 171x^2 + 108x + 27 = 0$

The first Differential

$D_1 = 6x^5 + 60x^4 + 228x^3 + 408x^2 + 342x + 108$

Factorization of D_1 by Vilokanam. (As $S_0 = S_e$ (x + 1) is a factor). In order to get the remaining expression A of D_1 we can use Adyamadyena followed by

Argumentation. Adyamadyena gives $\frac{6x^5}{x}$, $\frac{108}{1}$ the first and the last.

But the complete expression of A has also x^3 term as αx^3, x^2 term as βx^2 and x term as γx where α, β, and γ are to be determined.

$D_1 = (x + 1)(6x^4 + \alpha x^3 + \beta x^2 + \gamma x + 108)$

By Argumentation, by way of comparison of similar coefficients , one can write the following relations $108x + \gamma x = 342x \Rightarrow \gamma = 234$

Similarly $234x^2 + \beta x^2 = 408x^2 \Rightarrow \beta = 174$

And $174x^3 + \alpha x^3 = 228x^3 \Rightarrow \alpha = 54$

$D_1 = (x + 1)(6x^4 + 54x^3 + 174x^2 + 234x + 108)$

If (x + 1) is a factor of the given 6^{th} degree equation E, then E should be factorizable in terms of $(x + 1)^2$

Let us see if this can be carried out. The 6^{th} degree equation E is now written as $(x^2 + 2x + 1)$ (B). B should be in the form $(\alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \xi)$. The coefficient of x^4 and constant terms can be obtained by using Adyamadyena $\alpha = \frac{x^6}{x^2}$, $\xi = \frac{27}{1}$. The remaining coefficients β γ δ can be obtained by comparison of the similar terms on both sides i.e. by Argumentation as follows

$54x + \delta x = 108x \Rightarrow \delta = 54$

$\gamma x^2 + 135x^2 = 171x^2 \Rightarrow \gamma = 36$

$\beta x^3 + 126x^3 = 136x^3 \Rightarrow \beta = 10$

$E = (x + 1)^2 (x^4 + 10x^3 + 36x^2 + 54x + 27) = 0$

This factorization satisfies Gunita Samuccayah Sutram. $S_c = 512 = (1 + 1)(1 + 1)(1 + 10 + 36 + 54 + 27) = 512$. Hence $(x + 1)^2$ is once repeated factor of the given sixth degree equation E. If (x + 1) still repeats in the given equation E, then $(x + 1)^3$ is a factor. For this let us consider the second differential $D_2 = 30x^4 + 240x^3 + 684x^2 + 816x + 342$ As $S_0 = S_e$, it has (x + 1) as a factor. Using the above principles, D_2 can be written as $(x + 1)(30x^3 + 210x^2 + 474x + 342)$

This shows that (x + 1) will repeat 3 times in the given sixth degree equation. To verify the repetition of (x + 1) three times in the given 6^{th} degree equation, one has to factorise E with $(x + 1)^3$ as the three factors,

$E = (x + 1)^3 C$, C will have x^3, x^2, x and constant terms.

By Argumentation and comparison of the coefficients one can show that $C = x^3 + 9x^2 + 27x + 27$ which is verified by Gunita Samuccayah.

It is interesting to note that D_1 should be factorizable also in terms of $(x + 1)^2$. If so $D_1 = (x + 1)^2 F$

Where $F = 6x^3 + 48x^2 + 126x + 108$ which is similarly obtained by using the above principles.

This satisfies the Gunita Samuccayah Sutram

The third differential D_3 of the given sixth degree equation is
$D_3 = 120x^3 + 720x^2 + 1368x + 816$. By Vilokanam it can be clearly seen that $(x + 1)$ is not a factor as $S_0 \neq S_e$
$(x + 1)$ repeats only thrice in E.

It is easier to find the remaining roots of the given 6^{th} degree equation, by concentrating on the remaining third degree equation $x^3 + 9x^2 + 27x + 27 = C$.

Let us differentiate this expression successively. The first differential of the equation $D_1 = 3x^2 + 18x + 27 = 3(x^2 + 6x + 9) = 3(x + 3)(x + 3)$

The third degree equation should be expressible in terms of $(x +3)^3$

Here the third degree equation is itself $(x + 3)^3$

The sixth degree equation can be factorized as $(x + 1)^3 (x + 3)^3$. Similar procedure is adoptable for locating the repetition of the roots (the factors if any), for an equation any degree by successive differentiation method which is considered to be very elegant.

The Six roots of E are $-1, -1, -1, -3, -3$ and -3.

SECTION – 10

PURANA APURANABHYAM

I Cubics

1) $E = x^3 + 6x^2 - 37x + 30 = 0$ (Ref. Vilokanam Section)

a) By Purana Apuranabhyam Method.

Rewriting the equation E as $x^3 + 6x^2 = 37x - 30$ (By Paravartya)

Consider a perfect cube in which the first two terms of the equation E $(x^3 + 6x^2)$ occur as they are $(x+2)^3 = x^3 + 6x^2 + 12x + 8$. This is Purana Method. Substituting in the Standard equation $(x+2)^3$ for the first two terms $x^3 + 6x^2$ from the given equation $E = (x^3 + 6x^2 - 37x + 30) = 0$, $x^3 + 6x^2 = 37x - 30$
$(x+2)^3 = 37x - 30 + 12x + 8 = 49x - 22$

Let $(x+2) = y \therefore x = y - 2$
$\therefore y^3 = 49(y-2) - 22 = 0$

b) $y^3 - 49y + 120 = 0$ ———— (1) Solving cubic equation
1st step by Vilokanam
$S_c \neq 0 \therefore (y-1)$ is not a factor
$S_0 \neq S_c \therefore (y+1)$is not a factor
But a trial in Succession, shows that y = 3 satisfies the equation $\therefore (y-3)$ is a factor.

$\therefore$ equation (1) = (y – 3)A. A should have y^2, y and constant terms. Applying Adyamadyena Sutram, the first and the last terms of A are y^2 and – 40.
i.e $\left(\frac{y^3}{y} = y^2, \ \frac{120}{-3} = -40\right)$

Now the equation can be written as $(y-3)(y^2 + \alpha y - 40)$ where α is the coefficient of y and can be determined by applying Gunita Samuccayah Sutram.
$S_c = 72 = (1-3)(1 + \alpha - 40) \quad \therefore \alpha = 3$

$\because$ The Quadratic expression is $(y^2 + 3y - 40)$. This can be factorized further by applying Adyamadyena Sutram.

$y^2 + 3y - 40 = y^2 + 8y - 5y - 40 = (y+8)(y-5)$
The equation is factorized as $(y-3)(y-5)(y+8)$; but $y = (x+2)$
$y = 3 \Rightarrow x = 1$; $y = 5 \Rightarrow x = 3$; $y = -8 \Rightarrow x = -10$
$\therefore$ The given equation E is factorized as $(x-1)(x-3)(x+10)$

2) $E = x^3 + 9x^2 + 23x + 15 = 0$

a) <u>By Purana Apuranabhyam Method</u>

Rewriting the equation as $x^3 + 9x^2 = -23x - 15$. Consider the perfect cube in which the first two terms $x^3 + 9x^2$ of E occur as they are $(x + 3)^3 = x^3 + 9x^2 + 27x + 27$. This is Purana Method. Substituting in the Standard equation $(x + 3)^3$ for the first two terms $(x^3 + 9x^2)$ form the given equation the expression $-23x - 15$.

$\therefore\ (x + 3)^3 = -23x - 15 + 27x + 27 = 4x + 12 = 4(x + 3)$
Let $(x + 3) = y$
b) $y^3 = 4y \quad \Rightarrow \quad y = 0$ is one solution.
$y^2 = 4 \quad \Rightarrow \quad y = \pm 2$ are the other solutions.
$y = (x + 3)$
$y = 0 \Rightarrow x = -3; \quad y = 2 \Rightarrow x = -1; \quad y = -2 \Rightarrow x = -5$
$(x + 1)(x + 3)(x + 5)$ are the factors of the given equation.

3) $E = x^3 + 9x^2 + 24x + 16 = 0$

a) <u>By Purana Apuranabhyam Method</u>

Rewriting the equation as $x^3 + 9x^2 = -24x - 16$. Consider the perfect cube in which the first two terms of E, $x^3 + 9x^2$ occur as they are $(x + 3)^3 = x^3 + 9x^2 + 27x + 27$. This is Purana Method substituting in the standard equation $(x + 3)^3$ for the first two terms from the given equation $x^3 + 9x^2 = -24x - 16$.
$(x + 3)^3 = -24x - 16 + 27x + 27 = 3x + 11$
$(x + 3)^3 = 3(x + 3) + 2$
Let $(x + 3) = y$
b) $y^3 = 3y + 2; \quad y^3 - 3y - 2 = 0$ Solving the cubic equation in y
1^{st} step by Vilokanam
$S_c \neq 0 \quad (y - 1)$ is not a factor
$S_0 = S_c \Rightarrow (y + 1)$ is a factor

The other two roots can be determined by applying Adyamadyena Sutram followed by Gunita Samuccayah Sutram.

The equation in y can be factorized as $(y + 1)$ A. Where A should contain y^2, y and constant terms.

The first and last terms of A can be obtained by using Adyamadyena and they are y^2 and -2.

The equation in y can be written as $(y + 1)(y^2 + \alpha y - 2)$
By Gunita Samuccayah Sutram, α can be determined.
$S_c = -4 = (1 + 1)(1 + \alpha - 2); \quad \alpha \doteq -1$

This α can also be determined by Argumentation i.e. by comparing the like terms in $y^3 - 3y - 2 = 0$ and $(y + 1)(y^2 + \alpha y - 2)$
$(\alpha - 2) = -3;\ \alpha = -1$

Thus A, the Quadratic expression = $(y^2 - y - 2)$ and this can be further factorized by using Adyamadyena Sutram.
$y^2 - 2y + y - 2 \Rightarrow (y + 1)(y - 2)$
The equation in y can be factorized as $(y + 1)^2 (y - 2)$
$y = (x + 3) \Rightarrow x = y - 3$
If $y = -1 \Rightarrow x = -4 \Rightarrow (x + 4)$ is a factor
If $y = 2 \Rightarrow x = -1 \Rightarrow (x + 1)$ is a factor

Thus the given equation is factorised as $x^3 + 9x^2 + 24x + 16 = (x + 4)^2 (x + 1) = 0$
The repetition of the root can be also obtained by Successive Differentiation

The above problems are solved using general argumentation Method (Vedic Method)

1) $E = x^3 + 6x^2 - 37x + 30 = 0$
By Vilokanam
$S_c = 0 \Rightarrow (x - 1)$ is a factor
$S_0 \neq S_e \Rightarrow (x + 1)$ is not a factor
$\therefore$ E = (x – 1) A. A should have x^2, x and constant terms.

By Adyamadyena the first and last terms of A are $\frac{x^3}{x} = x^2,\ \frac{30}{-1} = -30$
$\therefore$ $E = (x - 1)(x^2 + \alpha x - 30)$ α is to be determined by comparing the x–coefficient on both sides

$-30 - \alpha = -37$
$-\alpha = -7 \Rightarrow \alpha = 7$
$\therefore$ $A = (x^2 + 7x - 30)$. A can be solved. By using differential relation
$D_1 = \pm \sqrt{Discriminant}$
$2x + 7 = \pm \sqrt{49 + 120}$
$2x + 7 = \pm 13, \Rightarrow x = -10, 3$
$\therefore$ $E = (x - 1)(x - 3)(x + 10)$

2) $E = x^3 + 9x^2 + 23x + 15 = 0$
By Vilokanam
$S_c \neq 0 \Rightarrow (x - 1)$ is not a factor
$S_0 = S_e \Rightarrow (x + 1)$ is a factor
$\therefore$ E = (x + 1) A. A should have x^2, x and constant terms.

By Adyamadyena the first and last terms of A are $\frac{x^3}{x} = x^2,\ \frac{15}{1} = 15$

$\therefore$ E = (x + 1) $(x^2 + \alpha x + 15)$ α is to be determined by comparing the x–coefficient on both sides

$15 + \alpha = 23$

$\alpha = 8$

$\therefore$ A = $(x^2 + 8x + 15)$. A can be solved. By using differential relation

$D_1 = \pm\sqrt{Discriminant}$

$2x + 8 = \pm\sqrt{64 - 60}$

$2x + 8 = \pm 2, \Rightarrow x = -5, -3$

$\therefore$ E = (x + 1) (x + 3) (x + 5)

This is verified by Gunita Samuccayah

$S_c = 1 + 9 + 23 + 15 = 48 = (1 + 1)(1 + 3)(1 + 5) = (2)(4)(6) = 48$

3) $E = x^3 + 9x^2 + 24x + 16 = 0$

By Vilokanam

$S_c \neq 0 \quad \Rightarrow \quad (x - 1)$ is not a factor

$S_0 = S_e \quad \Rightarrow \quad (x + 1)$ is a factor

$\therefore$ E = (x + 1) A. A should have x^2, x and constant terms.

By Adyamadyena the first and last terms of A are $\frac{x^3}{x} = x^2$, $\frac{16}{1} = 16$

$\therefore$ E = (x + 1) $(x^2 + \alpha x + 16)$ α is to be determined by comparing the x-coefficient on both sides

$16 + \alpha = 24 \quad \therefore \alpha = 8$

$\therefore A = (x^2 + 8x + 16) = (x + 4)^2 \qquad \therefore E = (x + 1)(x + 4)^2$

This is verified by Gunita Samuccayah

$S_c = 1 + 9 + 24 + 16 = 50 = (1 + 1)(1 + 4)(1 + 4) = (2)(5)(5) = 50$

II **Bi – Quadratics**

1. $E = x^4 + 8x^3 + 22x^2 + 24x + 9 = 0$

$x^4 + 8x^3 = -22x^2 - 24x - 9$ —— (1)

$\left(x + \frac{8}{4}\right)^4 = (x + 2)^4 = x^4 + 4x^3 2 + 6x^2 . 2^2 + 4x.2^3 + 2^4$

$= x^4 + 8x^3 + 24x^2 + 32x + 16$

Here, the value of $x^4 + 8x^3$ is carried out from the above equation (1).

$\left(x + \frac{8}{4}\right)^4 = -22x^2 - 24x - 9 + 24x^2 + 32x + 16$

$= 2x^2 + 8x + 7$

$= 2(x + 2)^2 - 1$

Considering $\left(x + \frac{8}{4}\right)$ as y or (x + 2) = y

$y^4 = 2y^2 - 1$

$y^4 - 2y^2 + 1 = 0$
$(y^2 - 1)^2 = 0, (y^2 - 1)(y^2 - 1) = 0$
$y = \pm 1, \pm 1$
$x = -1, -1, -3, -3$

2. $E = x^4 - 4x^3 - 23x^2 + 54x + 72 = 0$
$x^4 \quad 4x^3 = 23x^2 - 54x - 72$ ——— (1)
$\left(x - \frac{4}{4}\right)^4 = (x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$
The value of $(x^4 - 4x^3)$ from (1) is carried out to this equation
$(x-1)^4 = 23x^2 - 54x - 72 + 6x^2 - 4x + 1$
$= 29x^2 - 58x - 71$
Let $(x - 1)$ be y
$t_1 = y^4 - 29y^2 + 100 = 0$
By Vilokanam
$S_c \neq 0 \Rightarrow (x - 1)$ is not a factor
$S_0 \neq S_e \Rightarrow (x + 1)$ is not a factor
Let $f(y) = y^4 - 29y^2 = -100$ ——— (1)

	LHS f(y)	RHS −100	RHS – LHS
y = 1	– 28		– 72
y = 2	– 100		0
y = 3	– 180		80
y = – 1	– 28		– 72
y = – 2	– 100		0
y = – 3	– 180		80

$y = 2,\ y = -2$ are two solution s of the equation (1)
$E_1 = (y + 2)(y - 2)$ A. A should have y^2, y and constant terms by Adyamadyena first and last terms of A are y^2, -25
$\therefore\ E = (y^2 - 4)(y^2 + \alpha y - 25)$, α to be determed
By comparing y^2 coefficient on both sides, $\alpha = 0$
$\therefore\ E_1 = (y^2 - 4)(y^2 - 25) \Rightarrow E_1 = (y + 2)(y - 2)(y + 5)(y - 5)$
But $y = (x - 1) \Rightarrow x = (y + 1)$
If $y = 2 \Rightarrow x = 3,\ y = -2 \Rightarrow x = -1,\ y = 5 \Rightarrow x = 6, y - 5 \Rightarrow x = -4$
$\therefore\ E = (x + 1)(x - 3)(x + 4)(x - 6)$

3. By Poorana Apooranabhyam
$x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$
$x^4 - 12x^3 = -49x^2 + 78x - 40$ ——— (1)
$\left(x - \frac{12}{4}\right)^4 = (x-3)^4 = x^4 - 4x^3.3 + 6x^2.3^2 - 4x.3^3 + 3^4$
$= x^4 - 12x^3 + 54x^2 - 108x + 81$
The value of $(x^4 - 12x^3)$ from (1) is carried out to this equation.

$\left(x-\frac{12}{4}\right)^4 = -49x^2 + 78x - 40 + 54x^2 - 108x + 81$

$(x-3)^4 = 5x^2 - 30x + 41$

$= 5(x-3)^2 - 4$

Considering $(x - 3) = y$

$y^4 = 5y^2 - 4$

$y^4 - 5y^2 + 4 = 0$

$(y^2 - 1)(y^2 - 4) = 0$

$y = \pm 1, \pm 2$

$x = 1, 2, 4, 5$

$E = (x-1)(x-2)(x-4)(x-5) = 0$

The above problems are solved using general Argumentation Method (Vedic Method)

1. $E = x^4 + 8x^3 + 22x^2 + 24x + 9 = 0$

$S_c = 64$

The set of factors for the constant term 9 are $(\pm 3, \pm 3)(\pm 1, \pm 9)(\pm 9, \pm 1)$
Select a combination which explains the given equation.

$E = (x^2 + \alpha x + 3)(x^2 + \beta x + 3)$

x co–eff $3\alpha + 3\beta = 24$ $\alpha + \beta = 8$

x^3 co–eff $\alpha + \beta = 8$

x^2 co–eff $3 + 3 + \alpha\beta = 22,$ $\alpha\beta = 16$ $\alpha - \beta = 0$

$\alpha = \beta = 4$

Verification of x^2 coeff $3 + 3 + 16 = 22$

$E = (x^2 + 4x + 3)(x^2 + 4x + 3) = 0$

Verification by Gunita Samuccayah Sutra $S_c = 64 = (1 + 4 + 3)(1 + 4 + 3) = 64$

$\therefore E = (x+3)(x+1)(x+3)(x+1) = 0$

$E = (x+3)^2(x+1)^2$

$x = -3, -3, -1, -1$

Verification by Samuccayah Sutram $S_c = 64 = 4 \times 4 \times 2 \times 2 = 64$

2. $E = x^4 - 4x^3 - 23x^2 + 54x + 72 = 0$

The Set of factors for the constant term 72 are $(\pm 1, \pm 72), (\pm 3, \pm 24), (\pm 2, \pm 36), (\pm 4, \pm 18)(\pm 6, \pm 12)(\pm 8, \pm 9)$. Select a combination which explains the given equation.

$E = (x^2 + \alpha x - 3)(x^2 + \beta x - 24) = 0$

x Co–eff: $-24\alpha - 3\beta = 54$

x^3 Co–eff : $\alpha + \beta = -4$ $\alpha = -2$ $\beta = -2$

Verify x^2 Co–eff : $-24 - 3 + \alpha\beta = -27 + 4 = -23$

Given equation $E = (x^2 - 2x - 3)(x^2 - 2x - 24) = 0$

Verification by Gunita Samuccayah Sutram $S_c = 100 = (1-2-3)(1-2-24) = (-4)(-25) = 100$

The two Quadratic equations can be solved.

$x^2 - 2x - 3 = 0$ $\qquad$ $x^2 - 2x - 24 = 0$

$(x-3)(x+1) = 0$ $\qquad$ $(x-6)(x+4) = 0$

$x = 3, -1$ $\qquad$ $x = 6, -4$

$E = (x-3)(x+1)(x-6)(x+4) = 0$

$S_c = 100 = (-2)(2)(-5)(5) = 100$ Verified by Gunita Samuccayah Sutram.

3. $E = x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$

   $S_c = 0$

The set of factors for the constant term 40 are $(\pm 1, \pm 40)(\pm 2, \pm 20)(\pm 4, \pm 10)(\pm 5, \pm 8)$. Select a combination which explains the given equation.

$E = (x^2 + \alpha x + 5)(x^2 + \beta x + 8) = 0$

x Co–eff: $\quad 8\alpha + 5\beta = -78$

x^3 Co–eff: $\quad \alpha + \beta = -12$

Solving $\alpha = -6$; $\quad \beta = -6$

Verify x^2 Co–eff: $\quad 8 + 5 + \alpha\beta = 8 + 5 + 36 = 49$

Given equation $E = (x^2 - 6x + 5)(x^2 - 6x + 8) = 0$

Verification by Gunita Samuccayah Sutra $S_c = 0 = (1-6+5)(1-6+8) = 0$. verified

$x^2 - 6x + 5 = 0$ $\qquad$ $x^2 - 6x + 8 = 0$

$x = 1, 5$ $\qquad$ $x = 1, 4$

$\therefore\ E = (x-1)(x-5)(x-1)(x-4) = (x-1)^2(x-4)(x-5) = 0$

$S_c = 0 = (1-1)(1-5)(1-2)(1,-4) = 0$

Verified by Gunita Samuccayah Sutram.

A few more 4th degree equations are factorized using Argumentation

1. $E = x^4 - 6x^3 + 3x^2 + 22x - 6 = 0$

   $S_c = 14$

Consider all possibilities of factors of the constant term -6 and select a combination which explains the equation.

The set of factors for -6 are $(-3, 2)(-2, 3)(6, -1)(-1, 6), (1, -6)(3, -2)(2, -3)(-6, 1)$

$E = (x^2 + \alpha x + 1)(x^2 + \beta x - 6) = 0$

x^3 Co–eff: $\quad \alpha + \beta = -6$

x Co–eff: $\quad -6\alpha + \beta = 22$

$\alpha = -4, \qquad \beta = -2$

Verify x^2 Coeff $= -6 + 1 + \alpha\beta = -6 + 1 + 8 = 3$

Given equation $E = (x^2 - 4x + 1)\ (x^2 - 2x - 6) = 0$

Verification by Gunita Samuccayah Sutra $S_c = 14 = (1 - 4 + 1)(1 - 2 - 6) = 14$

The two Quadratics are further factorised using Differential relation.

$x^2 - 4x + 1 = 0$ $\quad\quad$ $x^2 - 2x - 6 = 0$

$D_1 = 2x - 4 = \pm\sqrt{16-4}$ $\quad\quad$ $D_1 = 2x - 2 = \pm\sqrt{4+24}$

$x = 2 \pm \sqrt{3}$ $\quad\quad$ $x = 1 \pm \sqrt{7}$

$E = (x - 2 + \sqrt{3})(x - 2 - \sqrt{3})(x - 1 + \sqrt{7})(x - 1 - \sqrt{7})$

Verification by Samuccayah Sutra $S_c = 14$

$(1 - 2 + \sqrt{3})(1 - 2 - \sqrt{3})(1 - 1 + \sqrt{7})(1 - 1 - \sqrt{7}) = 2 \times 7 = 14$

2. $E = 4x^4 - 4x^3 - 39x^2 + 36x + 27 = 0$

$S_c = 24$

The set of factors for the constant term 27 are $(\pm 27, \pm 1), (\pm 3, \pm 9)$

Select a combination which explains the given equation.

$E = (x^2 + \alpha x - 9)\ (4x^2 + \beta x - 3) = 0$

x Co–eff : $\quad -3\alpha - 9\beta = 36$

x^3 Co–eff : $\quad 4\alpha + \beta = -4$ $\quad\quad$ $\alpha = 0$ by Sunyamanyat as $\left(-\dfrac{9\beta}{1\beta} = \dfrac{36}{-4}\right)$

$\therefore\ \beta = -4$

Verify x^2 Co–eff : $\quad -36 - 3 + \alpha\beta = -36 - 3 + 0 = -39$

$E = (x^2 - 9)(4x^2 - 4x - 3) = 0 \Rightarrow (x + 3)(x - 3)(4x^2 - 4x - 3) = 0$

Verification by Gunita Samuccayah $S_c = 24 = (1 - 9)(4 - 4 - 3) = 24$

Quadratic Eq $4x^2 - 4x - 3 = 0$ can be solved by Differential relation.

$D_1 = 8x - 4 \quad = \pm\sqrt{16+48}$

$x = \dfrac{3}{2}, -\dfrac{1}{2}$

$E = (x + 3)(x - 3)\left(x - \dfrac{3}{2}\right)\left(x + \dfrac{1}{2}\right) = 0$

$E = (x + 3)(x - 3)\ (2x - 3)(2x + 1) = 0$

$S_c = 24 = 4(-2)(-1)(3) = 24$

Verified by Gunita Samuccayah Sutram.

3. $E = x^4 - 13x^3 + 42x^2 + 32x - 224 = 0$

$S_c = -162$

The set of factors for the constant term 224 are $(\pm 1, \pm 224)(\pm 2, \pm 112)(\pm 4, \pm 56)(\pm 7, \pm 32)\ (\pm 8, \pm 28)(\pm 14, \pm 16)$. Select a combination which satisfy the given equation.

$E = (x^2 + \alpha x - 14)\ (x^2 + \beta x + 16) = 0$

x coeff : $\quad 16\alpha - 14\beta = 32$

x^3 Coeff: $\quad \alpha + \beta = -13$

$\therefore\ \alpha = -5, \quad \beta = -8$

Verify x^2 Coeff: $\quad 16 - 14 + \alpha\beta = 42$

$16 - 14 + (-5)(-8) = 42$

$E = (x^2 - 5x - 14)\ (x^2 - 8x + 16) = 0$

Verification by Gunita Samuccayah Sutram

$S_c = -162 = = (1 - 5 - 14)(1 - 8 + 16) = -162$

The two Quadratic Equations can be solved

$x^2 - 5x - 14 = 0$ $\Rightarrow$ $x = -2, 7$

$x^2 - 8x + 16 = 0$ $x = 4, 4$

$E = (x + 2)(x - 7)(x - 4)^2 = 0$

$S_c = -162 = 3 \times (-6)(-3)(-3) = -162$

Verified by Gunita Samuccayah

Repetition of the roots can be also got from Successive differentiation.

4. $E = x^4 - x^3 - 5x^2 + 22x - 20 = 0$

$S_c = 0$

The set of factors for the constant term -20 are $(\pm 1, \pm 20)(\pm 2, \pm 10)(\pm 4, \pm 5)$. Select a combination which explains the given equation.

$E = (x^2 + \alpha x - 4)\ (x^2 + \beta x + 5) = 0$

x Co–eff : $5\alpha - 4\beta = 22$

x^3 Coeff : $\alpha + \beta = -1$

$\therefore\ \alpha = 2,\quad \beta = -3$

Verify x^2 Co–eff : $5 - 4 - 6 = -5$

Given equation $E = (x^2 + 2x - 4)\ (x^2 - 3x + 5) = 0$

Verification by Gunita Samuccayah Sutra $S_c = -3 = (1 + 2 - 4)(1 - 3 + 5) = -3$

The two Quadratic Equations can be solved.

$x^2 + 2x - 4 = 0$

$D_1 = 2x + 2 = \pm\sqrt{4 + 16} = \pm 2\sqrt{5}$

$x = -1 \pm \sqrt{5}$

$x^2 - 3x + 5 = 0$

$D_1 = 2x - 3 =$

$\pm\sqrt{9 - 20} = \pm\sqrt{11}i$

$x = \dfrac{3 \pm \sqrt{11}i}{}$

$$E = (x + 1 - \sqrt{5})(x + 1 + \sqrt{5})\left(x - \frac{3}{2} - \frac{\sqrt{11}i}{2}\right)\left(x - \frac{3}{2} + \frac{\sqrt{11}i}{2}\right)$$

$$S_c = -1\left(\frac{1}{4} + \frac{11}{4}\right) = -3$$

5. $E = x^4 + x^3 - 3x^2 - 6x + 18 = 0$

The set of factors for the constant term 18 are $(\pm 1, \pm 18), (\pm 2, \pm 9)(\pm 3, \pm 6)$. Select a combination which explains the given equation

$\therefore E = (x^2 + \alpha x + 3)\ (x^2 + \beta x + 6) = 0$

x Co–eff : $6\alpha + 3\beta = -6$

x^3 Co–eff : $\alpha + \beta = 1 \Rightarrow 3\alpha + 3\beta = 3$

$3\alpha = -9;\quad \alpha = -3,\quad \beta = 4$

Verify x^2 Co–eff : $6 + 3 - 12 = -3$

Given equation $E = (x^2 - 3x + 3)\ (x^2 + 4x + 6) = 0$

Verification by Gunita Samuccayah Sutram $S_c = 11 = 1\ (1 + 4 + 6) = 11$

The two quadratic equations can be solved.

$x^2 - 3x + 3 = 0$ $\quad$ $x^2 + 4x + 6 = 0$

$D_1 = 2x - 3 = \pm\sqrt{9 - 12}$ $\quad$ $D_1 = 2x + 4 = \pm\sqrt{16 - 24}$

$x = \frac{3 + \sqrt{3}i}{2}, \frac{3 - \sqrt{3}i}{2}$ $\quad$ $x = -2 \pm \sqrt{2}i$

$$E = \left(x - \left(\frac{3 + \sqrt{3}i}{2}\right)\right)\left(x - \left(\frac{3 - \sqrt{3}i}{2}\right)\right)\left[x + \left(2 + \sqrt{2}i\right)\right]\left[x + \left(2 - \sqrt{2}i\right)\right]$$

$S_c = 11 = \left(1 - \left(\frac{3 + \sqrt{3}i}{2}\right)\right)\left(1 - \left(\frac{3 - \sqrt{3}i}{2}\right)\right) = 1 \times 11$ Verified by Gunita

Samuccayah Sutram.

6. $E = x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$

$S_c = 20$

The set of factors for the constant term 36 are $(\pm 1, \pm 36)\ (\pm 2, \pm 18)\ (\pm 3, \pm 12)\ (\pm 4, \pm 9)$. Select a combination which explains the given equation

$E = (x^2 + \alpha x + 4)\ (x^2 + \beta x + 9) = 0$

x Co–eff : $4\beta + 9\alpha = -24$

x^3 Co–eff : $\alpha + \beta = -6$

By Sunyamanyat $\alpha = 0$ as $\frac{4\beta}{\beta} = \frac{-24}{-6}$

$\beta = -6$

Verify x^2 coeff: $4 + 9 + \alpha\beta = 4 + 9 + 0 = 13$

Given equation $E = (x^2 + 4)\ (x^2 - 6x + 9) = 0$

Verification by Gunita Samuccayah Sutram.

$S_c = 20 = (1 + 4)\ (1 - 6 + 9) = 20$.

The two Quadratic Equation can be solved.

$x^2 + 4 = 0$ $\quad$ $x^2 - 6x + 9 = 0$

$\Rightarrow x = \pm 2i$ $\quad$ $x = 3, 3$

$E = (x - 2i)\ (x + 2i)\ (x - 3)^2 = 0$

$S_c = 20 = (1 - 2i)\ (1 + 2i)\ (1 - 3)^2 = 5 \times 4 = 20$

Verified by Gunita Samuccayah Sutram.

7. $E = 16x^4 - 24x^2 - 16x - 3 = 0$

$S_c = -27$

The set of factors for the constant term 3 are $(\pm 1, \pm 3)$. Select a combination which satisfies the equation.

$E = (4x^2 + \alpha x - 3)\ (4x^2 + \beta x + 1) = 0$

x Co–eff : $\alpha - 3\beta = -16$

x^3 Co–eff : $4\beta + 4\alpha = 0$ $\alpha = -4;$ $\beta = 4$

Verify x^2 Co–eff : $4 - 12 + \alpha\beta = -24$

Given equation $E = (4x^2 - 4x - 3)(4x^2 + 4x + 1) = 0$

Verification by Gunita Samuccayah Sutram $S_c = -27 = (4 - 4 - 3)(4 + 4 + 1) = -27$

$4x^2 - 4x - 3 = 0$ $4x^2 + 4x + 1 = 0$

$D_1 = 8x - 4 = \pm\sqrt{16+48}$ $(2x+1)^2 = 0$

$x = \frac{3}{2}, -\frac{1}{2}$ $x = -\frac{1}{2}, -\frac{1}{2}$

$$E = \left(x + \frac{1}{2}\right)\left(x - \frac{3}{2}\right)\left(x + \frac{1}{2}\right)^2 = 0$$

$E = (2x + 1)(2x - 3)(2x + 1)^2 = 0 = (2x - 3)(2x + 1)^3$

$S_c = -27 = 3(-1)(3)(3) = -27$

Verified by Gunita Samuccayah Sutram.

8. $E = x^4 - 31x^2 + 42x + 72 = 0$

$S_c = 84$

The Set of factors for the constant term 72 are (± 1, ± 72), (± 2, ± 36), (± 3, ± 24), (± 4, ± 18), (± 6, ± 12), (± 8, ± 9).Select a combination which explains the given equation.

$E = (x^2 + \alpha x + 6)(x^2 + \beta x + 12) = 0$

x Co–eff : $12\alpha + 6\beta = 42$

x^3 Co–eff : $\alpha + \beta = 0$ $\Rightarrow \alpha = -\beta$

$\alpha = 7,$ $\beta = -7$

Verify x^2 Co–eff : $12 + 6 + \alpha\beta = 12 + 6 - 49 = -31$

Given equation $E = (x^2 + 7x + 6)(x^2 - 7x + 12) = 0$

Verification by Gunita Samuccayah Sutram

$S_c = 84 = (1 + 7 + 6)(1 - 7 + 12) = 84$

The two Quadratic Equations can be solved.

$x^2 + 7x + 6 = 0$ $x^2 - 7x + 12 = 0$

$x = -1, -6$ $x = 3, 4$

$E = (x + 1)(x + 6)(x - 3)(x - 4) = 0$

$S_c = 84 = 2 \times 7 \times (-2)(-3) = 84$. Verified by Gunita Samucayah Sutram.

9. $E = x^4 - 3x^2 - 6x - 2 = 0$

$S_c = -10$

The factors for the constant term 2 are (± 1, ± 2)

$E = (x^2 + \alpha x + 2)(x^2 + \beta x - 1) = 0$

x Co–eff : $-\alpha + 2\beta = -6$

x^3 Co–eff : $\alpha + \beta = 0$ $\Rightarrow \alpha = -\beta$

$\alpha = 2;$ $\beta = -2$

Verify x^2 Co–eff : $2 - 1 + \alpha\beta = 2 - 1 - 4 = -3$

Given equation $E = (x^2 + 2x + 2)(x^2 - 2x - 1) = 0$

Verification by Gunita Samuccayah Sutra $S_c = -10 = (1+2+2)(1-2-1) = -10$

The two Quadratic Equations can be solved by differential method.

$x^2 + 2x + 2 = 0$ $\qquad$ $x^2 - 2x - 1 = 0$

$2x + 2 = \pm\sqrt{4-8}$ $\qquad$ $2x - 2 = \pm\sqrt{4+4}$

$x = -1 \pm i$ $\qquad$ $x = 1 \pm \sqrt{2}$

$E = (x+1+i)(x+1-i)(x-1+\sqrt{2})(x-1-\sqrt{2}) = 0$

$S_c = -10 = (2+i)(2-i)(\sqrt{2})(-\sqrt{2}) = (4+1)(-2) = -10$

Verified by Gunita Samuccayah Sutram.

Swamiji's work using Purana Method are solved using Argumentation Method

1) $E = x^4 + 16x^3 + 86x^2 + 176x + 105 = 0$

A set of factors for 105 which satisfy the x^3, x^2, x coefficients are to be finally selected

Let $E = (x^2 + \alpha x + 3)(x^2 + \beta x + 35)$

Equating the coefficients of like terms

Coeff of x^3: $\alpha + \beta = 16$ ——— (1)

Coeff of x: $35\alpha + 3\beta = 176$

From (1) $\underline{3\alpha + 3\beta = 48}$

Subtracting $32\alpha = 128$

$\therefore \alpha = 4 \Rightarrow \beta = 12$

For these values x^2 coefficient is satisfied

$\therefore E = (x^2 + 4x + 3)(x^2 + 12x + 35) = 0$

$x^2 + 4x + 3 = 0$

By differential method.

$\therefore D_1 = 2x + 4 = \pm\sqrt{16-12}$ $\qquad$ $\therefore x = -1, -3$

$x^2 + 12x + 35 = 0$

$\therefore 2x + 12 = \pm\sqrt{144-140}$ $\qquad$ $\therefore x = -5, -7$

$x = -1, -3, -5$ and -7

$\therefore E = (x+1)(x+3)(x+5)(x+7) = 0$

This is verified by Gunita Samuccayah.

$S_c = 1 + 16 + 86 + 176 + 105 = (1+1)(1+3)(1+5)(1+7) = 384$

2) $E = x^4 - 16x^3 + 91x^2 - 216x + 180 = 0$

A set of factors for 180 which satisfy the x^3, x^2 and x coefficients are to be finally selected

Let $E = (x^2 + \alpha x + 10)(x^2 + \beta x + 18) = 0$

Equating the coefficients of like terms

Coeff of x^3: $\alpha + \beta = -16$ ——— (1)

Coeff of x: $18\alpha + 10\beta = -216$

From (1) $\underline{18\alpha + 18\beta = -288}$

Subtracting $-8\beta = 72$

$\therefore \beta = -9$ $\qquad$ $\alpha = -7$

For these values of α and β, the coefficient of x^2 is satisfied

$\therefore$ $E = (x^2 - 7x + 10)(x^2 - 9x + 18) = 0$

$x^2 - 7x + 10 = 0$

By differential method.

$\therefore$ $D_1 = 2x - 7 = \pm\sqrt{49 - 40}$

$2x = 7 \pm 3$ $\qquad\qquad \therefore$ $x = 5, 2$

$x^2 - 9x + 18 = 0$

$D_1 = 2x - 9 = = \pm\sqrt{81 - 72}$

$\therefore$ $2x = 9 \pm 3$ $\qquad\qquad \therefore$ $x = 6, 3$

$\therefore$ $x = 2, 3$ 5 and 6

$\therefore$ $E = (x - 2)(x - 3)(x - 5)(x - 6) = 0$

This is verified by Gunita Samuccayah.

$S_c = 1 - 16 + 91 - 216 + 180 = (1 - 2)(1 - 3)(1 - 5)(1 - 6) = 40$

3) $E = x^4 - 20x^3 + 137x^2 - 382x + 360 = 0$

A set of factors for 360 which satisfy the coefficients of x^3, x^2 and x are to be finally selected.

Let $E = (x^2 + \alpha x + 18)(x^2 + \beta x + 20) = 0$

Equating the coefficients of like terms

Coeff of x^3: $\alpha + \beta = -20$ ——— (1)

Coeff of x: $20\alpha + 18\beta = -382$

From (1) $\underline{20\alpha + 20\beta = -400}$

Subtracting $-2\beta = 18$

$\therefore$ $\beta = -9 \quad \Rightarrow \quad \alpha = -11$

For these values, the coefficient of x^2 is satisfied

$\therefore$ $E = (x^2 - 11\alpha + 18)(x^2 - 9x + 20) = 0$

$x^2 - 11\alpha + 18 = 0$

$\therefore$ $D_1 = 2x - 11 = \pm\sqrt{121 - 72}$

$2x = 11 \pm 7$

$\therefore$ $x = 9, 2$

$x^2 - 9x + 20 = 0$

$D_1 = 2x - 9 = = \pm\sqrt{81 - 80}$

$\therefore$ $2x = 9 \pm 1$ $\qquad\qquad \therefore$ $x = 5, 4$

$\therefore$ $x = 2, 4, 5, 9$

$\therefore$ $E = (x - 2)(x - 4)(x - 5)(x - 9) = 0$

This is verified by Gunita Samuccayah.

$S_c = 1 - 20 + 137 - 382 + 360 = (1 - 2)(1 - 4)(1 - 5)(1 - 9) = 96$

(Ref. Swamiji's book for Purana Method of these Bi–Quadratics)

Note : Purana Apuranabhyam method, though a bit hard in comparison with the Argumentation method, is helpful in finding out the repeated roots.

The two following problems can be taken for examples.

1) $E = x^3 - 3x^2 + 4 = 0$
By Paravartya in the given equation is $x^3 - 3x^2 = -4$
Let us consider the standard cube in which the first two terms of E occur as they are $(x-1)^3 = x^3 - 3x^2 + 3x - 1$. substitute for $x^3 - 3x^2$ in the standard equation, its value as -4
$\therefore\ (x-1)^3 = -4 + 3x - 1$
$\therefore\ (x-1)^3 = 3x - 5$
Let $(x-1)$ be y
$\Rightarrow y^3 = 3(y+1) - 5$
$y^3 - 3y + 2 = 0$
By Vilokanam as $S_c = 0$, $(y-1)$ is a factor
$\therefore\ (y^3 - 3y + 2) = (y-1)A$. A should have y^2, y and constant terms
$(y^3 - 3y + 2) = (y-1)(y^2 + \alpha y - 2)$. By comparing y coefficient $-2 - \alpha = -3 \Rightarrow \alpha = 1$
$\therefore\ (y^3 - 3y + 2) = (y-1)(y^2 + y - 2)$. By Adyamadyena Antyamantyena
$y^2 + y - 2 = y^2 + 2y - y - 2 = (y+2)(y-1)$
$(y^3 - 3y + 2) = (y-1)^2 (y+2)$
$\therefore\ E = (x-2)^2 (x+1)$
$(x-2)$ is once repeated factor.

2) $E = x^6 + 12x^5 + 57x^4 + 136x^3 + 171x^2 + 108x + 27 = 0$
By paravartya, the given equation is
$x^6 + 12x^5 = -57x^4 - 136x^3 - 171x^2 - 108x - 27$
Consider the standard Sixth degree equation in which the first two terms of E are the same as in E.
$(x+2)^6 = x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$
Substitute for $x^6 + 12x^5$ form the given equation, the expression
$(-57x^4 - 136x^3 - 171x^2 - 108x - 27)$
$\therefore\ (x+2)^6 = -57x^4 - 136x^3 - 171x^{-2} - 108x - 27 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$
$(x+2)^6 = 3x^4 + 24x^3 + 69x^2 + 84x + 37$
Let $(x+2)$ be $y \Rightarrow x = y - 2$
$y^6 = 3(y-2)^4 + 24(y-2)^3 + 69(y-2)^2 + 84(y-2) + 37$
$= 3(y^4 - 8y^3 + 24y^2 - 32y + 16) + 24(y^3 - 6y^2 + 12y - 8) + 69(y^2 - 4y + 4) + 84(y-2)$
$= 3y^4 - 24y^3 + 72y^2 - 96y + 48 + 24y^3 - 144y^2 + 288y - 192 + 69y^2 - 276y + 276 + 84y - 168 + 37$
$\Rightarrow y^6 - 3y^4 + 3y^2 - 1 = 0$
$= (y^2 - 1)^3 = 0$
$[(y+1)(y-1)]^3 = [(x+3)(x+1)]^3$
$(x+3)$ and $(x+1)$ are thrice repeated factors

Refer example 8 for the repeated factors using successive Differentiation Section $\div$ 9

SECTION – 11

A FEW ADDITIONAL PROBLEMS

1) $E = 4x^3 - 24x^2 + 23x + 18 = 0$

1st step Vilokanam is applied

The sum of the coefficients $S_c \neq 0$

Sum of the coefficients of even powers $S_e \neq$ Sum of the coefficients of odd powers, S_0

$(x - 1)$ & $(x + 1)$ are not factors.

But a trial with $x = 2$ Satisfies the given equation

$(x - 2)$ is a factor.

$\therefore$ $E = (x - 2)A$ where A should contain x^2, x and constant terms

The 1st and the last terms of A can be got by using Adyamadyena. They are $4x^2$, -9

$E = (x - 2)(4x^2 + \alpha x - 9)$, α can be determined.

By Gunita Samuccayah Sutram

$S_c = 21 = (1 - 2)(4 + \alpha - 9)$

$21 = -1(\alpha - 5)$

$(\alpha - 5) = -21$

$\alpha = -16$

By Argumentation i.e. by comparing the like terms on both sides.

$-9 - 2\alpha = 23$

$2\alpha = -32 \Rightarrow \alpha = -16$

This is verified by applying Gunita Samuccayah Sutram. As $S_c = 21 = -1$ $(4 - 16 - 9) = 21$

The given equation

$$4x^3 - 24x^2 + 23x + 18 = (x - 2)(4x^2 - 16x - 9)$$

The second factor which is Quadratic $(4x^2 - 16x - 9)$ can be further factorised by using the relation between the first differential and discriminant.

$$D_1 = \pm\sqrt{Discriminant}$$

$$8x - 16 = \pm\sqrt{256 + 144} \qquad \therefore\ x = \frac{9}{2}, \frac{-1}{2}$$

The given equation can be factorized as $(x - 2)\left(x - \frac{9}{2}\right)\left(x + \frac{1}{2}\right) = 0$

$= (x - 2)(2x - 9)(2x + 1) = 0$

2) $E = 32x^3 + 80x^2 - 122x - 35 = 0$

First step by Vilokanam

$S_c \neq 0 \Rightarrow (x - 1)$ is not a factor

$S_0 \neq S_e \Rightarrow (x + 1)$ is not a factor

Let $f(x) = 32x^3 + 80x^2 - 122x = 35$

	LHS f(x)	RHS = 35	RHS – LHS
x = 1	– 10		+ 45
x = – 1	170		– 135

By this trial one can expect that the roots lie between 1 and – 1
As a finer step one can try with 0.75, 0.5, 0.25, – 0.25, – 0.5, – 0.75

	LHS f(x)	RHS = 35	RHS – LHS
x = 0.75	– 33		+ 68
x = 0.5	– 37		72
x = 0.25	– 25		60
x = – 0.25	35		0
x = – 0.5	77		– 42

$\therefore \left(x+\frac{1}{4}\right)$ is one factor

To find out the other two factors, applying Adyamadyena, we can write down the given equation as $\left(x+\frac{1}{4}\right)$A. Where A should contain x^2, x and constant terms.

Applying Adyamadyena Antyamantyena / Argumentation the first and last terms of A are $32x^2$, – 140.

$\therefore$ Given equation $\left(x+\frac{1}{4}\right)(32x^2 + \alpha x - 140)$

α is to be determined by applying Gunita Samuccayah Sutram.

$S_c = 32 + 80 - 122 - 35$

$= \left(1+\frac{1}{4}\right)(32 + \alpha - 140)$

$-45 = \frac{5}{4}(\alpha - 108)$

$\therefore \alpha = 72$

The given equation is factorised is

$\left(x+\frac{1}{4}\right)(32x^2 + 72x - 140)$

α can also be determined by applying Argumentation, comparing like terms on both sides.

$\frac{\alpha}{4} - 140 = -122$

$\alpha = 72$

$\therefore\ 32x^3 + 80x^2 - 122x - 35$

$\left(x+\frac{1}{4}\right)(32x^2 + 72x - 140)$

This is also verified by Gunita Samuccayah

i.e. $S_c = -45 = \frac{5}{4}(32 + 72 - 140)$

$= -45$

The remaining Quadratic expression can be further factorized by applying the relation between first differential and discriminant.

$D_1 = \pm\sqrt{Discriminant}$

$64x + 72 = \pm\sqrt{5184 + 17920}$

$x = \frac{5}{4}, -\frac{7}{2}$

$\therefore$ The given equation E is factorised as $\left(x+\frac{1}{4}\right)\left(x-\frac{5}{4}\right)\left(x+\frac{7}{2}\right)$ or (4x + 1) (4x – 5) (2x + 7)

This is verified by Gunita Samuccayah

$S_c = -45 = (5)(-1)(9) = -45$

3) $E = 20x^3 + 41x^2 + 33x + 6 = 0$

First step by Vilokanam

$S_c \neq 0 \Rightarrow (x-1)$ is not a factor

$S_0 \neq S_e \Rightarrow (x+1)$ is not a factor

Let $f(x) = 20x^3 + 41x^2 + 33x = -6$

	LHS f(x)	RHS	RHS – LHS
x = 1	94	– 6	– 100
x = – 1	– 12	– 6	+ 6

By this trial one can expect that the roots lie between 1 and – 1

As a finer step one can try with 0.75, 0.5, 0.25, – 0.25, – 0.5, – 0.75

x = 0.75	56.25	– 62.25
x = 0.5	29.25	– 35.25
x = 0.25	11.125	– 17.125
x = – 0.25	– 6	0
x = – 0.5	– 8.75	2.75

$\left(x+\frac{1}{4}\right)$. is a factor of the given equation. the remaining factors of the given cubic equation can be obtained by applying argumentation or Gunita Samuccayah Sutram. By applying Argumentation one can write down the given equation as $20x^3 + 41x^2 + 33x + 6 = \left(x+\frac{1}{4}\right)A$. Where A should contain x^2, x and constant terms. The first and the last terms of A can be obtained by Adyamadyena as $20x^2$ and 24.

$E = \left(x+\frac{1}{4}\right)(20x^2 + \alpha x + 24)$

The value of α can be determined by applying Gunita Samuccayah or by Argumentation.

By Gunita Samuccayah

$S_c = 20 + 41 + 33 + 6 = 100$

$= \left(1+\frac{1}{4}\right)(20 + \alpha + 24)$

$100 = \frac{5}{4}(44 + \alpha)$

$\alpha = 36$.

α can be determined by Argumentation i.e. by comparing the like terms on both sides.

$24 + \frac{\alpha}{4} = 33$

$\alpha = 36$

$\therefore\ 20x^3 + 41x^2 + 33x + 6$

$= \left(x+\frac{1}{4}\right)(20x^2 + 36x + 24)$

This is also verified by Gunita Samuccayah

$S_c = 100 = \frac{5}{4}(80) = 100$

$\therefore\ E = \left(x+\frac{1}{4}\right)(20x^2 + 36x + 24) = 0$

The remaining Quadratic expression $A = (20x^2 + 36x + 24)$ can be further factorized by applying the relation between first differential and discriminant.

$40x + 36 = \pm\sqrt{1296 - 1920} = \pm\sqrt{-624}$

$\therefore\ x = -\frac{9}{10} \pm \frac{\sqrt{39}i}{10}$

$\therefore\ \left(x+\frac{1}{4}\right)\left[x-\left(-\frac{9}{10}+\frac{\sqrt{39}i}{10}\right)\right]\left[x-\left(-\frac{9}{10}-\frac{\sqrt{39}i}{10}\right)\right]$ are the factors of the given equation.

4) $E = x^3 + 5x^2 - 4x - 20 = 0$

1^{st} step by Vilokanam

$S_c \neq 0$ $(x - 1)$ is not a factor

$S_0 = S_e$ $(x + 1)$ is not a factor

But a trial in Succession, $x = 2$ satisfies the equation

$\therefore\ (x - 2)$ is a factor

The remaining factor can be obtained by Argumentation and by Adyamadyena.

The given equation can be factorized as $(x - 2)$ A where A should contain x^2, x and constant terms. The first and the last terms are got by applying Adyamadyena. They are x^2, 10

$\therefore\ E = (x - 2)(x^2 + \alpha x + 10)$

The coefficient of x can be determined by applying Gunita Samuccayah Sutram.

By Gunita Samuccayah Sutram

$S_c = (1-2)(1+\alpha+10) = -18$

$\therefore \alpha = 7$

By Argumentation also, α can be determined. i.e. by comparing the like terms on both sides of E and $(x-2)(x^2+\alpha x+10)$

$-2\alpha + 10 = -4 \Rightarrow \alpha = 7$

$\therefore (x^3+5x^2-4x-20) = (x-2)(x^2+7x+10)$

The Quadratic expression can be further factorized by using Adyamadyena Sutram or Differentiation method

$D_1 = 2x+7 = \pm\sqrt{49-40}$ $\therefore x = -2, -5$

The given equation is factorized as $(x-2)(x+2)(x+5) = 0$

This is verified by Gunita Samuccayah Sutram

$S_c = -18 = (1-2)(1+2)(1+6) = -18$

5) $E = x^3 - 27x + 54 = 0$

1^{st} step by Vilokanam

$S_c \neq 0$ $(x-1)$ is not a factor

$S_0 \neq S_e$ $(x+1)$ is not a factor

But a trial in Succession, with x = 2, 3 etc is carried out. x = 3 satisfies the given equation $(x-3)$ is a factor.

Let the remaining expression be A.

$E = x^3 - 27x + 54 = (x-3)A$. A should have x^2, x and constant terms

By Adyamadyena the first and last terms of A are x^2 and -18

$(x^3-27x+54) = (x-3)(x^2+\alpha x-18)$

Applying Gunita Samuccayah Sutram,

$S_c = 28 = -2(-17+\alpha)$ $\alpha = 3$

$E = x^3-27x+54 = (x-3)(x^2+3x-18)$

α can also be determined by applying argumentation i.e. comparing the like terms on both sides of E and $(x-3)(x^2+\alpha x-18)$

$-18-3\alpha = -27 \Rightarrow \alpha = 3$

$\therefore (x^3-27x+54) = (x-3)(x^2+3x-18)$

This is verified by applying Gunita Samuccayah Sutra

$S_c = 28 = -2(1+3-18) = 28$

The Quadratic equation $x^2 + 3x - 18 = 0$ is solved by using Differential Method $D_1 = \pm\sqrt{Discriminant}$

$D_1 = 2x+3 = \pm\sqrt{9+72}$

$x = 3, -6$

The equation is factorised as $(x-3)(x-3)(x+6)$

This is verified by Gunita Samuccayah as $S_c = 28 = (-2)(-2)(7) = 28$

Note : The factor (x – 3) repeats once. This can be also obtained by Successive Differentiation.

6) $E = x^3 + 6x^2 + 11x + 6 = 0$

1st step by Vilokanam

$S_c \neq 0$ (x – 1) is not a factor

$S_0 = S_e$ (x + 1) is a factor

The other two roots can be obtained by solving the remaining expression A. Applying Adyamadyena Sutram followed by Gunita Samuccayah Sutram.

$E = (x + 1) A.$ A should have x^2, x and constant terms

By Adyamadyena we get the first and last terms of A as x^2 and 6.

The given equation can be written as $(x + 1)(x^2 + \alpha x + 6)$ where α is the coefficient of x and can be determined by applying Gunita Samuccayah Sutram. By Gunita Samuccayah

$S_c = 24 = (1 + 1)(1 + \alpha + 6) = 2(7 + \alpha)$

$\alpha = 5$

A = the Quadratic expression is $(x^2 + 5x + 6)$.

α can also be determined by applying Argumentation i.e. by comparing the like terms on both sides of E $6 + \alpha = 11$ $\alpha = 5$

$x^3 + 6x^2 + 11x + 6 = (x + 1)(x^2 + 5x + 6)$

This can be further factorized by applying Adyamadyena Sutram (or) by applying the relation between first differential & Discriminant.

This is verified by Gunita Samuccayah $S_c = 24 = (2)(1 + 5 + 6) = 24$

Adyamadyena

For the expression $x^2 + 5x + 6$, the middle term can be split in two terms and Adyamadyena Gives the factorisation.

$\Rightarrow x^2 + 3x + 2x + 6 = (x + 2)(x + 6)$ (or)

$D_1 = \pm\sqrt{Discriminant}$

$2x + 5 = \pm\sqrt{25 - 24}$; $x = -3, -2$

(x + 2) (x + 3) are the two factors

The given equation can be factorized as (x + 1) (x + 2) (x + 3) = 0

$S_c = (24) = (2)(3)(4) = 24$

7) $x^3 + 21x^2 + 11x - 33 = 0$

1st step by Vilokanam ; The sum of the coefficients $S_c = 1 + 21 + 11 - 33 = 0$

(x – 1) is a factor.

$S_0 \neq S_e$ ∴ (x + 1) is not a factor.

The remaining roots of the given cubic equation can be obtained by applying Adyamadyena followed by Argumentation.

The given cubic equation can be factorized as $E = x^3 + 21x^2 + 11x - 33$ $(x - 1)$ A where A should have x^2, x and constant terms.

Applying Adyamadyena, the first and last term of the remaining expression A can be obtained as $\frac{x^3}{x} = x^2$ the first term and $\frac{-33}{-1} = 33$ as the last (constant) term.

$x^3 + 21x^2 + 11x - 33 = (x - 1)(x^2 + \alpha x + 33)$; $\quad A = (x^2 + \alpha x + 33)$

The value of α can be determined by Argumentation by comparing the x coefficients on both sides as follows.

$33x - \alpha x = 11x \qquad \alpha = 22$

A, the remaining Quadratic expression is $(x^2 + 22x + 33)$. This can be further factorized by applying the relation between first differential and discriminant.

$\Rightarrow D_1 = \pm\sqrt{Discriminant}$

$2x + 22 = \pm\sqrt{484 - 132} = \pm 4\sqrt{22}$; $\qquad x = -11 \pm 2\sqrt{22}$

$\therefore$ $(x - 1)(x + 11 + 2\sqrt{22})(x + 11 - 2\sqrt{22})$ are the factors of the given cubic equation.

8) $E = x^3 - 7x + 6 = 0$

By Vilokanam as $S_c = 0$, $(x - 1)$ is a factor.

$S_c \neq S_0$, $(x + 1)$ is not a factor.

$\therefore$ $E = (x - 1)$ A. A should have x^2, x and constant terms

Applying Adyamadyena, the first and last terms of A are x^2 and -6.

$\therefore$ $E = (x - 1)(x^2 + \alpha x - 6)$

By comparing the x coefficient on both sides

$-6 - \alpha = -7$

$-\alpha = -1 \Rightarrow \alpha = 1$

Verification by Gunita Samuccayah Sutram

$S_c = 0 = (1 - 1)(1 + 1 - 6) = 0$

$\therefore$ $E = (x - 1)(x^2 + x - 6)$

The quadratic expression A, $(x^2 + x - 6)$ can be solved by using Differential $D_1 = \pm\sqrt{Discriminant}$

$2x + 1 = \pm\sqrt{1 + 24}$

$2x + 1 = \pm 5 \Rightarrow x = -3, 2$

$\therefore$ $E = (x - 1)(x - 2)(x + 3) = 0$

$S_c = 0 = 0(-1)(4) = 0$

Verified by Gunita Samuccayah.

9) $E = x^3 + 3x^2 - 54x - 112 = 0$

By Vilokanam as $S_c \neq 0$, $(x - 1)$ is not a factor.

$S_o \neq S_e$, $(x + 1)$ is not a factor.

But a trial in Succession, $x = -2$ satisfies the equation, E.

$\therefore$ $(x + 2)$ is a factor of E.

$\therefore$ $E = (x + 2)A$. A should have x^2, x and constant terms.

Applying Adyamadyena, the first and last terms of A are x^2 and -56.

$\therefore$ $E\ (x + 2)\ (x^2 + \alpha x - 56)$

By comparing the x coefficient on both sides.

$-56 + 2\alpha = -54$ $\therefore \alpha = 1$

α can be determined by Gunita Samuccayah Sutram also as $S_c = -162 = (1 + 2)$
$(1 + \alpha - 56) \Rightarrow \alpha = 1$

Verification by Gunita Samuccayah Sutram.

$S_c = -162 = (1 + 2)(1 + 1 - 56) = -162$

$\therefore$ $E = (x + 2)(x^2 + x - 56) = 0$

The quadratic expression A, $(x^2 + x - 56)$ can be solved by using first Differential

$D_1 = \pm\sqrt{Discriminant}$

$\Rightarrow 2x + 1 = \pm\sqrt{1 + 224}$

$2x + 1 = \pm\sqrt{225} \Rightarrow 2x + 1 = \pm 15 \Rightarrow x = -8, 7$

$\therefore$ $E = (x + 2)(x + 8)(x - 7) = 0$

This is verified by Gunita Samuccayah as $S_c = -162 = (3)(9)(-6) = -162$

10) $E = x^4 - 27x^2 + 14x + 120 = 0$

By Vilokanam as $S_c \neq 0$, $(x - 1)$ is not a factor.

$S_o \neq S_e$, $(x + 1)$ is not a factor.

By Trial and Succession,

Let $f(x) = x^4 - 27x^2 + 14x = -120$

x value	L.H.S f(x)	R.H.S −120	R.H.S − L.H.S
1	−12		−108
2	−64		−56
3	−120		0
4	−120		0

By this trial one can obtain $(x - 3)$ and $(x - 4)$ are two factors of E.

$\therefore$ $E = (x - 3)(x - 4)A$
$= (x^2 - 7x + 12)A$. A should have x^2, x and constant terms.

Applying Adyamadyena, the first and last terms of A are x^2 and 10.

$\therefore$ $E\ (x^2 - 7x + 12)(x^2 + \alpha x + 10)$

By comparing the x coefficient on both sides.

$-70 + 12\alpha = 14 \Rightarrow 12\alpha = 84 \Rightarrow \alpha = 7$

α can be determined by Gunita Samuccayah Sutram also as $S_c = 108 = (1 - 7 + 12)$
$(1 + \alpha + 10) = 6(\alpha + 11) \Rightarrow \alpha = 7$

Verification by Gunita Samuccayah Sutram

$S_c = 108 = (1 - 7 + 12)(1 + 7 + 10)$

$\therefore\ E = (x^2 - 7x + 12)(x^2 + 7x + 10)$

The quadratic expression A, $(x^2 + 7x + 10)$ can be solved by using first Differential $D_1 = \pm\sqrt{Discriminant}$

$2x + 7 = \pm\sqrt{49 - 40}$

$2x + 7 = \pm 3 \Rightarrow x = -5, -2$

$\therefore\ E = (x - 3)(x - 4)(x + 2)(x + 5) = 0$

This is verified by Gunita Samuccayah as $S_c = 108 = (-2)(-3)(3)(6) = 108$

11) $E = x^5 - 10x^4 - 21x^3 + 382x^2 - 568x - 960 = 0$

By Vilokanam as $S_c \neq 0$, $(x - 1)$ is not a factor.

$S_o = S_e$, $(x + 1)$ is a factor.

But by Trial in Succession

Let $f(x) = x^5 - 10x^4 - 21x^3 + 382x^2 - 568x = 960$

x value	L.H.S f(x)	R.H.S 960	R.H.S – L.H.S
1	– 216		1176
2	96		864
3	600		360
4	960		0
5	960		0
6	624		336
7	336		624
8	960		0
9	3960		– 3000
10	11520		– 10560

By this trial one can obtain $(x - 4)(x - 5)(x - 8)$ are three factors of E.

$\therefore\ E = (x + 1)(x - 4)(x - 5)(x - 8)A$

$= (x^2 - 3x - 4)(x^2 - 13x + 40)A$

$= (x^4 - 16x^3 + 75x^2 - 68x - 160)A$. A should have x and constant terms.

By Adyamadyena the x coefficient is 1 and constant terms is 6.

$\therefore\ E = (x^4 - 16x^3 + 75x^2 - 68x - 160)(x + 6)$

$= (x + 1)(x - 4)(x - 5)(x - 8)(x + 6)$

This is verified by Gunita Samuccayah as $S_c = -1176 = (2)(-3)(-4)(-7)(7) = -1176$

12) $E = 2x^5 + x^4 - 12x^3 - 12x^2 + x + 2 = 0$
$f(x) = 2x^5 + x^4 - 12x^3 - 12x^2 + x = -2$

	L.H.S	R.H.S	Diff
$x = -1$	-2	-2	0
$x = -2$	-2	"	0
$x = 1$	-20	"	18

By trial
$(x + 1)$ and $(x + 2)$ are two factors of E.

Hence $E = (x + 1)(x + 2)A = (x^2 + 3x + 2)A$. A should contain x^3, x^2, x and a constant term as. By "Adyamadyena" the first and last terms of A are $2x^3$ and 1.

$\therefore\ A = 2x^3 + \alpha x^2 + \beta x + 1$
By Vilokanam $(x + 1)$ and $(x - 1)$ are not factors of A.
$E = (x^2 + 3x + 2)(2x^3 + \alpha x^2 + \beta x + 1)$

Comparing the coefficients of like terms on both sides.
x coeff : $2\beta + 3 = 1 \quad \therefore\ \beta = -1$
x^2 coeff : $1 + 2\alpha + 3\beta = -12$
$1 + 2\alpha - 3 = -12 \quad \therefore\ \alpha = -5$

Verify x^3 co-eff in E: $-12 = 4 + 3\alpha + \beta = 4 + 3(-5) + (-1) = -12$
Verification by Gunita Samuccayah Sutram $S_c = -18 = (1 + 3 + 2)(2 - 5 - 1 + 1)$
$= -18$
$\therefore\ E = (x^2 + 3x + 2)(2x^3 - 5x^2 - x + 1) = 0$
Consider $(2x^3 - 5x^2 - x + 1)$
By 'Vilokanam'
$2x^3 - 5x^2 - x = -1$

	L.H.S	R.H.S	Diff
		-1	R.H.S – L.H.S
$x = 1$	-4	"	$+3$
$x = -1$	-6	"	$+5$
$x = -2$	-34	"	$+33$
$x = -0.5$	-1	"	0

Since $(x + 0.5)$ is a factor of A
$(x + 0.5)B = A$
B should have x^2, x and a constant term.
By "Adyamadyena". The first and last terms are $2x^2$ and 2.
$B = (2x^2 + \alpha x + 2)$
$\therefore\ A = (x + 0.5)(2x^2 + \alpha x + 2) = 0$
Comparing the coefficients of like terms on both sides.

$0.5\alpha + 2 = -1$

$0.5\alpha = -3 \Rightarrow \alpha = \dfrac{-3}{0.5} = -6$

Verification by Gunita Samuccayah Sutram.

S_c (of A) = $(2-5-1+1) = -3 = (1+0.5)(2-6+2) = -3$

$A = (x+0.5)(2x^2 - 6x + 2) = 0$

$= (2x+1)(x^2 - 3x + 1) = 0$

$x^2 - 3x + 1 = 0$ is a Quadratic equation can be solved by differential relation.

$D_1 = \pm\sqrt{Discriminant} = 2x - 3 = \pm\sqrt{9-4}$

$\therefore\ x = \dfrac{3}{2} \pm \dfrac{\sqrt{5}}{2}$

$$E = (x+1)(x+2)(2x+1)\left(x - \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\left(x - \frac{3}{2} + \frac{\sqrt{5}}{2}\right) = 0$$

$$S_c = -18 = (2)(3)(3)\left(1 - \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\left(1 - \frac{3}{2} + \frac{\sqrt{5}}{2}\right) = -18$$

Verified by Gunita Samuccayah Sutram.

13) $E = x^3 + x^2 - 7x - 15 = 0$

$S_c = -20$

By trial 1 and – 1 are not factors

But x = 3 satisfies the equation

$\therefore$ (x – 3) is a factor

E = (x – 3)A, A should contain x^2,x and constant terms

By Adyamadyena, the first and last terms are x^2, 5

$E = (x-3)(x^2 + \alpha x + 5)$

Comparing the x co – efficient on both sides

$-3\alpha + 5 = -7$

$\alpha = 4$

$A = x^2 + 4x + 5$

This is verified by Gunita Samuccayah

$S_c = -20 = (1-3)(1+4+5) = -20$

$\therefore\ E\ (x-3)(x^2 + 4x + 5) = 0$

A can be solved by differential relation

$2x + 4 = \pm\sqrt{16-20}$

$x = -2 \pm i$

$\therefore\ E = (x-3)(x+2+i)(x+2-i) = 0$

$S_c = -20 = (-2)(3+i)(3-i) = -20$

Verified by Gunita Samuccayah Sutram

14) $x^3 + 63x - 316 = 0$

Let $f(x) = x^3 + 63x = 316$

By trial in Succession

	L.H.S	R.H.S	Diff R.HS. – L.H.S
x = 1	64	316	252
x = 2	134	"	182
x = 3	216	"	100
x = 4	316	"	0

$\therefore$ $x - 4$ is one factor

$\therefore$ $E = (x - 4)$ A. A should contain x^2, x and constant term.

By Adyamadyena, the first and last terms are x^2 and 79.

$\therefore$ $A = x^2 + \alpha x + 79$

$E = (x - 4)(x^2 + \alpha x + 79) = 0$

By comparing the x co-efficients on both sides.

$-4\alpha + 79 = 63$ $\therefore$ $\alpha = 4$

$E = (x - 4)(x^2 + 4x + 79) = 0$

Verification by Gunita Samuccayah Sutram

$S_c = -252 = (1 - 4)(1 + 4 + 79) = -252$

Solving the quadratic equation A by Differential Method as.

$2x + 4 = \pm\sqrt{16 - 316}$

$= \pm 10\sqrt{3}i$

$x = -2 \pm 5\sqrt{3}i$

$\therefore$ $E = (x - 4)(x + 2 + 5\sqrt{3}i)(x + 2 - 5\sqrt{3}i) = 0$

$S_c = -252 = (1 - 4)(1 + 2 + 5\sqrt{3}i)(1 + 2 - 5\sqrt{3}i) = (-3)(9 + 75) = -252$

Verified by Gunita Samuccayah Sutram.

15) $E = x^3 + 10x^2 + 29x + 20 = 0$

$S_c = 60$

By Vilokanam

$S_c \neq 0$ $\therefore$ $x - 1$ is not a factor

$S_0 = S_e = 30$ $\therefore$ $x + 1$ is a factor

$E = (x + 1)$ A, A should contain x^2, x and constant term.

By Adyamadyena the first and last terms of A are x^2, 20

$\therefore$ $A = x^2 + \alpha x + 20$

Comparing the x coeff on both sides of E.

$20 + \alpha = 29$

$\alpha = 9$

Verification by Gunita Samuccayah Sutram.

$S_c = 60 = (1 + 1)(1 + 9 + 20) = 60$

$\therefore$ $E = (x + 1)(x^2 + 9x + 20) = 0$

A can be solved by differential method as

$D_1 = \pm\sqrt{Discriminant}$

$2x + 9 = \pm\sqrt{81 - 80} = \pm 1$

$x = -4, -5$

$\therefore\ E = (x + 1)(x + 4)(x + 5) = 0$

$S_c = 60 = (1 + 1)(1 + 4)(1 + 5) = 60$

Verified by Gunita Samuccayah Sutram.

16) $E = x^3 - 15x - 126 = 0$

	L.H.S	R.H.S	Diff
x = 1	– 14	126	140
x = 2	– 22	"	148
x = 3	– 18	"	144
x = 4	4	"	122
x = 5	50	"	76
x = 6	126	"	0
x = 7	238	"	– 112

$\therefore$ (x – 6) is one factor of E

E = (x – 6) A where A should have x^2, x and constant terms

By Adyamadyena the first and last terms of A are x^2 and 21

$\therefore\ E = (x - 6)(x^2 + \alpha x + 21) = 0$

Comparing the x coeff on both sides

$-6\alpha + 21 = -15$

$\alpha = 6$

$E = (x - 6)(x^2 + 6x + 21) = 0$

Verification by Gunita Samuccayah Sutram.

$S_c = -140 = (1 - 6)(1 + 6 + 21) = -140$

$A = x^2 + 6x + 21 = 0$ can be solved by Differential Method as.

$D_1 = \pm\sqrt{Discriminant}$

$2x + 6 = \pm\sqrt{36 - 84}$

$= \pm\sqrt{-48}$

$x = -3 \pm 2\sqrt{3}i$

$\therefore\ E = (x - 6)(x + 3 - 2\sqrt{3}i)(x + 3 + 2\sqrt{3}i)$

$S_c = -140 = (1 - 6)(1 + 3 - 2\sqrt{3}i)(1 + 3 + 2\sqrt{3}i) = (-5)(16 + 12) = -140$

Verified by Gunita Samuccayah Sutram.

17) $E = x^3 + 2x^2 - 21x + 18 = 0$

$S_c = 0$ $\therefore$ (x – 1) is a factor

E = (x – 1) A, A should contain x^2, x and constant terms

By Adyamadyena, the first and last terms are x^2 and – 18

$\therefore\ E = (x - 1)(x^2 + \alpha x - 18) = 0$

Comparing the x coeff on both sides

$-\alpha - 18 = -21$ $\therefore \alpha = 3$

$\therefore\ E = (x - 1)(x^2 + 3x - 18) = 0$

Verification by Gunita Samuccayah Sutram

$S_c = 0 = (1 - 1)(1 + 3 - 18) = 0$

Solving the Quadratic Equation by Differential Method as

$D_1 = 2x + 3 = \pm\sqrt{9 + 72}$

$2x = -3 \pm 9$

$x = 3, -6$

$\therefore\ E = (x - 1)(x - 3)(x + 6) = 0$

$S_c = 0 = (1 - 1)(1 - 3)(1 + 6) = 0$

Verified by Gunita Samuccayah Sutram.

18) $E = x^3 + 3x^2 - 10x - 24 = 0$

$S_c = -30$

By Vilokanam 1 and – 1 are not factor.

But if $x = 3$ satisfies the equation

$\therefore\ (x - 3)$ is a factor

$E = (x - 3)$ A, A should contain x^2, x and constant terms

By Adyamadyena, the first and the last terms are x^2 and 8.

Comparing the x coefficient on both sides $\therefore\ E = (x - 3)(x^2 + \alpha x + 8)$

$8 - 3\alpha = -10$

$\therefore\ \alpha = 6$

$\therefore\ E = (x - 3)(x^2 + 6x + 8) = 0$

Solving the Quadratic equation by Differential Method as

$D_1 = 2x + 6 = \pm\sqrt{36 - 32}$

$\therefore\ x = -6 \pm 2 = -2, -4$

$\therefore\ E = (x - 3)(x + 2)(x + 4) = 0$

$S_c = -30 = (-2)(3)(5) = -30$ verified by Gunita Samuccayah Sutram.

19) $E = x^3 + 8x^2 + x - 42 = 0$

$S_c = -32$

By Vilokanam 1 and –1 are not factors.

But $x = 2$ satisfies the equation

$\therefore\ E = (x - 2)$ A where A should have x^2, x and constant terms .

$\therefore$ By Adyamadyena the first and last terms are x^2 and 21

$\therefore\ E = (x - 2)(x^2 + \alpha x + 21)$

By comparing x coefficient on both sides of E.

$21 - 2\alpha = 1 \Rightarrow \alpha = 10$

Verification by Gunita Samuccayah Sutram.

$S_c = -32 = (1 - 2)(1 + 10 + 21) = -32$

$\therefore\ E = (x - 2)(x^2 + 10x + 21) = 0$

The Quadratic Equation can be solved by differential method as

$D_1 = 2x + 10 = \pm\sqrt{100 - 84}$

$\therefore\ 2x = -10 \pm 4$

$x = -3, -7$

$E = (x - 2)(x + 3)(x + 7) = 0$
$S_c = -32 = (-1)(4)(8) = -32$
Verified by Gunita Samuccayah Sutram.

20) $E = x^3 + 2x^2 - 3 = 0$
$S_c = 0 \quad \therefore \; (x - 1)$ is a factor.
$E = (x - 1)$ A. A should contain x^2, x and constant terms.
By Adyamadyena, the first and the last terms are x^2 and 3.
Comparing the x coefficient on both sides of E.
$\therefore \; E = (x - 1)(x^2 + \alpha x + 3)$
$-\alpha + 3 = 0 \qquad \therefore \; \alpha = 3$
Verification by Gunita Samuccayah Sutram.
$S_c = 0 = (1 - 1)(1 + 3 + 3) = 0$
$\therefore \; E = (x - 1)(x^2 + 3x + 3) = 0$
The Quadratic Equation A can be solved by Differential Method as.
$2x + 3 = \pm\sqrt{9 - 12}$
$= \pm\sqrt{3}i$
$\therefore \; x = \frac{-3 \pm \sqrt{3}i}{}$

$$E = (x - 1)\left(x - \left(\frac{-3 + \sqrt{3}i}{2}\right)\right)\left(x - \left(\frac{-3 - \sqrt{3}i}{2}\right)\right) = 0$$

$S_c = 0 = (1 - 1)(2 + 3 - \sqrt{3}i)(2 + 3 + \sqrt{3}i) = 0$
Verified by Gunita Samuccayah Sutram.

21) $E = x^3 + 15x^2 + 71x + 105 = 0$
By Vilokanam, + 1 and – 1 are not factors
But $x = -3$ satisfies the equation
$\therefore \; x + 3$ is a factor
$E = (x + 3)$ A, A should contain x^2, x and constant terms.
By Adyamadyena, the first and the last terms of A are x^2 and 35.
Comparing the x coefficient on both sides of E.
$\therefore \; E = (x + 3)(x^2 + \alpha x + 35)$
$3\alpha + 35 = 71$
$\therefore \; \alpha = 12$
Verification by Gunita Samuccayah Sutram.
$S_c = 1 + 15 + 71 + 105 = 192 = (1 + 3)(1 + 12 + 35) = 192$
$\therefore \; E = (x + 3)(x^2 + 12x + 35) = 0$
The Quadratic Equation can be solved by Differential Method.
$2x + 12 = \pm\sqrt{144 - 140} = \pm 2$
$\therefore \; x = -5, \; -7$
$\therefore \; E = (x + 3)(x + 5)(x + 7) = 0$
$S_c = 192 = (4)(6)(8) = 192$
Verified by Gunita Samuccayah Sutram.

22) $E = x^3 + 7x^2 - 6x - 72 = 0$

By Vilokanam, + 1 and – 1 are not factors

But x = 3 satisfies the equation

$\therefore$ x – 3 is a factor

E = (x – 3) A. A should contain x^2, x and constant terms.

By Adyamadyena, the first and the last terms of A are x^2 and 24.

Comparing the x coefficient on both sides.

$\therefore\ E = (x - 3)(x^2 + \alpha x + 24)$

$-3\alpha + 24 = -6$

$\therefore\ \alpha = +10$

Verification by Gunita Samuccayah Sutram.

$S_c = 1 + 7 - 6 - 72 = -70 = (1 - 3)(1 + 10 + 24) = -70$

$\therefore\ E = (x - 3)(x^2 + 10x + 24) = 0$

The Quadratic Equation can be solved by Differential Method as.

$2x + 10 = \pm\sqrt{100 - 96} = \pm 2$

$\therefore\ x = -4,\ -6$

$\therefore\ E = (x - 3)(x + 4)(x + 6) = 0$

$S_c = -70 = (-2)(5)(7) = -70$

Verified by Gunita Samuccayah Sutram.

23) $E = x^3 - 24x + 23 = 0$

By Vilokanam, Sc = 0

$\therefore$ x – 1 is a factor

E = (x – 1) A, A should contain x^2, x and constant terms.

By Adyamadyena, the first and the last terms of A are x^2 and – 23.

Comparing the x coefficient on both sides.

$\therefore\ E = (x - 1)(x^2 + \alpha x - 23)$

$-23 - \alpha = -24$ $\therefore\ \alpha = 1$

Verification by Gunita Samuccayah Sutram.

$S_c = 0 = (1 - 1)(1 + 1 - 23) = 0$

$\therefore\ E = (x - 1)(x^2 + x - 23) = 0$

The Quadratic Equation can be solved by Differential Method as.

$2x + 1 = \pm\sqrt{1 + 92} = \pm\sqrt{93}$

$$\therefore\ x = \frac{-1 \pm \sqrt{93}}{2}$$

$$\therefore\ E = (x - 1)\left(x - \left(\frac{-1 + \sqrt{93}e}{2}\right)\right)\left(x - \left(\frac{-1 - \sqrt{9}_{}}{2}\right|\right| = 0$$

$\therefore\ E = (x - 1)(2x + 1 - \sqrt{93})(2x + 1 + \sqrt{93}) = 0$

$S_c = 0 = (1 - 1)(2 + 1 - \sqrt{93})(2 + 1 + \sqrt{93}) = 0$

Verified by Gunita Samuccayah Sutram.

SECTION – 12

COMPARISON BETWEEN THE CURRENT METHOD AND VEDIC METHOD

1. $E = x^3 - 3x^2 + 12x + 16 = 0$

Current Method

Comparing the given equation with the standard equation.

$ax^3 + 3bx^2 + 3cx + d = 0$

$h = \frac{-b}{a} = \frac{-(-1)}{1} = 1$

To remove the second term, roots of the given equation are to be diminished by 1.

1	1	–3	12	16
		1	–2	10
	1	–2	10	26
		1	–1	
	1	–1	9	
		1		
	1	0		

∴ The transformed equation (Cardon's Equation) is

$y^3 + 9y + 26 = 0$

Where $y = x - 1 \Rightarrow x = y + 1$

Let $y = \left(p^{\frac{1}{3}} + q^{\frac{1}{3}}\right)$

$\therefore\ y^3 = p + q + 3\,p^{\frac{1}{3}}q^{\frac{1}{3}}\left(p^{\frac{1}{3}} + q^{\frac{1}{3}}\right)$

$\Rightarrow y^3 - 3\,p^{\frac{1}{3}}q^{\frac{1}{3}}\,y - (p + q) = 0 \quad \because\ p^{\frac{1}{3}} + q^{\frac{1}{3}} = y$

Comparing this with Cardon's Equation

$-3\,p^{\frac{1}{3}}q^{\frac{1}{3}} = 9 \Rightarrow p^{\frac{1}{3}}q^{\frac{1}{3}} = -3$

$\therefore\ pq = -27$

$-(p + q) = 26 \Rightarrow (p + q) = -26$

$(p - q)^2 = (p + q)^2 - 4pq$

$= (-26)^2 - 4(-27)$

$= 676 + 108 = 784$

Vedic Method

$E = x^3 - 3x^2 + 12x + 16 = 0$

By Vilokanam

$S_c \neq 0 \Rightarrow (x - 1)$ is not a factor.

$S_e = S_0\ \therefore\ (x + 1)$ is a factor

$E = (x + 1)A$ Where A should contain x^2, x and constant terms

By Adyamadyena, the 1st and the last terms of A are x^2 and 16

$\therefore\ E = (x + 1)(x^2 + \alpha x + 16)$

α can be determined by Argumentation, Comparison of x terms on both sides.

$16 + \alpha = 12\ \therefore\ \alpha = -4$

$\therefore\ E = (x + 1)(x^2 - 4x + 16) = 0$

The Quadratic Equation $x^2 - 4x + 16 = 0$ can be solved by the Differential relation

$D_1 \quad 2x - 4 = \pm\sqrt{16 - 64}$

$2x - 4 = \pm 4\sqrt{3}i$

$x = 2 \pm 2\sqrt{3}i$

The factors are $(x + 1)(x - 2 - 2\sqrt{3}i)$ and $(x - 2 + 2\sqrt{3}i)$

$\therefore\ x = -1, 2 + 2\sqrt{3}i, 2 - 2\sqrt{3}i$

$\therefore\ p - q = 28$

$\underline{p + q = -26}$

$2p = 2,\ p = 1 \ = \ q = (-27)$

$\therefore\ p^{\frac{1}{3}} = (1)^{\frac{1}{3}} = 1, w, w^2$

Where $w = \dfrac{-1 - i\sqrt{3}}{2}$

But $p^3 q^3 = -3$

$\therefore\ q^{\frac{1}{3}} = \dfrac{-3}{p^{\frac{1}{3}}} = \dfrac{-3}{1}, \dfrac{-3}{w}\ \dfrac{-3}{w^2}$

$= -3, -3w^2 - 3w \qquad w^2 = \dfrac{1}{w}$

$\therefore$ The roots of Cardon's Equation are

$\therefore y = | p^{\bar{3}} + q^{\bar{3}}$

$= (1 - 3);\quad (w - 3w^2), (w^2 - 3w)$

$= -2;\quad \dfrac{-1 - i\sqrt{3}}{2} - \dfrac{3(-1 + i\sqrt{3})}{2};\ \dfrac{-1 + i\sqrt{3}}{2} - \dfrac{3(-1 - i\sqrt{3})}{2}$

$= -2;\quad \dfrac{-1 - i\sqrt{3} + 3 - 3i\sqrt{3}}{2};\ \dfrac{-1 + i\sqrt{3} + 3 + 3i\sqrt{3}}{2}$

$= -2;\quad \dfrac{+2 - 4i\sqrt{3}}{2};\ \dfrac{2 + 4i\sqrt{3}}{2}$

or $y = -2; (+1 - 2i\sqrt{3}); (1 + 2i\sqrt{3})$

$\therefore\ x = y + 1$

$\Rightarrow\ x = (-2 + 1), (+1 - 2i\sqrt{3} + 1), (1 + 2i\sqrt{3} + 1)$

$\therefore\ x = -1, (2 - 2i\sqrt{3}), (2 + 2i\sqrt{3})$

2. $E = x^3 - 3x^2 - 9x - 5 = 0$

Current Method

$x^3 - 3x^2 - 9x - 5 = 0$ ———— (1)

Comparing this with standard equation $ax^2 + 3bx^2 + 3cx + d = 0$

$a = 1,\ b = -1,\quad c = -3$ and $d = -5$

$\therefore\ h = \dfrac{-b}{a} = \dfrac{-(-1)}{1} = 1$

To remove the 2nd term of (1) roots of (1) are diminished by 1 as follows

Vedic Method

$E = x^3 - 3x^2 - 9x - 5 = 0$

$S_c = -16 \qquad \therefore\ (x - 1)$ is not a factor.

$S_0 = S_e \qquad \therefore\ (x + 1)$ is a factor.

$\therefore\ E = (x + 1)\,A.$ A should contain x^2, x and constant terms.

By Adyamadyena the first and last terms of A are x^2 and -5.

$E = (x + 1)(x^2 + \alpha x - 5) = 0$

1	1	− 3	− 9	− 5
		1	− 2	− 11
	1	− 2	− 11	− 16
		1	− 1	
	1	− 1	− 12	
		1		
	1	0		

∴ The transformed equation is

$y^3 - 12y - 16 = 0$ ——— (2)

Where $y = x - 1$ $\boxed{x = y + 1}$

Let $y = p^{\frac{1}{3}} + q^{\frac{1}{3}}$

$\Rightarrow y^3 = p + q + 3p^{\frac{1}{3}}q^{\frac{1}{3}}(p^{\frac{1}{3}} + q^{\frac{1}{3}})$

or $y^3 - 3p^{\frac{1}{3}}q^{\frac{1}{3}}y - (p + q) = 0$

Comparing with (2)

$p^{\frac{1}{3}} q^{\frac{1}{3}} = 4 \Rightarrow q^{\frac{1}{3}} = \frac{4}{p^{\frac{1}{3}}}$ —— (3)

∴ $pq = 64$ and $(p + q) = 16$

By using $(p - q)^2 = (p + q)^2 - 4pq$

$(p - q)^2 = 256 - 256 = 0$

or $(p - q) = 0$

∴ $p + q = 16 \Rightarrow p = 8 = q$

$p^{\frac{1}{3}} = (8)^{\frac{1}{3}} = 2, 2w, 2w^2$

and $q^{\frac{1}{3}} = \frac{4}{p^{\frac{1}{3}}}$ (By (3))

∴ $q^{\frac{1}{3}} = \frac{4}{2}, \frac{4}{2w}, \frac{4}{2w^2}$

$= 2, 2w^2, 2w$ $w = w^*$

∴ $y = (p^{\frac{1}{3}} + q^{\frac{1}{3}}) = (2 + 2)\ (2w + 2w^2),\ (2w^2 + 2w)$

$= 4,\ 2(w + w^2),\ 2(w^2 + w)$

$= 4,\ 2(-1),\ 2(-1)$ ∵ $w + w^2 + 1 = 0$

∴ $y = 4, -2, -2$

But $x = y + 1 = (4 + 1), (-2 + 1), (-2 + 1)$

∴ $x = 5, -1$ and -1

α can be determined by Argumentation comparison of x terms on both sides

$\alpha - 5 = -9$ $\alpha = -4$

∴ $E = (x + 1)(x^2 - 4x - 5) = 0$

The Quadratic equation can be solved by the differential relation.

$D_1 = 2x - 4 = \pm$

$x = 2x - 4 = \pm 6$

$x = 5, -1$

So the factors are $(x + 1)(x - 5)(x + 1)$

∴ $x = -1, -1, 5$

Verification of the result by Gunita Samuccayah

$S_c = -16 = 2 \times 2(-4) = -16$

As $(x + 1)$ repeats, it can be obtained from Successive Differentiation also

$D_1 = 3x^2 - 6x - 9 = 0$

$S_e = S_0$ ∴ $(x + 1)$ is a factor

Hence E should have $(x + 1)^2$ as a factor.

This can be verified by writing down

$E = (x + 1)^2 B$. B should have x and constant terms.

By Adyamadyena the first and the last terms of B are x and − 5

By Gunita Samuccayah factorisation is verified $-16 = 4 \times -4 = -16$

∴ E is factorizable in terms of $(x + 1)^2$

∴ $E = (x + 1)^2 (x - 5) = 0$

3. $E = x^4 - 15x^2 + 20x - 6 = 0$

Current Method

Let $x^4 - 15x^2 + 20x - 6 \equiv (x^2 + kx + \ell_1)(x^2 - kx + \ell_2)$

Equating the coefficients of like terms

Coeff of x^2: $\ell_1 + \ell_2 - k^2 = -15$

$\Rightarrow \ell_1 + \ell_2 = k^2 - 15$

Coeff of x: $k(\ell_2 - \ell_1) = 20$ & $\ell_1 \ell_2 = -6$

$\Rightarrow \ell_2 - \ell_1 = \frac{20}{\cdot}$

We know that

$(\ell_1 + \ell_2)^2 - (\ell_2 - \ell_1)^2 = 4\ell_1 \ell_2$

i.e., $k^4 - 30k^2 + 225 - \frac{400}{k^2} = 4(-6)$

i.e., $k^6 - 30k^4 + 225k^2 + 24k^2 - 400 = 0$

or $k^6 - 30k^4 + 249k^2 - 400 = 0$

i.e., $(k^2)^3 - 30k^4 + 249k^2 - 400 = 0$

By inspection if $k^2 = 16$, the equation is satisfied.

$k^2 = 16 \Rightarrow k = 4$

$\ell_1 + \ell_2 - 16 = -15$

$\ell_1 + \ell_2 = 1$ and $l_2 - l_1 = 5$

$\Rightarrow 2\ell_2 = 6$

$\ell_2 = 3$

$\ell_1 = -2$

Given equation is $(x^2 + kx + \ell_1)(x^2 - kx + \ell_2) = 0$

$\Rightarrow (x^2 + 4x - 2)(x^2 - 4x + 3) = 0$

$x^2 + 4x - 2 = 0$

$\Rightarrow x = \frac{-4 \pm \sqrt{16+8}}{}$

$= -2 \pm \sqrt{6}$

and $x^2 - 4x + 3 = 0$

$x = \frac{4 \pm \sqrt{16-2}}{2} - \frac{4 \pm \sqrt{4}}{2} \quad 4 \pm 2$

$x = 3, 1$

$x = 1, 3, -2, \pm\sqrt{6}$

Vedic Method

$E = x^4 - 15x^2 + 20x - 6 = 0$

By Vilokanam

$S_c = 0$ $(x - 1)$ is a factor.

$E = (x - 1)$ A. A should contain x^3, x^2, x and constant terms.

The 1[st] and last terms of A can be obtained by applying Adyamadyena

These are x^3 and 6

$E = (x - 1)(x^3 + \alpha x^2 + \beta x + 6) = 0$

α and β are obtained by Argumentation

$-\beta + 6 = 20 \qquad \beta = -14$

$-14 - \alpha = -15 \quad \alpha = 1$

$E = (x - 1)(x^3 + x^2 - 14x + 6) = 0$

We have to solve the Cubic Equation $(x^3 + x^2 - 14x + 6) = 0$

By Vilokanam $(x + 1)$, $(x - 1)$ are not factors.

But a trial in Succession,

$x = 3$, satisfies the Cubic equation

$(x - 3)$ is a factor

$E = (x - 1)(x - 3)$ B

$(x^2 - 4x + 3)$B B should have x^2, x and constant terms. The first and last terms of B can be determined by Adyamadyena. They are x^2 and -2.

$(x^2 - 4x + 3)(x^2 + \alpha x - 2) = 0$

Comparing coefficients of x we get

$3\alpha + 8 = 20 \therefore \alpha = 4$

$(x^2 - 4x + 3)(x^2 + 4x - 2) = 0$

are the two factors of E

Wherein we have determined already

$x^2 - 4x + 3 = (x - 1)(x - 3)$

Solving the second Quadratic equation by differential relation

$2x + 4 = \pm\sqrt{16+8}$

$2x + 4 = \pm 2\sqrt{6} \quad \therefore x = -2 \pm \sqrt{6}$

The 4[th] degree equation is factorised as

$(x - 1)(x - 3)(x + 2 \pm \sqrt{6})$

$(x - 1)(x - 3)(x + 2 + \sqrt{6})(x + 2 - \sqrt{6})$

$\therefore x = +1, +3, \quad -2 \pm \sqrt{6}$

4. $E = x^4 + 5x^3 - 20x^2 - 60x + 144 = 0$

Current Method

$x^4 + 5x^3 - 20x^2 - 60x + 144 = 0$ — (1)

$h = \dfrac{-a_1}{na_0} \quad -5$

To remove the 2^{nd} term of (1) first we multiply the roots with 4 and then diminish the roots by – 5

Multiplying the roots of (1) with 4

$y^4 + 5(4)x^3 - 20(4)^2x^2 - 60(4)^3x + (4)^4(144) = 0$

$y^4+20x^3-320x^2-3840x+36864=0$ — (2)

When $y = 4x \Rightarrow \boxed{x = \frac{y}{4}}$ —— (4)

Now diminishing the roots of (2) by – 5

1	20	– 320	–3840	36864
	–5	– 75	1975	9325
1	15	– 395	–1865	46189
	–5	– 50	2225	
1	10	– 445	360	
	–5	– 25		
1	5	– 470		
	–5			
1	0			

∴ The transformed equation is

$z^4 - 470z^2 + 360z + 46189 = 0$ —— (3)

Where $z = y + 5 \Rightarrow \boxed{y = z - 5}$ — (B)

Let

$Z^4 - 470z^2 + 360z + 46189 = (z^2 + kz + \ell_1)(z^2 - kz + \ell_2)$ ———— (4)

On comparing the like terms

$\ell_2 + \ell_1 - k^2 = -470$ and $k(\ell_2 - \ell_1) = 360$ and $\ell_1 \ell_2 = 46189$

$\Rightarrow \ell_2 + \ell_1 = k^2 - 470$ and $\ell_2 - \ell_2 = \dfrac{360}{k}$

Eliminating ℓ_1 and ℓ_2 by using

$(\ell_2 + \ell_1)^2 - (\ell_2 - \ell_1)^2 = 4\ell_1\ell_2$

Vedic Method

$E = x^4 + 5x^3 - 20x^2 - 60x + 144 = 0$

$f(x) = x^4 + 5x^3 - 20x^2 - 60x = -144$

By Vilokanam

$S_c \neq 0$ ∴ (x – 1) is not a factor.

$S_e \neq S_e$ ∴ (x – 1) is not a factor.

Let $f(x) = x^4 + 5x^3 - 20x^2 - 60x = -144$

	LHS	RHS	DIFF
	f(x)	-144	RHS – LHS
x = 2	- 144		0
x = 3	- 144		0

∴ (x – 2), (x – 3) are two factors

$E = (x - 2)(x - 3) A$

$= (x^2 - 5x + 6)A$ A should have x^2, x and constant terms

By Adyamadyena the first and last terms are x^2, 24.

$E = (x^2 - 5x + 6)(x^2 + \alpha x + 24)$

α is to be determined by argumentation

By comparing x coefficients on both sides

$6\alpha - 120 = -60 \qquad \alpha = 10$

∴ $A = x^2 + 10x + 24$

This is verified by Gunita Samuccayah

$S_c = 70 = (1 - 5 + 6)(1 + 10 + 24) = 70$

A is further solved by using the Differential relation

$2x + 10 = \pm\sqrt{100 - 96}$

$2x + 10 = \pm 2$

$x = -4, -6$

The factors are

$(x - 2)(x - 3)(x + 4)(x + 6)$

$$(k^2-470)^2-\left(\frac{360}{k}\right)^2=4(46189)$$

$$k^4-940k^2+220900-\frac{129600}{k^2}\quad 184756$$

$k^6-940k^4+36144k^2-129600=0$

This is a cubic in k^2

By inspection $k^2=4 \quad \therefore k=2$

By using this

$\ell_2+\ell_1=k^2-470=-466$

and $\ell_2-\ell_1=\dfrac{360}{k}=180$

Adding $2\ell_2=-286 \Rightarrow \ell_2=-143$

and $\ell_1=\ell_2-180=-323$

substituting ℓ_1, ℓ_2 and k values in (4) the two quadratic equations are

$z^2+2z-323=0$ ———————— I

and $z^2-2z-143=0$ ———————— II

Solving Ist Quadratic equation

$z^2+2z-323=0$

$$z=\frac{-2\pm\sqrt{4+1292}}{2}=\frac{-2\pm 36}{2}$$

or $z=-1\pm 18$

$=17,-19$

Solving the IInd Quadratic equation

$z^2-2z-143=0$

$$z=\frac{2\pm\sqrt{4+572}}{2}=\frac{2\pm 24}{2}$$

$=1\pm 12$

$\therefore\ z=13,-11$

$\therefore\ z=13, 17, -11, -19$

But $y=(z-5)$ (from (b))

i.e. $y=(13-5), (17-5), (-11-5)$ and $(-19-5)$

or $y=8, 12, -16$ and -24

But $x=\dfrac{y}{4}$ (from (A))

i.e., $x=\dfrac{8}{4}, \dfrac{12}{4}, \dfrac{-16}{4}$ and $\dfrac{-24}{4}$

$\therefore\ x=2, 3, -4$ and -6

The continuation of the Lecture Notes – III(a) will be published shortly as Lecture Notes III(b)