Vedic Mathematics

Lecture Notes – 4

Square, Cube, Expansion, Roots (Numbers and Polynomials) and Equation (continued)

By

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VEDIC MATHEMATICS
OR SIXTEEN SIMPLE MATHEMATICAL FORMULAE

Sixteen Sutras and Their Corollaries

Sūtras

1. एकाधिकेन पूर्वेण
   Ekādhikena Pūrveṇa (also a corollary)
2. निखिलं नवतश्चरमं दशतः
   Nikhilaṃ Navataścaramaṃ Daśataḥ
3. ऊर्ध्वतिर्यग्भ्याम्
   Ūrdhva-tiryagbhyāṃ
4. परावर्त्य योजयेत्
   Parāvartya Yojayet
5. शून्यं साम्यसमुच्चये
   Śūnyaṃ Sāmyasamuccaye
6. (आनुरूप्ये) शून्यमन्यत्
   (Ānurūpye) Śūnyamanyat
7. संकलनव्यवकलनाभ्याम्
   Saṅkalana-vyavakalanābhyam (also a corollary)
8. पूरणापूरणाभ्याम्
   Pūraṇāpūraṇābhyāṃ
9. चलनकलनाभ्याम्
   Calana-Kalanābhyām
10. यावदूनम्
    Yāvadūnam
11. व्यष्टिसमष्टिः
    Vyaṣṭisamaṣṭiḥ
12. शेषाण्यङ्केन चरमेण
    Śeṣāṇyaṅkena Carameṇa
13. सोपान्त्यद्वयमन्त्यम्
    Sopāntyadvayamantyam
14. एकन्यूनेन पूर्वेण
    Ekanyūnena Pūrveṇa
15. गुणितसमुच्चयः
    Guṇitasamuccayaḥ

Sub-Sūtras or Corollaries

1. आनुरूप्येण
   Ānurūpyeṇa
2. शिष्यते शेषसंज्ञः
   Śiṣyate Śeṣasaṃjñaḥ
3. आद्यमाद्येनान्त्यमन्त्येन
   Ādyamādyenāntya-mantyena
4. केवलैः सप्तकं गुण्यात्
   Kevalaiḥ Saptakaṃ Guṇyāt
5. वेष्टनम्
   Veṣṭanaṃ
6. यावदूनं तावदूनम्
   Yāvadūnaṃ Tāvadūnaṃ
7. यावदूनं तावदूनीकृत्य वर्गं च योजयेत्
   Yāvadūnaṃ Tāvadūnīkṛtya Vargañca Yojayet
8. अन्त्ययोर्दशकेऽपि
   Antyayordaśake'pi
9. अन्त्ययोरेव
   Antyayoreva
10. समुच्चयगुणितः
    Samuccayaguṇitaḥ
11. लोपनस्थापनाभ्याम्
    Lopanasthāpanābhyām
12. विलोकनम्
    Vilokanam
13. गुणितसमुच्चयः समुच्चयगुणितः
    Guṇitasamuccayaḥ
    Samuccayaguṇitaḥ

(Editor of the original book on Vedic Mathematics)

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Part – I

Squares, Square Roots, Cubes and General Expansions, Cube Roots and Higher Roots

Section–A

Squares and Square Roots (Numbers and Polynomials in One Variable)

Squares

In the current system squaring of a number or polynomial is worked out by multiplying the number or polynomial by itself whereas in the Vedic Method <u>different methods</u> are applied in general.

(1) Urdhva Tiryak Method of multiplication.

(2) Using the Yavadunam Thavadunikrutya Vargancha Yogayet Sutram, which is easier than the Urdha Tiryak. This makes use of deficiencies or excesses over the bases (convenient bases) and the answer is derived from them.

(3) Using the duplex i.e., Dwandvayoga method.

Out of the three, the third one i.e., the duplex method and the first method are considered to be applicable to numbers containing any number of digits, and polynomials also while the second method as it is not suitable for polynomials. The Duplex Dwandwayoga concept is explained in the text.

A number can be converted to vinculum form if necessary when both the above two methods 1 and 3 are applicable in the vinculum form also.

(4) Squares of decimals are also worked out on the basis of the above methods.

It is noticed that the Vedic Method is much simpler in writing down the result as is very clear from the working in the above manner.

The simplicity of working by the Vedic Method of the squaring of a number is very significantly noticed in case of more digited numbers.

(5) By application of Anurupyena sutram, the square of the number can be evaluated by considering the geometrical progression, which is explained in the text.

In the current method one can attempt to write down the given number in terms of two, three etc., and then use the general formula as $(a + b + c + \cdots)^2$

$= a^2 + b^2 + c^2 + \quad 2ab + 2ac + 2bc \cdots\cdots$

$97^2 = (\underset{a}{90} + \underset{b}{7})^2$; or $(\underset{a}{100} - \underset{b}{3})^2 = (937)^2 = (\underset{a}{1000} - \underset{b}{63})^2$ or $(\underset{a}{900} + \underset{b}{30} + \underset{c}{7})^2$

Anurupyena Sutram

By applying Anurupyena Sutram, squares can be determined as follows. The given number is grouped into two parts a and b. Then the Anurupyena sutram can be applied in the following manner to obtain square of the given number. The steps are as follows in order of three terms.

(1) To put down the square of the first group i.e., a^2 as the first term.

(2) This is followed by the ratio $\frac{b}{a} \times a^2 = ab$ as the second term.

(3) To write down the value of $ab \times \frac{b}{a} = b^2$ as the third term.

The three terms are in G.P. or in other words these terms are worked out in succession to have the same geometrical ratio $\frac{b}{a}$.

(4) To write down the three terms in order and under the second term the same value of it is written again and added up. Finally the sum of all these terms is the square of the number.

$$\begin{array}{l} a^2 + ab + b^2 \\ \underline{\quad\quad ab \quad\quad} \\ a^2 + 2ab + b^2 \end{array}$$

Yavadunam thavadunikrutya vargamca Yojayet Sutram:

Means "whatever the extent of its deficiency from a base, lessen it from the given number still further to that very extent, and also set up the square of that deficiency". (The same holds good for excess also with addition).

For every number a base is considered, such as 10 or its powers, multiples or sub multiples. The deficiency or excess of the number from the considered base is significant in working out the answer. The working details are explained below for 7^2, 18^2, 89^2, 21^2, and so on including squares of many digitied numbers. At first deficiency or excess, over the base, of the given number is shown as – or + accordingly. The deficiency can be worked out by applying the Sutram Nikhilam Navatahcharamamdastah where as the excess can be worked out by vilokanam in relation to the base. Making use of these, the answer is worked out in two parts. The method makes use of a notable point that it is easier to square deficiency or excess over a base than the number itself.

In the first part we have to lessen the deficiency from the given number or increase the given number by excess over the base as the case may be. The second part consists of the square of the deficiency or excess. Depending on the base considered, the provision in the second part is one less than the number of digits in the base. Any digit (s) more than the provision in the second part will be carried over to the first part.

Examples:

1. $(7)^2$

Current Method

$7 \times 7 = 49$

Vedic Method

Number = 7

Base = 10

Deficiency = −3

First part = 7 − 3 = 4

Second part = $(-3)^2$ = 9

Answer = 4 / 9 = 49

2. $(18)^2$

Current Method

```
 18
 18
144
18
324
```

Vedic Method

Number = 18

Base = 10

Number − base = 8

Excess = 8

First part = 18 + 8 = 26

Second part = $8^2 = 64$

Answer = 26 / $_6$4 = 324

3. $(89)^2$

Current Method

```
  89
  89
 801
712
7921
```

Vedic Method

Number = 89

Base = 100

Deficiency = −11

First part = 89 − 11 = 78

Second part = $(-11)^2 = 121$

Answer = 78 / $_1$21 = 7921

4. $(21)^2$

Current Method

```
 21
 21
 21
42
441
```

Vedic Method

Number = 21

Base = 10

Excess = 11

First part = 21 + 11 = 32

Second part = $11^2 = 121$

Answer = 32 / $_{12}$1 = 441

5. $(103)^2$

Current Method

```
  103
  103
  309
 000
103
10609
```

Vedic Method

Number = 103
Base = 100
Excess = 3
First part = 103 + 3 = 106
Second part = 3^2 = 9
Answer = 106 / 09 = 10609

6. $(997)^2$

Current Method

```
   997
   997
  6979
 8973
8973
994009
```

Vedic Method

Number = 997
Base = 1000
Deficiency = −3
First part = 997 − 3 = 994
Second part = $(-3)^2$ = 9
Answer = 994 / 009 = 994009

7. $(1014)^2$

Current Method

```
   1014
   1014
   4056
  1014
 0000
1014
1028196
```

Vedic Method

Number = 1014
Base = 1000
Excess = 14
First part = 1014 + 14 = 1028
Second part = 14^2 = 196
Answer = 1028 / 196 = 1028196

8. $(9983)^2$

Current Method

```
     9983
     9983
    29949
   79864
  89847
 89847
 99660289
```

Vedic Method

Number = 9983
Base = 10000
Deficiency = –17
First part = $(-17)^2 = 289$
Second part = 9983 – 17 = 9966
Answer = 9966 / 0289 = 99660289

9. $(10015)^2$

Current Method

```
      10015
      10015
      50075
     10015
    00000
   00000
  10015
  100300225
```

Vedic Method

Number = 10015
Base = 10000
Excess = 15
First part = $15^2 = 225$
Second part = 10015 + 15 = 10030
Answer = 10030 / 0225 = 1000300225

10. $(99998)^2$

Current Method

```
      99998
      99998
     799984
    899982
   899982
  899982
 899982
 9999600004
```

Vedic Method

Number = 99998
Base = 100000
Deficiency = –2
First part = $(-2)^2 = 4$
Second part = 499998 – 2 = 99996
Answer = 99996/ 00004 = 9999600004

11. $(100012)^2$

Current Method	Vedic Method
100012	Number = 100012
100012	Base = 100000
200024	Excess = 12
100012	First part = 12^2 = 144
000000	Second part = 100012 + 12 = 100024
000000	Answer=100024/00144
000000	=10002400144
100012	
10002400144	

12. $(999982)^2$

Current Method	Vedic Method
999982	
999982	Number = 999982
1999964	Base = 1000000
7999856	Deficiency = −18
8999838	First part = $(-18)^2 = 324$
8999838	Second part = 999982 − 18 = 999964
8999838	Answer = 999964/000324 =999964000324
8999838	
999964000324	

With Convenient Base

One can use a convenient base also. For example we can call it as a working base (WB) as distinguished from the theoretical base (TB) which is 10 or powers of 10. Working base can be multiples or Sub multiples of the theoretical base. The deficiency or excess are now to be worked out with reference to the working base. In the final answer however one has to consider multiplication of the first part with the ratio of working base to theoretical base, in case of multiple base or division in case of sub multiples ex 1, 2, 3, 4, 5.

Examples:

1. $(23)^2$

Current Method

```
  23
  23
  69
 46
 529
```

Vedic Method

Number = 23

Theoretical base is 10

Working base is 20

Excess = 3

First part = $3^2 = 9$

Second part = 23 + 3 = 26

The fist part is multiplied by
Where WB = Working base and
TB = Theoretical base

$\frac{WB}{TB} = \frac{20}{10}$

Answer = 26 / 9
× 2

= 52 / 9 = 529

2 $(313)^2$

Current Method

```
  313
  313
  939
 313
939
97969
```

Vedic Method

Number = 313

Theoretical base is 100

Working base is 300

Excess = 13

$\frac{WB}{TB} = \frac{300}{100}$

First part = $13^2 = 169$

Second part = 313 + 13 = 326

Ans.: 326 / 69,
× 3 / 1

= 978 / 1 69

= 97969

3. $(500019)^2$

Current Method

```
       500019
       500019
      4500171
     500019
    000000
   000000
  000000
 2500095
 250019000361
```

Vedic Method

Number = 500019

Theoretical Base is 1000000

Working Base is 1000000/2 = 500000

Excess = 19

First part = $19^2 = 361$

Second part = 500019 + 19 = 500038

$\frac{WB}{TB} = \frac{500000}{1000000}$

Ans.: $\frac{500038}{2}$ / 000361

=250019000361

4. $(6999984)^2$

Current Method

```
        6999984
        6999984
        2799936
      55999872
     62999856
    62999856
   62999856
  62999856
 41999904
 489997760 00256
```

Vedic Method

Number 6999984

Theoretical Base is 1000000

Working Base is 7000000

$\frac{WB}{TB} = \frac{700000}{1000000} = 7$ Deficiency = -16

First part = $(-16)^2 = 256$

Second part = 6999984−16 = 6999968

Ans.: 6999968 / 000256
× 7

= 48999776/000256

= 48999776000256

In case of big numbers one may sometimes get, depending on the base considered, the deficiency or excess very large numbers. In such case the second part of the answer can be obtained by Urdhva multiplication or by the application of 'Duplex' method by repeating the process of squaring at this stage. This is shown in example 5. The Urdhva multiplication is simpler than a number of repeated squaring processes.

5. $(885679)^2$

Current Method

```
      885679
      885679
    --------
     7971111
    6199753
   5314074
  4428395
 7085432
7085432
------------
784427191041
```

Vedic Method

Number = 885679

Theoretical Bas is 100000

Working Base is 900000

$\frac{WB}{TB} = 9$

Deficiency = – 14321

First part = $(-14321)^2$ = 205091041

Second part = 885679 – 14321

= 871358

Ans.: 871358 × 9 / $_{2050}$91041

= 7842222 / $_{2050}$91041

= 784427291041

(1) First part = $(14321)^2$

By Urdhva Tiryagbhyam

```
            1 4 3 2 1
            1 4 3 2 1
  -------------------
  2 0 5 0 9 1 0 4 1
   1 2 3 2 2 1
```

∴ $(14321)^2$ = 205091041

(2) By repeating the above process of Squaring.

$(14321)^2$

Base is 10000

Excess = 4321

First part = $(4321)^2$ = 18671041

Second part 14321 + 4321 = 18642

Answer = 18642 / $_{1867}$1041

= 205091041

$(4321)^2$

Theoretical Base is 1000
Working Base is 4000
Excess = 321

First part = $(321)^2$ = 103041
Second part = 4321 + 321 = 4642

Ans.: 4642 / $_{103}$041
× 4

= 18568 / $_{103}$041

= 18671/041
= 18671041

$(321)^2$

Theoretical Base is 100
Working Base is 300
Excess = 21
First part = $(21)^2$ = 441
Second part = 321+ 21 = 342

Ans.: 342 / $_4$41
×3

= 1026 / $_4$41

= 103041

$(21)^2$

Theoretical Base is 10
Working Base is 20
Excess = 1
First part = 1^2 = 1
Second part = 21 + 1 = 22

Ans.: 22 / 1
×2

= 44 / 1 = 441

Proof:

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$\quad = a(a \pm 2b) + b^2$

Special case

Square of Numbers ending in 5

It can be solved by the application of Ekadhikena Purvena. The answer is divided into two parts. The second part consists of square of last digit 5 i.e. 25. The first part consists of multiplication of the left out in the number, with that obtained by adding 1 to it. This is shown in the following examples.

Examples:

1. $(45)^2$

Current Method

```
  45
  45
 ---
 225
180
----
2025
```

Vedic Method

First part = 4 × (4 + 1) = 4 + 5 = 20

Second part = 5^2 = 25

Answer = 20/25 = 2025

2. $(165)^2$

Current Method

```
   165
   165
  ----
   825
  990
 165
 -----
 27225
```

Vedic Method

165^2 = 16 × 17 / 25

= 272 / 25

= 27225

3. $(225)^2$

Current Method

```
  225
  225
 ----
 1125
 450
450
-----
50625
```

Vedic Method

$225^2 = 22 \times 23 / 25$

$= 506 / 25$

$= 50625$

In case of multiplication of large numbers, the first part still consists of large numbers inspite of deletion of 5. In such case one can also apply vinculum coupled with Urdhva (or Urdhva Method) as the case may be.

4. $(891425)^2$

Current Method

```
      891425
      891425
     -------
     4457125
    1782850
   3565700
   891425
 8022825
7131400
------------
794638530625
```

Vedic Method

891425 = 89142 × 89143 / 25

= 7946385306 / 25

89142 × 89143 = $1\ \bar{1}\ \bar{1}\ 1\ 4\ 2 \times 1\ \bar{1}\ \bar{1}\ 1\ 4\ 3$

(Vinculum)

$1\ \bar{1}\ \bar{1}\ 1\ 4\ 2$

$1\ \bar{1}\ \bar{1}\ 1\ 4\ 3$

$1\ \bar{2}\ \bar{1}\ 4\ 7\ \bar{6}\ \bar{2}\ 5\ 3\ 0\ 6$

$\bar{1}\ \ 2\ \ 2$

$1\ \bar{2}\ \bar{1}\ 4\ 7\ \bar{6}\ \bar{2}\ 5\ 3\ 0\ 6 = 7946385306$

The same result is obtained by Urdhva Tiryak application 89142× 89143

	8	9	1	4	2			
	8	9	1	4	3			
79	4	6	3	8	5	3	0	6
15	10	9	11	5	2	2		

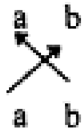
Squaring by Duplex Method (Straight Squaring in Combination with Duplex method)

Squaring can be processed using Duplex (D) or Dwandwayoga process. The Dwandwa concept is defined for single digit or more than one digit as follows. This is again explained on the basis of Urdhva Tiryagbhyam.

1. For single digit, D = Square

$= a^2$

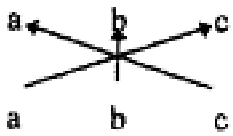
2. For two digit number, D = 2(product of two digits)



$= 2ab$

3. For three-digit number,

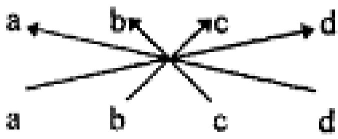
D = 2(product of first digit and last digit) + Square of the Middle Digit.



$= 2ac + b^2$

4. For four digit number,

D = 2(product of first digit and last digit) + 2(Product of second digit and last but one digit)



$2ad + 2bc$

and so on

Examples for a few Duplexes as are given below (as per the model diagrams shown above).

$D(5) = 5^2 = 25$

$D(23) = 2(2)(3) = 12$

$D(321) = 2(3)(1) + 2^2 = 10$

D(4289) = 2(36) + 2(s16) = 72 + 32 = 104

$D(72986) = 2(42) + 2(16) + 81 = 84 + 32 + 81 = 197$

$D(193789426) = 2(6) + 2(18) + 2(12) + 2(63) + 64$

$= 12 + 36 + 24 + 126 + 64 = 262$

A few illustrations showing the square calculation by Duplex method is given below.

Def: We here under explain/illustrate the concept of "Sum of combination of Duplexes" of a number for obtaining the square.

Considering $(a\ b\ c)^2$ by the Duplex method

abc

$\underline{a^2 + 2ab + 2ac + b^2 + 2bc + c^2}$

D(a)/D(ab)/D(abc)/D(bc)/D(c)

$a^2+2ab+2ac+b^2+2bc+c^2$

For example take $(123)^2$
Sum of combination of Duplexes of 123 is
D (1) / D (12) / D (123) / D (23) / D (3)
= 1 / 4 / 10 / 12 / 9
= 1 / 4 / $_{1}0$ / $_{1}2$ / 9
= 15129

Squares: Square of a number is Sum of combination of Duplexes of that number.
In showing the final result, the provision for each Duplex calculation is only one digit where as the remaining digits are to be carried over to the next left.

Examples:

1. $(247)^2$

Current Method

```
  247
  247
 ----
 1729
 988
494
-----
61009
```

Vedic Method using Duplex

247^2 =
D(2)/ D(24)/ D(247)/ D(47)/ D(7)

= 4/ $_{1}6$/ $_{4}4$/ $_{5}6$/ $_{4}9$

= 61009

2. $(3143)^2$

Current Method

```
   3143
   3143
  -----
   9429
 12572
 3143
9429
-------
9878449
```

Vedic Method using Duplex

3143^2 =
D(3)/ D(31)/ D(314)/ D(3143)/ D(143)/ D(43)/ D(3)

= 9/ 6/ $_{2}5$/ $_{2}6$/ $_{2}2$/ $_{2}4$/ 9

= 9878449

3. $(1244897)^2$

Current Method

```
      124897
      124897
     -------
      874279
    1124073
    999176
   499588
  249794
 124897
-----------
 15599260609
```

Vedic Method using Duplex

124897^2 =

D(1)/D(12)/D(124)/D(1248)/D(12489) /D(124897)/ D(24897)/ D(4897)/D(897) /D(97)/ D(7)

=

1/ 4/ $_{1}2$/ $_{3}2$/ $_{6}6$/ $_{11}4$/ $_{16}4$/ $_{20}0$/ $_{19}3$/ $_{12}6$/ $_{4}9$

= 15599260609

One can get the square of a very big number by working out Duplex of its vinculum, as given in example 4.

4. $(987657869)^2$

Current Method

```
          987657869
          987657869
         8888920821
        5925947214
       7901262952
      6913605083
     4938289345
    5925947214
   6913605083
  7901262952
 3888920821
975468066197621161
```

Vedic Method using Duplex to be determine the square when the number is expressed in Vinculum

$987657869 = 10\ \bar{1}\ \bar{2}\ \bar{3}\ \bar{4}\ \bar{2}\ \bar{1}\ \bar{3}\ \bar{1}$

$(1\,0\ \bar{1}\ \bar{2}\ \bar{3}\ \bar{4}\ \bar{2}\ \bar{1}\ \bar{3}\ \bar{1})^2 =$

$D(1)/D(10)/D(10\ \bar{1})/D(10\ \bar{1}\ \bar{2})$
$/D(10\bar{1}\bar{2}\bar{3})/\ D(10\bar{1}\bar{2}\bar{3}\bar{4})/\ D(10\bar{1}\bar{2}\bar{3}\bar{4}\bar{2})$
$/D(10\bar{1}\bar{2}\bar{3}\bar{4}\bar{2}\bar{1})D(10\bar{1}\bar{2}\bar{3}\bar{4}\bar{2}\bar{1}\bar{3})$
$/D(10\bar{1}\bar{2}\bar{3}\bar{4}\bar{2}\bar{1}\bar{3}\bar{1})/\ D(0\bar{1}\bar{2}\bar{3}\bar{4}\bar{2}\bar{1}\bar{3}\bar{1})$
$/D(\bar{1}\bar{2}\bar{3}\bar{4}\bar{2}\bar{1}\bar{3}\bar{1})D(\bar{2}\bar{3}\bar{4}\bar{2}\bar{1}\bar{3}\bar{1})$
$D(\bar{3}\bar{4}\bar{2}\bar{1}\bar{3}\bar{1})\ \ D(\bar{4}\bar{2}\bar{1}\bar{3}\bar{1})D(\bar{2}\bar{1}\bar{3}\bar{1})$
$D(\bar{1}\bar{3}\bar{1})\ \ D(\bar{3}\bar{1})\ \ D(\bar{1})$

$= 1/0/\bar{2}/\bar{4}/\bar{5}/\bar{4}/6/{}_18/{}_23/{}_32/{}_38/{}_36/{}_34$
$/{}_34/{}_21/{}_10/{}_11/6/1$

$= 1\,0\,\bar{2}\,\bar{4}\,\bar{5}\,\bar{4}\,8\,0\,6\,6\,1\,9\,7\,6\,2\,1\,1\,6\,1$
$= 9\,7\,5\,4\,6\,8\,0\,6\,6\,1\,9\,7\,6\,2\,1\,1\,6\,1$

Squares of the decimals can also be calculated by Duplex, ignoring the decimal, square of the number is determined by Duplex. But one has to count twice the number of decimal points given in the original number and fix up the decimal accordingly from the right in the answer.

5. $(23.345)^2$

Current Method	Vedic Method using Duplex
23.345	$(23.345)^2 =$
23.345	
116725	D(2)/ D(23)/ D(233)/ D(2334)/ D(23345)
93380	/ D(3345)/ D(345)/ D(45)/ D(5)
70035	
70035	$= 4/ {}_12/ {}_21/ {}_34/ {}_53/ {}_54/ {}_46/ {}_40/ {}_25$
46690	
544.989025	= 544.989025

Anurupyena Sutram for Squaring

Square of a two duplex number.

Let us consider $(11)^2$.The two digits bear the ratio l. Starting with the square of the first group, i.e.1, write down the Geometrical progression series upto 3 terms with the ratio 1. In this case such series consists of 1 1 1.

To the middle term, the same is added. (here 1). The answer for $(11)^2$ is

```
1   1   1
    1
```

Proof: $(ab)^2$, the ratio is $\frac{b}{a}$

1) $(ab)^2 =$ $\left(a^2, / \frac{a^2b}{a} \Rightarrow ab / ab\frac{b}{a} = b^2\right) = a^2/ab/b^2$

2) Starting with a^2

write down the three terms in GP with the ratio $\frac{b}{a}$

3) Adding the same term to the middle term ab we get $a^2 + 2ab + b^2$

This can be applied to write down the square of any number Let us consider 36^2.The ratio between the two digits is 6:3 = 2:1.

$\therefore$ 36^2 can be expanded as.

```
9,   9×2   18×2
      1     1
     18
___  ___   ___
 9   36    36
12   ̇9     6
```

$(36)^2 = 1\ 2\ 9\ 6$

This can be extended to any number containing more than two digits as well.

In all these cases, one has to rewrite the number in terms of two group units so that the above method can be carried out. For example in the case 383^2, we can construct two groups as (3) (83) or (38) (3). In both the cases the above method for two digit number is applied. Details are as follows.

This GP is continued to the next part (third part), thus completing the three parts. Next step is to put down below the second (middle) part, the same value of the second part as it is. Then addition is performed. While doing addition, the provision for the last two parts is dependent on the number of digits in the second group. All the remaining digits are carried out to the next left part.

(1) $(383)^2$

Current Method

```
    383
    383
   1149
  3064
   6889
 1149
 146689
```

Vedic Method

$(383)^2$

a b

Grouping 1: 3 (83)

$9,\ 9\times\frac{83}{3},\ 249\times\frac{93}{3}$

$a^2,\ a^2\frac{b}{a}\ \left(a^2\frac{b}{a}\right)\frac{b}{a}$

$(383)^2 = 9 \quad 249 \quad 6889$

$ 249$

$ 14 \quad 66 \quad 89$

$(383)^2 = 146689$

a b

Grouping 2: (38) 3

$a^2 \qquad \frac{a^2 b}{a}$

$38^2, \qquad \frac{38^2\times 3}{38}$

$\left(\frac{a^2 b}{a}\right)\frac{b}{a} \quad \frac{114\times 3}{38}$

$383^2 = 1444 \quad 114 \quad 9$

$ 114$

$ 1444 \quad 228 \quad 9$

$\therefore\ (383)^2 = 146689$

One more example is given below.

(2) $(897864)^2$

Current Method

```
      897864
      897864
     3591456
    5387184
   7182912
  6285048
 8080776
7182912
806159762496
```

Vedic Method

$(897864)^2$

a few groupings can also be worked out

Grouping1: (897) (864) — a = 897, b = 864

$\frac{b}{a} = \frac{864}{897}$

a^2 ab b^2

ab

```
804609    775008    746496
          775008
804609   1550016    746496
```

Ans = 806159762496

Grouping 2: (89) (7864) — a = 89, b = 7864

$\frac{b}{a} = \frac{7894}{89}$

a^2 ab b^2

ab

```
7921    699896    61842496
        699896
8061     5976       2496
```

Grouping 3: (8978) (64) — a = 8978, b = 64

$\frac{b}{a} = \frac{64}{8978}$

a^2 ab b^2

ab

```
80604484    574592    4096
            574592

80615976      24       06
```

$\therefore (8978642)^2 = 806159762496$

Further groupings also can be tried (89786) (4) and 8 (97864)

Squares of Polynomials

1) Urdhva tiryak $x^2 + 5x - 2$

$x^2 + 5x - 2$

Left to Right

1) $x^4 + 10x^3 + 21x^2 - 20x + 4$ same when multiplied right to left also. Refer to Lecture Notes I on multiplication

2) Dwandwayoga

$x^2 + 5x - 2$

$D(x^2) = x^4$

$D(x^2, 5x) = 10x^{-3}$

$D(x^2,\ 5x, -2) = -4x^2 + 25x^2 = 21x^2$

$D(5x,\ 2) = -20x^{'}$

$D(-2) = 4$ $\qquad x^4 + 10x^2 + 21x^2 - 20x + 4$

2) Urdhva Tiryak

$3x^2y^3 + xy^4 + \ + y^2z^3 + 10$

$\underline{3x^2y^3 + xy^4 + \ + y^2z^3 + 10}$

$9x^4y^6 + 6x^3y^7 + \ + 6x^2y^5z^3 + x^2y^8 + 60x^2y^3 + 2xy^6z^3 + 20xy^4 + y^4z^6 + 20y^2z^3 + 100$

Dwandwayoga

$D(3x^2y^3) = 9x^4y^6$

$D(3x^2y^3, xy^4) = 6x^3y^7$

$D(3x^2y^3, xy^4, y^2z^3) = 6x^2y^5z^3 + x^2y^8$

$D(3x^2y^3, xy^4, y^2z^3, 10) = 60x^2y^3 + 2xy^6z^3$

$D(xy^4, y^2z^3, 10) = 20xy^4 + y^4z^6$

$D(y^2z^3, 10) = 20y^2z^3$

$D(10) = 100$

3) Urdhva Tiryak

$xyz - xy^2 + 2xy - x^2z$

$x^2y^2z^2 - 2x^2y^3z - 4x^2y^2z + x^2y^4 - 2x^3yz^2 - 4x^2y^3 + 2x^3y^2z + 4x^2y^2 - 4x^3yz + x^4z^2$

Dwandwayoga

$D(xyz) = x^2y^2z^2$

$D(xyz, -xy^2) = -2x^2y^3z$

$D(xyz, -xy^2, 2xy) = 4x^2y^2z + x^2y^4$

$D(xyz, -xy^2, 2xy - x^2z) = -2x^3yz^2 - 4x^2y^3$

$D(-xy^2, 2xy, -x^2z) = -2x^3y^2z + 4x^2y^2$

$D(2xy, -x^2z) = -4x^3yz$

$D(-x^2z) = x^4z^2$

The Urdhva Tiryak and dwandwayoga are the most general for multiplication not only for numbers but also polynomials for any degree and having many variables as well. These are much simpler and elegant when compared to the Current Method.

xyz	$-$	xy^2	$+$	$2xy$	$-$	x^2z	(1st row)
xyz	$-$	xy^2	$+$	$2xy$	$-$	x^2z	(2nd row)
$-x^3yz^2$	$+$	x^3y^2z	$-$	$2x^3yz$	$+$	x^4z^2	Multiplication of 1st row with a^2z
$2x^2y^2z$	$-$	$2x^2y^3$	$+$	$4x^2y^2$	$-$	$2x^3yz$	Multiplication of 1st row with 2xy
$-x^2y^3z$	$+$	x^2y^4	$-$	$2x^2y^3$	$-$	$2x^3yz$	Multiplication of 1st row with xy^2
$x^2y^2z^2$	$-$	x^2y^3z	$+$	$2x^2y^2z$	$-$	x^3yz^2	Multiplication of 1st row with xy^3

$x^2y^2z^2 - 2x^3yz^2 + 2x^3y^2z - 4x^3yz + x^4z^2 + 4x^2y^2z - 2x^2y^3z - 4x^2y^3 + x^2y^4 + 4x^2y^2$

Note: It is interesting to note that in the current method, one has to pick – up the similar like terms and add (Refer example). Where as in the vedic methods it is, clearly seen that there is no necessity for such grouping.

For Example: Repetition of the term in the current system is clearly seen where as in Vedic method it is automatically clubbed during the process. As such there is no necessity for search for like terms.

Method gives directly the result

Vedic Method $x^2y^2z^2 - 2x^3y^2z$.______ by Urdhva Tiryak

This is same

1) $\begin{pmatrix} xyz \\ xyz \end{pmatrix} = x^2y^2z^2$

2) $\begin{pmatrix} xyz & -xy^2 \\ xyz & -xy^2 \end{pmatrix} = -2x^2y^3z$

3) $\begin{pmatrix} xyz & -xy^2 & 2xy \\ xyz & -xy^2 & 2xy \end{pmatrix} = 4x^2y^2z + x^2y^4$

4) $\begin{vmatrix} xyz - xy^2 & 2xy - x^2z \\ xyz - xy^2 & 2xy - x^2z \end{vmatrix} = -2x^3yz^2 - 4x^2y^3$

5) $\begin{matrix} -xy^2 & 2xy & -x^2z \\ -xy^2 & 2xy & -x^2z \end{matrix} = 2x^3y^2z + 4x^2y^2$

6) $\begin{pmatrix} 2xy & -x^2z \\ 2xy & -x^2z \end{pmatrix} - 4x^3yz$

7) $\begin{pmatrix} -x^2z \\ -x^2z \end{pmatrix} = x^4z^2$

or in the form of Duplexes

$D(xyz) + D(xyz, -xy^2) + D(xyz, -xy^2, 2xy) + D(xyz, -xy^2, 2xy, -x^2z) + D(-xy^2, 2xy, -x^2z) + D(2xy, -x^2z) + D(-x^2z)$

Square Roots (Vargamula) – Numbers

The procedure is similar to the straight division method. This is combined with dwandwayoga (duplex) process. A few first principles are significant in working out the square roots.

The relationship between square root and square is that if the square root contains n digits then the square must contain 2n or 2n–1 digits.

Conversely if given number has n digits, then square root contain $\frac{n}{2}$ or $\frac{n+1}{2}$ digits.

<u>Determination of square roots:</u>

The given number is arranged into groups of two digits from right to left. And if a digit is left out, it is also counted as a single group by itself. The working details are illustrated with examples.

Step(1):

Group the number consisting of two digits starting from right to left as 10 56 25

This group is necessary to frame the left hand most group. After this is over, it can be written 10: 5 6 2 5

Step(2):

Consider the square root of left most group i.e. (first group) such that the square of 1,2,...9 which fits nearly the considered group is obtained. Thus 3 becomes the first digit in the answer. The remainder 1 is placed between the left most group and the following digit. This gives rise to the gross dividend as 15.

```
6 |10 :  5 6 2 5
  |    1
  |3 :
   Q1
```

Step(3):

Common divisor is framed as twice the first digit of the answer. ($Q_1 = 3$)

i.e., $2 \times Q_1 = 6$

With the help of this divisor, the following steps are carried out.

Step(4):

The first gross dividend 15 is divided by the divisor 6. Quotient is 2 and the remainder is 3. This remainder generates the next gross dividend as 36

```
CD =  6   10:  5 6 2 5
            :  1 3
          3:   2
          Q1   Q2
```

Step(5):

From the gross dividend 36, we have to subtract duplex of Q_2, the second digit in the answer. After subtraction, (36 – 4) the value is 32 which is the new dividend.

Step(6):

Dividing above new dividend 32 by 6 we get 5 as the third digit of the answer (Q_3) and 2 as the remainder resulting in a gross dividend 22.

```
10:  5  6  2  5
        1  3  2
3 :     2  5
Q1      Q2 Q3
```

Step(7):

From this gross dividend 22 we have to deduct the duplex of (2, 5) (Q_1Q_2), which is equal to 20 . The new dividend thus formed is 2 (22 – 20 = 2)

Step(8):

The division of 2 by 6 gives the quotient Q_4 as zero with the remainder 2, resulting in gross dividend 25, From this gross dividend duplex of (2, 5, 0) ($Q_2Q_3Q_4$) has to be subtracted.

25-D(250) = 25 –25 =0

```
6 | 10:   5  6  2  5  0
  |   :   1  3  2  2  0
  |-----
   3 :    2  5  0
   Q1     Q2 Q3 Q4
```

The new dividend is 00.

Step(9):

This new dividend 00 is divided by 6 giving the quotient as 0 and remainder also 0

```
6  10:    5  6  2  5
    :   1  3  2  2  0
  3 :   2  5  .0  0
                 Q5
```

∴ Square root is 325

We can also confirm by squaring 325 = 105625

This has three groups. Hence the sq. root consists of 3 digits.

Placement of the decimal:

The question of placement of decimal is considered in relation to the number of groups in the given considered number i.e. the decimal point is placed in the answer after the number of digits equal to the number of groups considered in the given number (in case this is not a perfect square).

∴ 325.00 is the answer.

The following are some of the examples.

In the case of a perfect square after the decimal point there will be zero in the quotient and remainder also but from one particular stage onwards ie after the required number of digits in the sq roots are reached..

For the square roots of decimal numbers; the working is same as the above but the placement of the decimal in the answer is dependent on the decimal in the given number. The following are the principles that are adopted.

1) If the number starts with a decimal then the square root also starts with decimal.
2) If the decimal in the given problem is after n^{th} group then decimal in the square root.Starts after n^{th} digit in the answer

Straight Division method:
Application of Dwandwayoga(duplex) process:

Examples:

1. $\sqrt{105625}$

Current Method

3	105625	325
	9	
62	156	
	124	
645	3225	
	3225	
	0	

∴ 105625 is perfect square and its square root is 325.

Vedic Method

6	10 : 5 6 2 5 0
	1 3 2 2 0
	3 : 2 5 0 0

2. $\sqrt{27415696}$

Current Method

5	27415696	5236
	25	
102	241	
	204	
1043	3756	
	3129	
10466	62796	
	62796	
	0	

Vedic Method

	27:	4	1	5	6	9	6	
10		2	4	7	3	3	3	0
	5 :	2	3	6	0	0	0	

∴ 27415696 is a perfect square and its square root is 5236.

3. $\sqrt{602064369}$

Current Method

2	602064369	24537
	4	
44	202	
	176	
485	2606	
	2425	
4903	18143	
	14709	
49067	343469	
	343469	
	0	

Vedic Method

	6:	0	2	0	6	4	3	6	9
4	2	4	6	8	9	8	4	4	0
	2:	4	5	3	7 .	0	0	0	0

∴ 602064369 is a perfect square and its square root is 24537.

4. $\sqrt{236882881}$

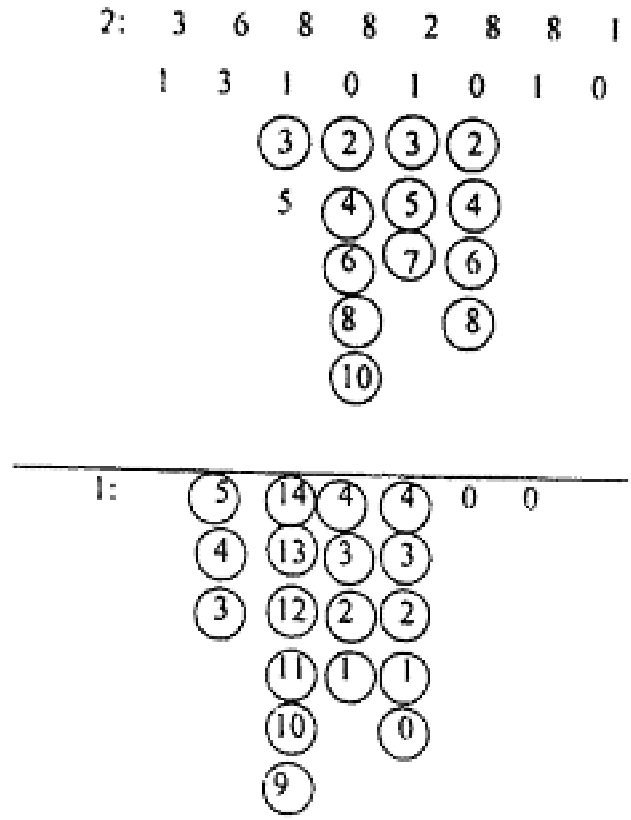
Current Method

1	236882881	15391
	1	
25	136	
	125	
303	1188	
	909	
3069	27928	
	27621	
30781	30781	
	0	

Vedic Method

	2 :	3	6	8	8	2	8	8	1
2	1	3	5	10	7	8	1	0	0
	1 :	5	3	9	1	0	0	0	0

By Reduction Process:



Two digit method:

30	236	8	8	2	8	8	1
	225	11	28	9	8	1	0
	15	3	9	1	0	0	0

∴ 236882881 is a perfect square and its square root is 15391.

5. $\sqrt{4020025}$

Current Method

2	4020025	2005
	4	
40	002	
	0	
400	200	
	0	
4005	20025	
	20025	
	0	

Vedic Method

	4 : 0 2 0 0 2 5
	/ / / / / /
4	0 0 2 0 0 2 0
	2 : 0 0 5 . 0 0 0

∴ 4020025 is a perfect square and its square root is 2005.

6. $\sqrt{81414529}$

Current Method

9	81414529	9023
	81	
18	0041	
	0	
1802	4145	
	3604	
18043	54129	
	54129	
	0	

Vedic Method

	81 : 41 45 29
	81 : 4 1 4 5 2 9
18	0 4 5 0 1 0 0
	9 : 0 2 3 . 0 0 0

∴ 81414529 is a perfect square and its square root is 9023.

7. $\sqrt{17.7241}$

Current Method

4	17.7241	4.21
	16	
82	172	
	164	
841	841	
	841	
	0	

Vedic Method

	17. : 7 2 4 1
8	1 1 0 0 0
	4. : 2 1 0 0

∴ 17.7241 is a perfect square and its square root is 4.21.

8. $\sqrt{45249}$

Current Method

2	45249	212.718
	4	
41	052	
	41	
422	1149	
	844	
4247	30500	
	29729	
42541	77100	
	42541	
425428	3455900	
	3403424	
	0	

Vedic Method

	4 : 5 2 4 9 0 0
4	0 1 3 2 7 8
	2 : 1 2. 7 1 8

4	4 5 2 4 9 0 0
4	0 1 3 2 3 0
	(7) (4)
	(8)
	2 1 2. 7 (2) (10) 3
	1(9)
	8

∴ 45249 is not a perfect square and its square root in decimal is 212.718

9. $\sqrt{9008763}$

Current Method

3	9008763	3001.4599
	9	
60	000	
	0	
600	087	
	0	
6001	8763	
	6001	
60024	276200	
	240096	
600285	3610400	
	3001425	
6002909	60897500	
	54026181	
6002909	687141900	
	540262701	
60029189	146879109	

Vedic Method

One digit method

	9	0	0	8	7	6	3	0	0
6	3	0	0	0	2	3	0	2	0
	3	0	0	1	4	6	0	2	

Two digit method

	900	8	7	6	3	0	0	0
60	30	0	8	27	36	21	2	32
	30	0	1	4	6	0	1	4

Three digit Method

	90087	6	3	0	0	0	0	0	0
600	90000	87	276	362	12	92	272	282	404
	300	1	4	6	0	1	4	4	6

Square root of 9008763 = 300.4601446

10. $\sqrt{32173}$

Current Method

1	32173	179.368
	1	
27	221	
	189	
349	3273	
	3141	
3583	13200	
	10749	
35866	245100	
	215196	
358728	2990400	
	2869824	
	120576	

Vedic Method

	3:	2	1	7	3	0 0
2	2	8	14	15	18	26
	1:	7	9	3	6	8

	3	2	1	7	3
2		2	0	0	1
			..8	.14	
	1	11	16	10	
		10	15		
		9	14		
		8	13		
		(7)	12		
			11		
			10		
			9		

Two digit Method:

	321	7	3	0	0	0	0
34	289	32	21	30	8	31	28
					42		
	17	(9)	(3)	7	(8)	3	
				(6)			

Sqrt of 32173 is 179.3683 (upto 4 decimals)

∴ 2736.3361 is a perfect square and its square root is 52.31

11. $\sqrt{2736.3361}$

Current Method

5	2736.3361	52.31
	25	
102	236	
	204	
1043	3233	
	3129	
10461	10461	
	10461	
	0	

Vedic Method

	27 :	3	6.	3	3	6	1		
10		2	3	2	1	0	0	0
	5 :	2 .	3	1	0	0	0		

12. $\sqrt{0.0423}$

Current Method

2	0423	.20566
	4	
40	23	
	0	
405	2300	
	2025	
4106	27500	
	24636	
41126	286400	
	246756	
	39644	

Vedic Method

	04:	2	3	0	0	0	0
4	4	0	2	3	2	3	2
				..6	..11	..14	
	2	0	5	7	8	12	1
				(6)	7	11	
					(6)	10	
						(9)	

Sqrt of (0.0423) is 0.205669 (upto 6 decimals)

13. $\sqrt{0.000623}$

Current Method

0	.000623	.024959
40	0	
	006	
	04	
44	223	
	176	
489	4700	
	4401	
4985	29900	
	24925	
49909	475000	
	449181	
	48319	

Vedic Method

	.0006	: 2		3		0	0
4	.004	2	2	3	2		
			..6	..11	..14		
	2	5	11	9	2		
		(4)	10	8			
			(9)	7			
				(6)			

Two digit Method

	.00062		0	0	0	0
	3					
4	576	4	3	11	2	
8		7	8			
				..5		
				0		
	24	9	6	0	1	
					0	
				$\bar{1}$		

$\Rightarrow 0.02496\bar{1} = 0.024959$

14. $\sqrt{0.32567}$

Current Method

5	.32567	.57067
	25	
107	756	
	749	
1140	770	
	0	
11406	77000	
	68436	
114127	856400	
	798889	
	57511	

Vedic Method

	32	5	6	7	0	0	0
10	25	7	5	7	7	6	2
				..17	..16		
	5	7	0	7	8	6	
				(6)	(7)		

0.057067 is the Square root of (0.32567)

15. $\sqrt{602064369}$

	6:	0	2	0	6	4	3	6	9
	2	4	6	8	9	8	4	4	0
	2:	4	5	3	7	0	0	0	0

Sqrt of (602064369) is 24537
2881) is 15391

16. $\sqrt{236882881}$

	2:	3	6	8	8	2	8	8	1
2	1	3	5	10	7	8	1	0	0
	1:	5	3	9	1	0	0	0	0

Sqrt of (23688

Square Roots of Polynomials

The method of solving the roots can be extended to polynomials as well. Here interestingly the square root of the term can be written in a form without remainder and hence need not be carried to the next term.

1) $\sqrt{4x^4+12x^3+29x^2+30x+25}$

It can be written in two ways. Ascending and descending order for final evaluation.

(1) Decreasing order: The given problem is to find out the square root of $4x^4 + 12x^3 + 29x^2 + 30x + 25$

The working details are in the form of steps with the concept of formation of a divisor, intermediate dividends, (ID) new dividends (ND), and Duplex Urdhva Tiryak sutram.

Step 1: First quotient

First quotient is square root of first terms. i.e. $2x^2$

In the answer Q_1 is the first term.

Step 2: Deriving the common divisor from Q_1

Common Divisor = $2\times Q_1 = 4x^2$ This is considered as the common divisor (CD) throughout the working.

Step 3: To divide the second term $12x^3$ with the common divisor to get the next quotient Q_2

I.e. $\frac{12x^3}{4x^2} = 3x$

Step 4: From the given dividend $29x^2$, one has to subtract duplex of $Q_2 = (9x^2)$

Then the result is divided by CD $4x^2$, the divisor, to get the quotient Q_3.

i.e. $\frac{20x^2}{4x^2} = 5$

Step 5: From the given dividend we have to subtract the duplex of (Q_2, Q_3) i.e. $2Q_2Q_3$.

Dividing the result by the CD $4x^2$ we get the quotient Q4 = $0x$

Step 6: The next dividend is considered for evaluation of Q_5.

From the given dividend subtract Duplex of (Q_2, Q_3, Q_4) $(3x, 5, \frac{0}{x^{-1}})$

The result is zero.

Dividing the result by the CD $4x^2$ we get the quotient Q_5 as $\frac{0}{x^{-2}}$

The square root of the given dividend is $\pm$ $(2x^2 + 3x + 5)$ in this example.

$25 + 30x + 29x^2 + 12x^3 + 4x^4 + 0$

Common Divisor 10 $25 + 30x + 29x^2 + 12x^3 + 4x^4$

Step1: $\sqrt{25} = 5\ (Q_1)$

5 is the first term in the answer.

Step2: Common Divisor = $2\,(Q_1) = 10$

Step3: $\frac{30x}{10} = 3x\,(Q_2)$

Step4: $29x^2 - D\,(3x) = 20x^2$

$= \frac{20x^2}{10} = 2x^2\,(Q_3)$

Step5: $12x^3 - D\,(3x, 2x^2) = 0$

$\frac{0}{10} = 0 \quad (Q_4)$

Step6: $4x^4 - D\,(3x, 2x^2, 0) = 0$

$\frac{0}{10} = 0\,(Q_5)$

$\therefore$ $(5 + 3x + 2x^2)$ is the exact square root of the expression.

Note: If the given polynomial has got a perfect square root, the evaluation of square roots in either ascending power series or descending powers series, gives the same result. Otherwise it is different as seen in the examples 5 and 6

(2) $\sqrt{9x^{10} + 6x^9 + x^8 + 12x^7 + 4x^6 + 36x^5 + 16x^4 + 24x^2 + 36}$

$6x^5 \,\big|\, 9x^{10} + 6x^9 + x^8 + 12x^7 + 4x^6 + 36x^5 + 16x^4 + 0 \quad + 24x^2 + 0 \quad + 36$

$\pm \;(3x^5 + x^4 + 0x^3 + 2x^2 + 0x + 6$

Square root is $\pm (3x^5 + x^4 + 2x^2 + 6)$. This is an exact Square.

Step1: $\sqrt{9x^{10}} = 3x^5 \quad (Q_1)$

Step2: Common Divisor $= 2 \times 3x^5 = 6x^5$

Step3: $Q_2 = \frac{6x^9}{6x^5} = x^4$

Step4: $x^8 - D\,(x^4) = 0$

$Q_3 = \frac{0}{6x^5} = 0$

Step5: $12x^7 - D\,(x^4, 0) = 12x^7$

$Q_4 = \frac{12x^7}{6x^5} = 2x^2$

Step6: $4x^6 - D\,(x^4, 0, 2x^2) = 0$

$= 4x^6 - 4x^6$

$= 0$

$Q_5 = \frac{0x^6}{6x^5} = 0$

<u>Step7:</u> $36x^5 - D(x^4, 0, 2x^2, 0) = 36x^5$

$Q_6 = \frac{36x^5}{6x^5} = 6$

<u>Step8:</u> $16x^4 - D(x^4, 0, 2x^2, 0, 6) = 0$

$Q_7 = \frac{0x^4}{6x^6} = \frac{0}{x}$

<u>Step9:</u> $0 - D(x^4 + 0 + 2x^2 + 0 + 6 + 0) = 0$

$Q_8 = \frac{0x^4}{6x^6} = \frac{0}{x^2}$

<u>Step10:</u> $24x^2 - D(x^4 + 0 + 2x^2 + 0 + 6 + 0 + 0) = 0$

$Q_9 = \frac{0}{6x^5} = 0$

<u>Step11:</u> $0 - D(x^4 + 0 + 2x^2 + 0 + 6 + 0 + 0 + 0) = 0$

$\frac{0}{6x^5} = 0 \ (Q_{10})$

<u>Step12:</u> $36 - D(x^4 + 0 + 2x^2 + 0 + 6 + 0 + 0 + 0 + 0)$

$= 36 - 36$

$= 0$

$\frac{0}{6x^5} = 0 \ (Q_{11})$

Square root $\pm \ (3x^5 + x^4 + 2x^2 + 6)$

(3) $\sqrt{x^6 - 10x^5 + 29x^4 - 26x^3 + 34x^2 - 12x + 9}$

$2x^3 \qquad x^6 - 10x^5 + 29x^4 - 26x^3 + 34x^2 - 12x + 9$

$\pm \left(x^3 - 5x^2 + 2x - 3 + \frac{0}{x} + \frac{0}{x^2} + \frac{0}{x^3}\right)$

This is an exact Square and its square root is $x^3 - 5x^2 + 2x - 3$

<u>Step1:</u> $\sqrt{x^6} = x^3 \ (Q_1)$

<u>Step2:</u> Common Divisor $= 2 \times Q_1$

$= 2x^3$

<u>Step3:</u> $\frac{-10x^5}{2x^3} = -5x^2 \ (Q_2)$

<u>Step4:</u> $29x^4 - D(-5x^2)$

$= 29x^4 - 25x^4$

$= 4x^4$

$\frac{4x^4}{2x^3} = 2x\ (Q_3)$

Step5: $-26x^3 - D(-5x^2, 2x)$

$= -26x^3 - (-20x^3)$

$= -6x^3$

$\frac{-6x^3}{2x^3} = -3\ (Q_4)$

Step6: $34x^2 - D(-5x^2, 2x, -3)$

$= 34x^2 - 34x^2$

$= 0$

$\frac{0}{2x^3} = 0\ (Q_5)$

Step:7 $-12x - D(-5x^2, 2x, -3, 0)$

$= -12x + 12x = 0$

$\frac{0}{2x^3} = 0\ (Q_6)$

Step8: $9 - D(-5x^2, 2x - 3, 0, 0)$

$= 9 - 9 = 0$

$\frac{0}{2x^3} = 0\ (Q_7)$

$\therefore$ Square root is $\pm\ (x^3 - 5x^2 + 2x - 3)$

(4) Obtain $\sqrt{4x^4 + 2x^3 + x^2 + 2x + 6}$ as a power series in x (decreasing)

$$4x^2 \,\Big|\, 4x^4 + 2x^3 + x^2 + 2x + 6$$

$$\pm\ \left(2x^2 + \frac{1}{2}x + \frac{3}{16} + \frac{29}{64x} + \frac{1411}{1024x^2} + \ldots\ldots.\right)$$

Step1: $\sqrt{4x^4} = 2x^2\ (Q_1)$

Step2: Common Divisor $= 2 \times 2x^2 = 4x^2$

Step3: $\frac{2x^3}{4x^2} = \frac{1}{2}x\ (Q_2)$

Step4: $x^2 - D\left(\frac{1}{2}x\right)$

$= x^2 - \frac{x^2}{4}$

$$= \frac{3x^2}{4}$$

$$\frac{3x^2}{4} \times \frac{1}{4x^2} = \frac{3}{16} = (Q_3)$$

Step5: $2x - D \mid \frac{1}{2}x, \frac{3}{16}$

$$= 2x - \frac{3x}{16} \qquad \frac{29x}{16}$$

$$\frac{29x}{16} \times \frac{1}{4x^2} - \frac{29}{64x} \quad (Q_4)$$

Step6: $6 - D \mid \frac{1}{2}x, \frac{3}{16}, \frac{29}{64x}$

$$= 6 - \frac{125}{256}$$

$$= \frac{1411}{256}$$

$$\frac{1411}{256} \times \frac{1}{4x^2} - \frac{1411}{1024x^2} \quad (Q_5)$$

The given expression has an imperfect square root

$$\pm \left(2x^2 + \frac{1}{2}x + \frac{3}{16} + \frac{29}{64x} + \frac{1441}{1024x^2} + ... \right.$$

(5) Obtain $\sqrt{9x^4 + 6x^3 + 5x^2 + 3x + 4}$
in decreasing power series of x.

$$6x^2 \;\Big|\; 9x^4 + 6x^3 + 5x^2 + 3x + 4$$

$$\pm\left(3x^2 + x + \frac{2}{3} + \frac{5}{18x} + \frac{1}{2x^2} - \frac{37}{162x^3} + \cdots\right)$$

Step1: $\sqrt{9x^4} = 3x^2 (Q_1)$

Step2: Common Divisor $= 2 \times 3x^2 = 6x^2$

Step3: $\frac{6x^3}{6x^2} = x\,(Q_2)$

Step4: $5x^2 - D(x)$

$$= 5x^2 - x^2$$

$$= 4x^2$$

$$\frac{4x^2}{6x^2} = \frac{2}{3} \ (Q_3)$$

<u>Step5:</u> $3x - D\left(x, \frac{2}{3}\right)$

$$= 3x - \frac{4x}{3}$$

$$= \frac{5x}{3}$$

$$\frac{5x}{3} \times \frac{1}{6x^2} = \frac{5}{18x} \ (Q_4)$$

<u>Step6:</u> $4 - D\left(x, \frac{2}{3}, \frac{5}{18x}\right)$

$$= 4 - 1 = 3$$

$$\frac{3}{6x^2} = \frac{1}{2x^2} \ (Q_5)$$

<u>Stpe7:</u> $0 - D\left(x, \frac{2}{3}, \frac{5}{18x}, \frac{1}{2x^2}\right)$

$$= 0 - \frac{37}{27x}$$

$$= \frac{-37}{27x}$$

$$\frac{-37}{27x} \cdot \frac{1}{6x^2} = \frac{-37}{162x^3} \cdot (Q_6)$$

The square root of the expression in decreasing power series in

$$\pm\left(3x^2 + x + \frac{2}{3} + \frac{5}{18x} + \frac{1}{2x^2} - \frac{37}{162x^3} + \ldots\right)$$

<u>increasing power series of x</u>

$$4 \,\big|\, 4 + 3x + 5x^2 + 6x^3 + 9x^4$$

$$\pm\left(2 + \frac{3}{4}x + \frac{71x^2}{64} + \frac{555x^3}{512} + \frac{25163x^4}{16384} + \ldots\right)$$

<u>Step1:</u> $\sqrt{4} = 2 \ (Q_1)$

<u>Step2:</u> Common Divisor $= 2 \times Q_1 = 2 \times 2 = 4$

<u>Step3:</u> $\frac{3x}{4} = \frac{3}{4}x \ (Q_2)$

<u>Step4:</u> $5x^2 - D\left(\frac{3}{4}x\right)$

$$= 5x^2 - \frac{9x^2}{16}$$

$$= \frac{71x^2}{16}$$

$$\frac{71x^2}{16} \times \frac{1}{4} = \frac{71x^2}{64} \quad (Q_3)$$

<u>Step5:</u> $6x^3 - D\left(\frac{3}{4}x, \frac{71x^2}{64}\right)$

$$= 6x^3 - \frac{213x^3}{128}$$

$$= \frac{555x^3}{128}$$

$$\frac{555x^3}{128} \times \frac{1}{4} = \frac{555x^3}{512} \quad (Q_4)$$

<u>Step6:</u> $9x^4 - D\left(\frac{3}{4}x, \frac{71x^2}{64}, \frac{555x^3}{512}\right)$

$$= 9x^4 - \frac{11701x^4}{4096}$$

$$= \frac{25163}{4096}x^4$$

$$\frac{25163}{4096}x^4 \times \frac{1}{4} = \frac{25163}{16384}x^4 \quad (Q_5)$$

The Square root of expression in increasing power series is

$$\pm\left(2 + \frac{3}{4}x + \frac{71x^3}{512} + \frac{25163x^4}{16384} + \dots\right)$$

This is evidently different from the square root obtained from decreasing power series

6) Obtain $\sqrt{x^4 + 6x^3 + 13x^2 + 14x + 16}$ in decreasing power series of x

$$2x^2 \;\Big|\; x^4 + 6x^3 + 13x^2 + 14x + 16$$

$$\pm\left(x^2 + 3x + 2 + \frac{1}{x} + \frac{3}{x^2} - \frac{11}{x^3} + \frac{53}{2x^4} + \dots\right)$$

<u>Step1:</u> $\sqrt{x^4} = x^2 \quad (Q_1)$

<u>Step2:</u> Common Divisor $= 2x^2$

Step3: $\frac{6x^3}{2x^2} = 3x \ (Q_2)$

Step4: $13x^2 - D(3x)$

$13x^2 - 9x^2$

$= 4x^2$

$\frac{4x^2}{2x^2} = 2 \ (Q_3)$

Step5: $14x - D(3x, 2)$

$= 14x - 12x$

$= 2x$

$\frac{2x}{2x^2} = \frac{1}{x} \ (Q_4)$

Step6: $16 - D\left(3x, 2, \frac{1}{x}\right)$

$= 16 - 10 = 6$

$\frac{6}{2x^2} = \frac{3}{x^2} \ (Q_5)$

Step7: $0 - D\left(3x, 2, \frac{1}{x}, \frac{3}{x^2}\right)$

$= \frac{-22}{x}$

$\frac{-22}{x} \times \frac{1}{2x^2} = \frac{-11}{x^3} \ (Q_6)$

Step8: $0 - D\left(3x, 2, \frac{1}{x}, \frac{3}{x^2}, -\frac{11}{x^3}\right)$

$= \frac{53}{x^2}$

$\frac{53}{x^2} \ \frac{1}{2x^2} = \frac{53}{2x^4} \ (Q_7)$

The square root of the expression in decreasing power series is

$$\pm\left(x^2 + 3x + 2 + \frac{1}{x} + \frac{3}{x^2} - \frac{11}{x^3} + \frac{53}{2x^4} + ...\right)$$

increasing power series of x

$$8 \;\big|\; 16 + 14x + 13x^2 + 6x^3 + x^4$$

$$\pm\left(4 + \frac{7x}{4} + \frac{159x^2}{128} + \frac{423x^3}{2048} - \frac{20741}{131072}x^4 + ...\right)$$

Step1: $\sqrt{16} = 4 \ (Q_1)$

Step2: Common Divisor = $2 \times 4 = 8$

Step3: $\frac{14x}{8} = \frac{7x}{4} (Q_2)$

Step4: $13x^2 - D\left(\frac{7x}{4}\right)$

$= 13x^2 - \frac{49x^2}{16}$

$= \frac{159x^2}{16}$

$\frac{159x^2}{16} \times \frac{1}{8} = \frac{159x^2}{128} \; (Q_3)$

Step5: $6x^3 - D\left(\frac{7x}{4}, \frac{159x^2}{128}\right)$

$= 6x^2 - \frac{1113x^3}{256}$

$= \frac{423x^3}{256}$

$\frac{423x^3}{256} \times \frac{1}{8} = \frac{423x^3}{2048} \; (Q_4)$

Step6: $x^4 - D\left(\frac{7x}{4}, \frac{159x^2}{128}, \frac{423x^3}{2048}\right)$

$= x^4 - \frac{37125}{16384}x^4$

$= -\frac{20741}{16384}x^4$

$-\frac{20741}{131072}x^4 \; (Q_5)$

The square to of the expression in increasing power series is

$$\pm\left(4 + \frac{7x}{4} + \frac{159x^2}{128} + \frac{428x^3}{2048} - \frac{20741}{131072}x^4 + \ldots\ldots\right)$$

7) Square Root of $x^2 + 8x + 16$

2x	$x^2 + 8x + 16$
	$x + 4 + 0 + 0$
	$Q_1 \; Q_2 \; Q_3$

1) $\sqrt{x^2} = x \longrightarrow Q_1$

2) Common Divisor = 2x

3) $\frac{8x}{2x} = 4 \longrightarrow Q_2$

4) $16 - D(4) = 16 - 16 = 0$

$\frac{0}{2x} = 0 \longrightarrow Q_3$

5) $0 - D(Q_2, Q_3) = 0 - 0 = 0$

$\frac{0}{2x} = 0 \longrightarrow Q_4$

$\therefore \sqrt{x^2 + 8x + 16} = \pm (x + 4)$

8) Square Root of $5x^2 + 7x + 9$

$2\sqrt{5x}$	$5x^2 \quad + 7x \quad + 9$
	$\pm \sqrt{5}x + \frac{7}{2\sqrt{5}} + \frac{131}{40\sqrt{5}}x^{-1} - \frac{917}{400\sqrt{5}}x^{-2}$
	$Q_1 \quad Q_2 \quad Q_3 \quad Q_4$

1) $\sqrt{5x^2} = \pm\sqrt{5}x \longrightarrow Q_1$

2) Common Divisor = $2\sqrt{5x}$

3) $(7x) \times \left(\frac{1}{2\sqrt{5}x}\right) = \frac{7}{2\sqrt{5}} \longrightarrow Q_2$

4) $9 - D\left(\frac{7}{2\sqrt{5}}\right) = 9 - \frac{49}{20} = \frac{131}{20} \times \frac{1}{2\sqrt{5}x} = \frac{131}{40\sqrt{5}}x^{-1} \longrightarrow Q_3$

5) $0 - D(Q_2, Q_3) = 0 - \frac{917}{400}x^{-1} - \frac{917}{400}x^{-1} \times \frac{1}{2\sqrt{5}x} = \frac{-917}{800\sqrt{5}}x^{-2} \longrightarrow Q_4$

6) $0 - D(Q_2, Q_3, Q_4) = 0 - \left(\frac{7}{2\sqrt{5}}\right)\left(-\frac{917}{800\sqrt{5}}x^{-2}\right) - \left(\frac{131}{40\sqrt{5}}x^{-1}\right)^2$

$= + \frac{6419}{8000}x^{-2} - \frac{1161}{8000}x^{-2} = \frac{5258}{8000}x^{-2} \times \frac{1}{2\sqrt{5}x} = \frac{2629}{8000\sqrt{5}}x^{-3} \longrightarrow Q_5$

$\therefore$ Square Root $= \pm \sqrt{5}x + \frac{7}{2\sqrt{5}} + \frac{131}{40\sqrt{5}}x^{-1} - \frac{917}{800\sqrt{5}}x^{-2} + \frac{2629}{8000\sqrt{5}}x^{-3}$

9) Square Root of $4x^8 + 12x^6 + 17x^4 + 12x^2 + 4$

$4x^4$	$4x^8 + 12x^6 + 17x^4 + 12x^2 + 4$
	$2x^4 + 3x^2 + 2 + 0 + 0 + 0 + 0$
	$Q_1 \quad Q_2 \quad Q_3 \quad Q_4 \quad Q_5 \quad Q_6 \quad Q_7$

1) $\sqrt{4x^8} = 2x^4 \longrightarrow Q_1$

2) Common Divisor = $4x^4$

3) $12x^6 \times \frac{1}{4x^4} = 3x^2 \longrightarrow Q_2$

4) $17x^4 - D(3x^2) = 17x^4 - 9x^4 = 8x^4 \times \frac{1}{4x^4} = 2 \longrightarrow Q_3$

5) $12x^2 - D(Q_2, Q_3) = 12x^2 - 12x^2 = (0) \times \left(\frac{1}{4x^4}\right) = 0 \longrightarrow Q_4$

6) $4 - D(Q_2, Q_3, Q_4) = 4 - 2(3x^2)(0) - (2)^2 = (0) \times \left(\frac{1}{4x^4}\right) = 0 \longrightarrow Q_5$

7) $0 - D(Q_2, Q_3, Q_4, Q_5) = 0 - 2(3x^4)(0) - 2(2)(0) = (0) \times \left(\frac{1}{4x^4}\right) = 0 \longrightarrow Q_6$

8) $0 - D(Q_2, Q_3, Q_4, Q_5, Q_6) = 0 - 2(3x^4)(0) - 2(2)(0) - (0)^2 = (0) \times \left(\frac{1}{4x^4}\right) = 0$

Square Root = $\pm\ (2x^4 + 3x^2 + 2)$

$\because$ Q_4, Q_5, Q_6, Q_7 are all zeroes it is a perfect square.

10) $f(x) = 5x^6 + 3x^5 + 2x^4 + x^3 - x^2 + 7x - 2$

$5x^6 + 3x^5 + 2x^4 + x^3 - x^2 + 7x - 2$

$$\pm\ \sqrt{5}\,x^3 + \frac{3}{2\sqrt{5}}x^2 + \frac{31}{40\sqrt{5}}x + \frac{107}{400\sqrt{5}} - \frac{10245}{16000\sqrt{5}}x^{-1} + \frac{584104}{160000\sqrt{5}}x^{-2} - \frac{6409909}{3200000\sqrt{5}}x^{-3} + \frac{2218811}{32\cdot 10^7\sqrt{5}}$$

$Q_1 \quad Q_2 \quad Q_3 \quad Q_4 \quad Q_5 \quad Q_6 \quad Q_7 \quad Q_8$

1) $\sqrt{5x^6} = \pm\sqrt{5}x^3 \longrightarrow Q_1$

2) Common Divisor = $2\sqrt{5}x^3$

3) $(3x^5) \times \left(\frac{1}{2\sqrt{5}x^3}\right) = \frac{3}{2\sqrt{5}}x^2 \longrightarrow Q_2$

4) $2x^4 - D(Q_2) = 2x^4 - \left(\frac{3}{2\sqrt{5}}x^2\right)^2 = 2x^4 - \frac{9}{20}x^4 = \left(\frac{31}{20}x^4\right) \times \left(\frac{1}{2\sqrt{5}x^3}\right) = \frac{31}{40\sqrt{5}}x \longrightarrow$

5) $x^3 - D(Q_2, Q_3) = x^3 - 2\left(\frac{3}{2\sqrt{5}}x^2\right)\left(\frac{31x}{40\sqrt{5}}\right) = x^3 - \frac{93}{200}x^3 = \frac{107}{200}x^3$

$\frac{107}{200}x^3 \times \frac{1}{2\sqrt{5}x^3} = \frac{107}{400\sqrt{5}} \longrightarrow Q_4$

6) $-x^2 - D(Q_2, Q_3, Q_4) = -x^2 - 2\left(\frac{3}{2\sqrt{5}}x^2\right)\left(\frac{107}{400\sqrt{5}}\right) - \left(\frac{31}{40\sqrt{5}}x\right)^2$

$-x^2 - \frac{321}{2000}x^2 - \frac{961}{8000}x^2 = -\frac{10245}{8000}x^2 \times \frac{1}{2\sqrt{5}x^3} = \frac{-10245}{16000\sqrt{5}}x^{-1} \longrightarrow Q_5$

7) $7x - D(Q_2, Q_3, Q_4, Q_5) = 7x - 2\left(\frac{3}{2\sqrt{5}}x^2\right)\left(\frac{-10245}{16000\sqrt{5}}x^{-1}\right) - 2\left(\frac{31}{40\sqrt{5}}x\right)\left(\frac{107}{400\sqrt{5}}\right) = \frac{58410[illegible]}{8000[illegible]}$

$$\frac{584101}{80000} \times \frac{1}{2\sqrt{5}x} = \frac{584101}{160000\sqrt{5}}x^{-2} \longrightarrow Q_6$$

8) $-2 - D(Q_2, Q_3, Q_4, Q_5, Q_6) = -2 - 2\left(\frac{3}{2\sqrt{5}}x^2\right)\left(\frac{584101}{16000\sqrt{5}}x^{-2}\right) - 2\left(\frac{31}{40\sqrt{5}}x\right)$

$$\left(\frac{-10245}{16000\sqrt{5}}x^{-1}\right) - \left(\frac{107}{400\sqrt{5}}\right)^2 = \frac{-6409909}{1600000}$$

$$\frac{-6409909}{1600000} \times \frac{1}{2\sqrt{5}x^3} = \frac{-6409909}{3200000\sqrt{5}}x^{-3} \longrightarrow Q_7$$

9) $-D(Q_2, Q_3, Q_4, Q_5, Q_6, Q_7)$

$$= -2\left(\frac{3}{2\sqrt{5}}x^2\right)\left(\frac{-6409909}{3200000\sqrt{5}}x^{-3}\right) - 2\left(\frac{31}{40\sqrt{5}}x\right)\left(\frac{584101}{160000\sqrt{5}}x^{-2}\right)$$

$$-2\left(\frac{107}{400\sqrt{5}}\right)\left(\frac{-10245}{16000\sqrt{5}}x^{-1}\right) = \frac{2218811}{16 \times 10^6}x^{-1}$$

$$\frac{2218811}{16 \times 10^6} \times \frac{1}{2\sqrt{5}x^3} = \frac{2218811}{32 \times 10^6 \times \sqrt{5}}x^{-4} \longrightarrow Q_8$$

Square Root =

$$\left([illegible] + \frac{3}{2\sqrt{5}}x^2 + \frac{31}{40\sqrt{5}}x + \frac{107}{400\sqrt{5}} - \frac{10245}{16000\sqrt{5}}x^{-1} + \frac{584101}{160000\sqrt{5}}x^{-2} - \frac{6409909}{3200000\sqrt{5}}x^{-3} + \frac{2218811}{32000000\sqrt{5}}x^{-4} + \text{------}\right)$$

11) $f(x) = [illegible] + 2x^3 + x^2 + 2x + 4 + \frac{2}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{1}{x^4}$

$2x^2$ | [illegible] $+ 4 + \frac{2}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{1}{x^4}$

$x^2 + x + 0 + \frac{1}{x} + \frac{1}{x^2} + 0 + 0 + 0 + 0$

$Q_1 \quad Q_2 \quad Q_3 \quad Q_4 \quad Q_5 \quad Q_6 \quad Q_7 \quad Q_8 \quad Q_7$

1) $\sqrt{x^4} = x^2 \longrightarrow Q_1$

2) Common Divisor : $2x^2$

3) $2x^3 \times \frac{1}{2x^2}$ [illegible] $\longrightarrow Q_2$

4) $x^2 - D(Q_2)$ [illegible] $= 0 \times \frac{1}{2x^2} = 0 \longrightarrow Q_3$

5) $2x - D(Q_2, Q_3) = 2x - 2(x)(0) = 2x \times \frac{1}{2x^2} = \frac{1}{x}$ ⟶ Q_4

6) $4 - D(Q_2, Q_3, Q_4) = 4 - 2(x)\left(\frac{1}{x}\right) - (0)^2 = 2 \times \frac{1}{2x^2} = \frac{1}{x^2}$ ⟶ Q_5

7) $\frac{2}{x} - D(Q_2, Q_3, Q_4, Q_5) = \frac{2}{x} - 2(x)\left(\frac{1}{x^2}\right) - 2(0)\left(\frac{1}{x}\right) = 0 \times \frac{1}{2x^2} =$ ⟶ Q_6

8) $\frac{1}{x^2} - D(Q_2, Q_3, Q_4, Q_5, Q_6) = \frac{1}{x^2} - 2(x)(0) - 2(0)\left(\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)^2 = 0 \times$

$0 \times \frac{1}{2x^2} = 0$ ⟶ Q_7

9) $\frac{2}{x^3} - D(Q_2, Q_3, Q_4, Q_5, Q_6, Q_7) = \frac{2}{x^3} - 2(x)(0) - 2(0)(0) - 2\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\right) = 0$

$0 \times \frac{1}{2x^2} = 0$ ⟶ Q_8

10) $\frac{1}{x^4} - D(Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8) = \frac{1}{x^4} - 2(x)(0) - 2(0)(0) - 2\left(\frac{1}{x}\right)(0) - \left(\frac{1}{x^2}\right)^2 = 0$

$0 \times \frac{1}{2x^2} = 0$ ⟶ Q_9

11) $0 \times \frac{1}{x^5} - D(Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9) = 0 \times \frac{1}{x^5} - 2(x)(0) - 2(0)(0)$

$2\left(\frac{1}{x}\right)(0) + 2\left(\frac{1}{x^2}\right)(0) = 0$

$0 \times \frac{1}{2x^2} = 0$ ⟶ Q_{10}

(SR) Square Root = $\left(x^2 + x + 0 + \frac{1}{x} + \frac{1}{x^2}\right)$

$D(SR) = x^4 + 2x^3 + x^2 + 2x + 2 + 2 + \frac{2}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{1}{x^4}$

12) Square root of $x^2 + 2x + 2$

$$\begin{array}{l|l} 2x & x^2 + 2x + 2 + 0 + 0 \\ \hline & x + 1 + \frac{1}{2x} - \frac{1}{2x^2} + \frac{3}{8x^3} + \ldots \\ & Q_1 \quad Q_2 \quad Q_3 \quad Q_4 \quad Q_5 \end{array}$$

Verification:

$D(Q_1\, Q_2\, Q_3\, Q_4\, Q_5)$

$\Rightarrow$ $D(Q_1) + D(Q_1\,Q_2) + D(Q_1\,Q_2\,Q_3) + D(Q_1\,Q_2\,Q_3\,Q_4) + D(Q_1\,Q_2\,Q_3\,Q_4\,Q_5) +$
$D(Q_2\,Q_3\,Q_4\,Q_5) + D(Q_3\,Q_4\,Q_5) + D(Q_4\,Q_5) + D(Q_5)$

$D(Q_1) = Q_2\,x^2$

$D(Q_1\,Q_2) = 2\,Q_1Q_2 = 2x$

$D(Q_1\,Q_2\,Q_3) = 2\,Q_1\,Q_3 + Q_2^2 = 2(x)\left(\frac{1}{2x}\right) + 1 = 2$

$D(Q_1\,Q_2\,Q_3\,Q_4) = 2\,Q_1\,Q_4 + 2\,Q_2\,Q_3 = 2(x)\left(-\frac{1}{2x^2}\right) + 2(1)\frac{1}{(2x)} = 0$

$D(Q_1\,Q_2\,Q_3\,Q_4\,Q_5) = 2\,Q_1\,Q_5 + 2\,Q_2\,Q_4 + Q_3^2 = \frac{3}{4x^2} - \frac{1}{x^2} + \frac{1}{4x^2} = 0$

$D(Q_2\,Q_3\,Q_4\,Q_5) = 2\,Q_2\,Q_5 + 2\,Q_3\,Q_4 = \frac{3}{4x^3} - \frac{2}{4x^3} = \frac{1}{4x^3}$

$D(Q_3\,Q_4\,Q_5) = \frac{3}{8x^4} + \frac{1}{4x^4} \quad \frac{5}{8x^4}$

$D(Q_4\,Q_5) = 2\,Q_4\,Q_5 = 2\left(\frac{-1}{2x^2}\right)\left(\frac{3}{8x^3}\right) \quad \frac{-3}{8x^4}$

$D(Q_5) = Q_5^2 = \frac{9}{64x^6}$

A few points to be noted:

If the given polynomial has n terms then the square root will have $\frac{n}{2}$ (or) $\frac{n+1}{2}$ (even or odd respectively) terms, if it is a perfect square. After completing the working of the square roots $\frac{n}{2}$ or $\frac{n+1}{2}$ terms, if the square root contains non–vanishing terms continuously, then one has to treat it as an imperfect square. If it shows continuously vanishing terms after the required number then it represents a perfect square.

If it is a perfect square, the square root will be the same whether it is worked out either in ascending or descending series of x. It is not so in case of imperfect Square.

1) Perfect square
   a) If (a) DQ's (Duplex of quotients) gives the given expression exactly. Without any additional term
   b) Following enbloc Duplexes are vanishing, then the given expression is a perfect square

2) If the condition 'b' is not satisfied i.e. enbloc Duplexes are not vanishing, then the given expression is not perfect Square

3) In case of imperfect square one can proceed to obtain the square root of the given polynomial to contain terms having x^{-1}, x^{-2}, x^{-3},……. etc upto a desired inverse power of x.(descending order) or incase of increasing power series square roots contain increasing powers of x.

4) Correctness of the terms of the given polynomial including the zero coefficients of the other terms

13) $1 + 2x + x^2$

Ascending order

$2x \mid 1 + 2x + x^2.$

$1 \quad x \quad 0$

Square root = $1 + x$

Deseending order

Square root $x^2 + 2x + 1$

$2x \mid x^2 + 2x + 1 + 0 +$

n = 3 ∴ Square root will have t two terms if it is a perfect square.

$x + 1 + \frac{0}{2x} - \frac{0}{2x^2} \quad \frac{0}{2x^3}$

$Q_1 \quad Q_2 \quad Q_3 \quad Q \quad Q_5$

Verification:

$D(Q_1\, Q_2\, Q_3\, Q_4\, Q_5) = D(Q_1) + D(Q_1\, Q_2) + D(Q_1\, Q_2\, Q_3) + D(Q_1\, Q_2\, Q_3\, Q_4) + D(Q_1\, Q_2\, Q_3\, Q_4\, Q_5) + D(Q_2\, Q_3\, Q_4\, Q_5) + D(Q_3\, Q_4\, Q_5) + D(Q_4\, Q_5) + D(Q_5)$

$D(Q_1) = Q_1^2 = x^2$

$D(Q_1\, Q_2) = 2\, Q_1\, Q_2 = 2x$

$D(Q_1\, Q_2\, Q_3) = 2\, Q_1\, Q_3 + Q_2^2 = 0 + 1 = 1$

$D(Q_1\, Q_2\, Q_3\, Q_4) = 2\, Q_1\, Q_4 + 2\, Q_2\, Q_3 = 0$

$D(Q_1\, Q_2\, Q_3\, Q_4\, Q_5) = 2\, Q_1\, Q_5 + 2\, Q_2\, Q_4 + Q_3^2 = 0$

$D(Q_2\, Q_3\, Q_4\, Q_5) = 2\, Q_2\, Q_5 + 2\, Q_3\, Q_4 = 0$ (ascending order)

$D(Q_3, Q_4\, Q_5) = 2\, Q_3\, Q_5 + Q_4^2 = 0$

$D(Q_4\, Q_5) = 2\, Q_4 Q_5 = 0$

$D(Q_5) = 2\, Q_5^2 = 0$

$D(Q_5) = Q_5^2 = 0$

14) Consider the Square Root of $9x^4 - 12x^3 - 20x^2 + 16x + 16$

$6x^2 \mid 9x^4 - 12x^3 - 20x^2 + 16x + 16 + 0x^{-1} + 0x^{-2} + 0x^{-3}$

$3x^2 \quad - 2x \quad - 4 \quad + 0 \quad + 0 + \quad 0 \quad + 0 \quad + 0$

$Q_1 \quad Q_2 \quad Q_3 \quad Q_4 \quad Q_5 \quad Q_6 \quad Q_7 \quad Q_8$

<u>**Step1**</u>: $\sqrt{9x^4} = 3x^2\ (Q_1)$

<u>**Step2**</u>: Common Divisor = $2\, Q_1 = 6x^2$

Step3: $-12x^3 \times \frac{1}{6x^2} = -2x\ (Q_2)$

Step4: $-20x^2 - D(Q_2) = -20x^2 - 4x^2$; $-24x^2 \quad \frac{1}{6x^2} \mid = -4(Q_3)$

Step5: $16x - D(Q_2\,Q_3) = 16x - 2(-2x)(-4)$; $0 \mid \frac{1}{6x^2} \quad 0(Q_4)$

Step6: $16x - D(Q_2\,Q_3\,Q_4) = 16 - 2(-2x)(0) - (-4)^2$; $0 \times \left(\frac{1}{6x^2}\right) = 0(Q_5)$

$\therefore$ The Square Root is $(3x^2 - 2x - 4)$

Suppose we continue the problem considering $0x^{-1}$, then

Step7: $0x^{-1} - D(Q_2\,Q_3\,Q_4\,Q_5) = 0x^{-1} - 2(-2x)(0) - 2(-4)(0)$; $0 \quad 6x^2 \quad = 0(Q_6)$

Step8: $0x^{-2} - D(Q_2\,Q_3\,Q_4\,Q_5\,Q_6) = 0x^{-2} - 2(-2x)(0) - 2(-4)(0) + (0)^2$; $0 \left(\frac{1}{6x^2}\right) = 0(Q_7)$

Step9: $0x^{-3} - D(Q_2\,Q_3\,Q_4\,Q_5\,Q_6\,Q_7) = 0x^{-3} - 2(-2x)(0) - 2(-4)(0) - 2\ (0.0)^4$; $0 \quad \frac{1}{6x^2}$

$0(Q_8)$ One can see that all Q's are becoming zeroes. This is a clear indication that the given expression is a perfect square one can also verify by squaring the answer (by duplex method)

Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8
$3x^2$	$-2x$	-4	0	0	0	0	0

$Q_1 = 3x^2$ $\quad Q_2 = -2x$ $\quad Q_3 = 4$ $\quad$ others are zero

$D(Q_1) = 9x^4$	enbloc terms
$D(Q_1\,Q_2) = -12x^3$	$D(Q_2\,Q_3\,Q_4\,Q_5\,Q_6\,Q_7\,Q_8) = 0$
$D(Q_1\,Q_2\,Q_3) = -20x^2$	$D(Q_3\,Q_4\,Q_5\,Q_6\,Q_7\,Q_8) = 0$
$D(Q_1\,Q_2\,Q_3\,Q_4) = 16x$	$D(Q_4\,Q_5\,Q_6\,Q_7\,Q_8) = 0$
$D(Q_1\,Q_2\,Q_3\,Q_4\,Q_5) = 16$	$D(Q_5\,Q_6\,Q_7\,Q_8) = 0$
$D(Q_1\,Q_2\,Q_3\,Q_4\,Q_5\,Q_6) = 0$	$D(Q_6\,Q_7\,Q_8) = 0$
$D(Q_1\,Q_2\,Q_3\,Q_4\,Q_5\,Q_6\,Q_7) = 0$	$D(Q_7\,Q_8) = 0$
$D(Q_1\,Q_2\,Q_3\,Q_4\,Q_5\,Q_6\,Q_7\,Q_8) = 0$	$D(Q_8) = 0$

$\therefore$ $(3x^2 - 2x - 4)^2$ = given Dividend.

15) Now let us consider the of $9x^4 - 12x^3 - 20x^2 + 16x + 13$

$6x^2 \,\lfloor\, \underline{9x^4 - 12x^3 - 20x^2 + 16x + 13}$

$3x^2 \;-\; 2x \;-\; 4 \;+\; 0 \;-\; \frac{\cdot}{2}x$

$Q_1 \quad Q_2 \quad Q_3$

Step1: $\sqrt{9x^4} = 3x^2\ (Q_1)$

Step2: Common Divisor $= 2\,Q_1 = 6x^2$

Step3: $12x^3\left(\frac{1}{6x^2}\right) = -2x\ (Q_2)$

Step4: $-20x^2 - D(Q_2)\ ;\ -24x^2\left(\frac{1}{6x^2}\right) = -4(Q_3)$

Step5: $16x - D(Q_2\ Q_3)\ ;\ 0\left(\frac{1}{6x^2}\right) = 0(Q_4)$

Step6: $13 - D(Q_2\ Q_3\ Q_4)\ ;\ -3\left(\frac{1}{6x^2}\right) = -\frac{1}{2}x^{-2}\ (Q_5)$

Now at this stage if one starts verifying the result then it is as follows

$3x^2 - 2x - 4 + 0 - \frac{1}{2}x^{-2}$

$Q_1\quad Q_2\quad Q_3\quad Q_4\quad Q_5$

	Enbloc terms
$D(Q_1) = 9x^4$	$D(Q_2\ Q_3\ Q_4\ Q_5) = 2x^{-1}$
$D(Q_1\ Q_2) = -12x^3$	$D(Q_3\ Q_4\ Q_5) = 4x^{-2}$
$D(Q_1\ Q_2\ Q_3) = -20x^2$	$D(Q_4\ Q_5) = 0$
$D(Q_1\ Q_2\ Q_3\ Q_4) = 16x$	$D(Q_5) = \frac{1}{4}x^{-4}$
$D(Q_1\ Q_2\ Q_3\ Q_4\ Q_5) = 13$	

All these enbloc terms clearly show that the given expression is not a perfect square Continuing further with the evaluation of square root one gets the following results:

$$6x^2 \,\underline{|\ 9x^4 - 12x^3 - 20x^2 + 16x + 13 + 0x^{-1} + 0x^{-2} + 0x^{-3}}$$

$$3x^2 - 2x - 4 + 0x^{-1} - \frac{1}{2}x^{-2} - \frac{1}{3}x^{-3} - \frac{8}{9}x^{-4} - \frac{28}{27}x^{-5}$$

$$Q_1\quad Q_2\quad Q_3\quad Q_4\quad Q_5\quad Q_6\quad Q_7\quad Q_8$$

Step7: $0x^{-1} - D(Q_2\ Q_3\ Q_4\ Q_5) = 0x^{-1} - 2(-2x)\left(-\frac{1}{2}x^{-2}\right) - 2(0x^{-1}) = -2x^{-1}$

$;\ -2x^{-1}\left(\frac{1}{6x^2}\right) = -\frac{1}{3}x^{-3}\ (Q_6)$

Step8: $0x^{-2} - D(Q_2\ Q_3\ Q_4\ Q_5\ Q_6) = 0x^{-2} - 2(-2x)\left(-\frac{1}{3}x^{-3}\right) - 2(-4)\left(-\frac{1}{2}x^{-2}\right) + (0x^{-1})^2$

$= -\frac{4}{3}x^{-2} - 4x^{-2}\ ;\ -\frac{16}{3}x^{-2}\left(\frac{1}{6x^2}\right) = -\frac{8}{9}x^{-4}$

Step9: $0x^{-3} - D(Q_2\ Q_3\ Q_4\ Q_5\ Q_6\ Q_7) = 0x^{-3} - 2(-2x)\left(-\frac{8}{9}x^{-4}\right) - 2(-4)\left(-\frac{1}{3}x^{-3}\right) - 2(0x^{-1})\left(-\frac{1}{2}x^{-2}\right)$

$= -\frac{32}{9}x^{-3} - \frac{8}{3}x^{-3} = -\frac{56}{9} \times \frac{1}{6x^2}x^{-3}\ (Q_8)$

Problems from British Authors Book

16) Find the Square root of $x^6 + 3x^5 + 4x^4 + 2x^3 + x^2 + 5x + 1$

$$2x^3 \,\underline{|\; x^6 + 3x^5 + 4x^4 + 2x^3 + x^2 + 5x + 1}$$

$$x^3 + \frac{3}{2}x^2 + \frac{7}{8}x - \frac{5}{16} + \frac{75}{128}x^{-1} + \frac{485}{256}x^{-2} - \frac{843}{1024}x^{-3} + \ldots$$

$Q_1 \quad Q_2 \quad Q_3 \quad Q_4 \quad Q_5 \quad Q_6 \quad Q_7$

One can extend the square root to any required power of x in case of imperfect squares.

Step1: $\sqrt{x^6} = x^3$ (Q_1)

Common Divisor = $2x^3$

Step2: $(3x^5)\left(\frac{1}{2x^3}\right) = \frac{3}{2}x^2$ (Q_2)

Step3: $4x^4 - D(Q_2) = 4x^4 - \left(\frac{3}{2}x^2\right)^2 = \frac{7}{4}x^4$

$\frac{7}{4}x^4\left(\frac{1}{2x^3}\right) = \frac{7}{8}x$ (Q_3)

Step4: $2x^3 - D(Q_2\,Q_3) = 2x^3 - (2)\left(\frac{3}{2}x^2\right)\left(\frac{7}{8}x\right) = \left(-\frac{5}{8}x^3\right)\left(\frac{1}{2x^3}\right) = \frac{-5}{16}$ (Q_4)

Step5: $x^2 - D(Q_2\,Q_3\,Q_4) = x^2 - (2)\left(\frac{3}{2}x^2\right)\left(-\frac{5}{16}\right) - \left(\frac{7}{8}x\right)^2 = x^2 + \frac{15}{16}x^2 - \frac{49}{64}x^2 = \frac{75}{64}.$

$\frac{75}{64}x^2\left(\frac{1}{2x^3}\right) = \frac{75}{128}x^{-1}$ (Q_5)

Step6: $5x - D(Q_2\,Q_3\,Q_4\,Q_5) = 5x - 2\left(\frac{3}{2}x^2\right)\left(\frac{75}{128}x^{-1}\right) - 2\left(\frac{7}{8}x\right)\left(-\frac{5}{16}\right) = 5x - \frac{225}{128}x + \frac{35}{64} = \frac{485x}{128}$

$\frac{485}{128}x \times \frac{1}{2x^3} = \frac{485}{256}x^{-2}$ (Q_6)

Step7: $- D(Q_2\,Q_3\,Q_4\,Q_5\,Q_6) = 1 - (2)\left(\frac{3}{2}x^2\right)\left(\frac{485}{256}x^{-2}\right) - (2)\left(\frac{7}{8}x\right)\left(\frac{75}{128}x^{-1}\right) - \left(-\frac{5}{16}\right)^2$

$= 1 - \frac{1455}{256} \quad \frac{525}{512} \quad \frac{25}{256} \quad \frac{2973}{512}$

$-\frac{2973}{512}\left(\frac{1}{2x^3}\right) = -\frac{2973}{1024}x^{-3}$ (Q_7)

one can extend the square root to any required power of x in case of imperfect squares.

Consider the increasing powers of x

$2 \,\lfloor\, 1 + 5x + x^2 + 2x^3 + 4x^4 + 3x^5 + x^6$

$\pm \quad 1 + \frac{5}{2}x - \frac{21}{8}x^2 + \frac{121}{16}x^3 + \frac{2605}{128}x^4 + \frac{18491}{256}x^5 - \frac{67403}{512}x^6 + \ldots$

$Q_1 \quad Q_2 \quad Q_3 \quad Q_4 \quad Q_5 \quad Q_6 \quad Q_7$

<u>Step1</u>: $\sqrt{1} = \pm 1$

Common Divisor = $2Q_1 = 2$

<u>Step2</u>: $(5x)\left(\frac{1}{2}\right) = \frac{5}{2}x\ (Q_2)$

<u>Step3</u>: $x^2 - D(Q_2) = x^2 - \frac{25}{4}x^2 = -\frac{21}{4}x^2 \times \frac{1}{2} = \frac{-21}{8}x^2\ (Q_3)$

<u>Step4</u>: $2x^3 - D(Q_2\,Q_3) = 2x^3 - 2\left(\frac{5}{2}x\right)\left(-\frac{21}{8}x^2\right) = 2x^3 + \frac{105}{8}x^3; \frac{121}{8}x^3 \times \frac{1}{2} = \frac{121}{16}x^3\ (Q_4)$

<u>Step5</u>: $4x^4 - D(Q_2\,Q_3\,Q_4) = 4x^4 - 2\left(\frac{5}{2}x\right)\left(\frac{121}{16}x^3\right) - \left(-\frac{21}{8}x^2\right)^2 = 4x^4 - \frac{605}{16}x^4 - \frac{441}{64}x^4$

$-\frac{2605}{64}x^4 \times \frac{1}{2} = -\frac{2605}{128}x^4\ (Q_5)$

<u>Step6</u>: $3x^5 - D(Q_2\,Q_3\,Q_4\,Q_5) = 3x^5 - 2\left(\frac{5}{2}x\right)\left(-\frac{2605}{128}x^4\right) - 2\left(-\frac{21}{8}x^2\right)\left(\frac{121}{16}x\right.$

$= 3x^5 + \frac{13025}{128}x^5 + \frac{2541}{64}x^5; \frac{18491}{128}x^5\left(\frac{1}{2}\right) = \frac{18491}{256}x^5\ (Q_6)$

<u>Step7</u>: $x^6 - D(Q_2\,Q_3\,Q_4\,Q_5\,Q_6) = x^6 - 2\left(\frac{5}{2}x\right)\left(\frac{18491}{256}x^5\right) - 2\left(-\frac{21}{8}x^2\right)\left(-\frac{2605}{128}x^4\right) - \left(\frac{121}{16}x^3\right.$

$= x^6 - \frac{92455}{256}x^6 - \frac{54705}{512}x^6 - \frac{14641}{256}x^6; \frac{-268385}{512}x^6 \times \frac{1}{2} = -\frac{268385}{1024}x^6\ (Q_7)$

Square Root $\pm \left(1 + \frac{5}{2}x - \frac{21}{8}x^2 + \frac{121}{16}x^3 - \frac{2605}{128}x + \frac{5082}{256}x^5 - \frac{268385}{1024}x^6 + \ldots\right)$

17) Find the square root of $9 + 12x + 4x^2$

$6x \,\lfloor\, 9 + 12x + 4x^2$

$\pm 3 + 2x + 0$

$Q_1 \quad Q_2 \quad Q_3$

<u>Step1</u>: $\sqrt{9} = \pm 3\ (Q_1)$

Divisor = $2(Q_1) = 6$

<u>Step2</u>: $(12x)\left(\frac{1}{6}\right) = 2x\ (Q_2)$

<u>Step3</u>: $4x^2 - D(Q_2) = 0 \times \frac{1}{6} = 0\ (Q_3)$

Square Root is $(2x + 3)$

18) Consider the example where the powers of x are written in the descending order.
$f(x) = 3x^5 + 2x^4 + x^3 + 4x^2 + 5x + 1$

<u>1st step</u>: is to obtain the square root of the first term $3x^5 = \sqrt{3x^5} = \pm\sqrt{3}x^{5/2}$
This is the first quotient Q_1.

$2\sqrt{3}x^{5/2}$ (Divisor)	$3x^5$	$+ \; 2x^4$	$+ \; x^3$	$+ \; 4x^2$	$+ \; 5x$	$+ \; 1$
	$\sqrt{3}x^{5/2}$	$+ \; \frac{x^{3/2}}{\sqrt{3}}$	$+ \; \frac{x^{1/2}}{3\sqrt{3}}$	$+ \; \frac{17x^{-1/2}}{9\sqrt{3}}$	$+ \; \frac{50x^{-3/2}}{27\sqrt{3}}$	$- \; \frac{53x^{-5/2}}{162\sqrt{3}}$
	Q_1	$+ \; Q_2$	$+ \; Q_3$	$+ \; Q_4$	$+ \; Q_5$	$+ \; Q_6$

<u>2nd step</u>: is to write down the divisor i.e. $2Q_1 = 2\sqrt{3}x^{5/2}$

<u>3rd step</u>: is to divide the second term $2x^4$ by the divisor to get Q_2.

$$\frac{2x^4}{2\sqrt{3}x^{5/2}} = \frac{x^{3/2}}{\sqrt{3}} = Q_2$$

<u>4th step</u>: From here onwards the duplexes of the quotients starting from Q_2, are taken in order and subtracted from the corresponding terms of the f(x) to get the new dividends ND.

eg From the third term x^3 subtract $D(Q_2)$
Where D denotes duplex

$$x^3 - D(Q_2) = x^3 - \left(\frac{x^{3/2}}{\sqrt{3}}\right)^2 = x^3 - \frac{x^3}{3} = \frac{2x^3}{3} \text{ND}$$

The ND is divided by the Divisor to get the quotient Q_3.

$$\frac{2x^3}{3} \div \text{Divisor} = \left(\frac{2x^3}{3}\right)\left(\frac{1}{2\sqrt{3}x^{5/2}}\right) = \frac{x^{1/2}}{3\sqrt{3}} = Q_3.$$

<u>5th step</u>: From the 4th term $4x^2$ subtract duplex of $(Q_2 . Q_3)$.

$$4x^2 - D(Q_2 . Q_3) = 4x^2 - \frac{2x^{3/2}x^{1/2}}{\sqrt{3} \; 3\sqrt{3}} = 4x^2 - \frac{2x^2}{9} = \frac{34x^2}{9} = \text{ND}$$

ND is divided by the divisor to get Q_4

$$\frac{34x^2}{9} \div \textit{Divisor} = \left(\frac{34x^2}{9}\right)\left(\frac{1}{2\sqrt{3}x^{5/2}}\right) = \frac{17x^{-1/2}}{9\sqrt{3}}$$

<u>6th step</u>: From 5th term 5x subtract the duplex $(Q_2 . Q_3 . Q_4)$ $5x - [D(Q_2 . Q_4) + D(Q_3)]$

$$5x - \left[\left(\frac{2x^{3/2}}{\sqrt{3}}\right)\left(\frac{17x^{-1/2}}{9\sqrt{3}}\right) + \frac{x}{27}\right] = \frac{100x}{27} ND$$

Now ND is divided by the divisor to get Q_5.

$$\frac{100x}{27}\left(\frac{1}{2\sqrt{3}x^{3/2}}\right) = \frac{50x^{-1/2}}{27\sqrt{3}} = Q_5$$

<u>7th Step</u>: From the constant term 1 subtract D of $(Q_2.Q_3.Q_4.Q_5)$.

1 – Duplex of $(Q_2.Q_3.Q_4.Q_5)$.

$= 1 - [D(Q_2.Q_5) + D(Q_3.Q_4)]$

$$= 1 - \left[\left(\frac{2x^{3/2}}{\sqrt{3}}\right)\left(\frac{50x^{-3/2}}{27\sqrt{3}}\right) + \left(\frac{2x^{1/2}}{3\sqrt{3}}\right)\left(\frac{17x^{-1/2}}{9\sqrt{3}}\right)\right] = \frac{-53}{81} \text{ND}$$

When ND is divided by the divisor we get Q_6.

$$-\frac{53}{81}\left(\frac{1}{2\sqrt{3}x^{3/2}}\right) = \frac{-53}{162\sqrt{3}} x^{-3/2} = Q_6$$

Consider the dividend terms to be extended as $0x^{-1} + 0x^{-2} + 0x^{-3}$ Let us consider the $0x^{-1}$ to obtain Q_7.

<u>8th step</u>: $0x^{-1} - D(Q_2.Q_3.Q_4.Q_5.Q_6)$

$$0x^{-1} - \frac{2x^{3/2}}{\sqrt{3}}\left(\frac{-53}{162}\right)\frac{x^{-3/2}}{\sqrt{3}} - \left(\frac{2x^{1/2}}{3\sqrt{3}}\right)\left(\frac{50^{-3/2}}{27\sqrt{3}}\right) - \frac{289x^{-1}}{243}$$

$(2Q_2Q_6)$ $(2Q_3Q_5)$ $(Q_4)^2$

$$0x^{-1} + \frac{53}{243}x^{-1} - \frac{100x^{-1}}{243} - \frac{289x^{-1}}{243} = \frac{-112x^{-1}}{81} \text{ ND}$$

ND is divided by the divisor to get Q_7.

$$\frac{-112}{81}\frac{x^{-1}}{2\sqrt{3}x^{3/2}} = \frac{-56}{81\sqrt{3}}x^{-5/2} = Q_7.$$

<u>9th step</u>: Similarly $0x^{-2} - D(Q_2.Q_3.Q_4.Q_5.Q_6.Q_7)$

$$= 0x^{-2} - 2\frac{x^{3/2}}{\sqrt{3}}\frac{(-56)x^{-5/2}}{81\sqrt{3}} - 2\frac{x^{1/2}}{3\sqrt{3}}\frac{(-53)}{162\sqrt{3}}x^{-3/2} - 2\left(\frac{17x^{-1/2}}{9\sqrt{3}}\right)\left(\frac{50x^{-3/2}}{27\sqrt{3}}\right)$$

$(2Q_2Q_7)$ $(2Q_3Q_6)$ $(2Q_4Q_5)$

$$= 0x^{-2} + \frac{112x^{-2}}{243} + \frac{53x^{-2}}{729} - \frac{(34)(50)x^{-2}}{729} = \frac{336x^{-2}}{729} - \frac{1700x^{-2}}{729} + \frac{53x^{-2}}{729} = \frac{1311}{729}x^{-2}\text{ND}$$

ND is divided by divisor to get Q_8

$$= \frac{-1311}{729}x^{-2}\left(\frac{1}{2\sqrt{3}x^{5/2}}\right) = \frac{-1311x^{-9/2}}{1458\sqrt{3}} = Q_8$$

<u>10^{th} step:</u> To get Q_9: $0x^{-3} - Q(Q_2\ Q_3\ Q_4\ Q_5\ Q_6\ Q_7\ Q_8)$

$$Q_9 = 0x^{-3} - 2\left(\frac{1}{\sqrt{3}}x^{3/2}\right)\left(\frac{-1311}{1458\sqrt{3}}x^{-9/2}\right) - 2\left(\frac{1}{3\sqrt{3}}x^{1/2}\right)\left(\frac{-56}{81\sqrt{3}}x^{-7/2}\right) - 2\left(\frac{17}{9\sqrt{3}}x^{-1/2}\right)$$

$$\left(\frac{-53}{162\sqrt{3}}x^{-5/2}\right) - \left(\frac{50}{27\sqrt{3}}x^{-3/2}\right)^2 = 0 + \frac{1311}{2187}x^{-3} + \frac{112}{729}x^{-3} + \frac{901}{2187}x^{-3} - \frac{2500}{2187}x^{-3}$$

$$= \frac{48}{2187}x^{-3} \times \frac{1}{2\sqrt{3}}x^{-5/2} = \frac{24}{2187\sqrt{3}}x^{-11/2}$$

$\therefore$ Square Root of $3x^5 + 2x^4 + x^3 + 4x^2 + 5x + 1$

$$= \pm\left(\sqrt{3}x^{5/2} + \frac{x^{3/2}}{\sqrt{3}} + \frac{+x^{1/2}}{3\sqrt{3}} + \frac{17x^{-1/2}}{9\sqrt{3}} + \frac{50x^{-3/2}}{27\sqrt{3}} - \frac{53x^{-5/2}}{162\sqrt{3}} + \frac{-56x^{-7/2}}{81\sqrt{3}}\ \frac{-1311}{1458\sqrt{3}}x^{-9/2} + \ldots\right)$$

Q_1 Q_2 Q_3 Q_4 Q_5 Q_6 Q_7 Q_8

<u>Verifying the Square Root by Duplex Method:</u>

$Q_1 = \sqrt{3}x^{5/2}$ $\quad Q_2 = \frac{1}{\sqrt{3}}x^{3/2}$ $\quad Q_3 = \frac{1}{3\sqrt{3}}x^{1/2}$

$Q_4 = \frac{17}{9\sqrt{3}}x^{-1/2}$ $\quad Q_5 = \frac{50}{27\sqrt{3}}x^{-3/2}$ $\quad Q_6 = \frac{-53}{162\sqrt{3}}x^{-5/2}$

$Q_7 = \frac{-56}{81\sqrt{3}}x^{-7/2}$ $\quad Q_8 = \frac{-1311}{1458\sqrt{3}}x^{-9/2}$ $\quad Q_9 = \frac{24}{2187\sqrt{3}}x^{-11/2}$

Note: If the given dividend contains n terms, for a perfect square, the square root should contains $\frac{n}{2}$ or $\frac{n+1}{2}$ terms depends on n is even or odd respectively. In the present case, it is a perfect square, the Square root should contain only 3 terms and 4^{th} onwards should vary. The fact that the 4^{th} term is existing, clearly indicates that the given expression is not a perfect square. Hence one can extend the evaluation of square root to any order of our choice for example an extension to six quotations will give the following results.

Verification by taking 6 terms

$D(Q_1\ Q_2\ Q_3\ Q_4\ Q_5\ Q_6) =$

$DQ_1 + D(Q_1\ Q_2) + D(Q_1\ Q_2\ Q_3) + D(Q_1\ Q_2\ Q_3\ Q_4) + D(Q_1\ Q_2\ Q_3\ Q_4\ Q_5) + D(Q_1\ Q_2\ Q_3\ Q_4\ Q_5\ Q_6) + D(Q_2\ Q_3\ Q_4\ Q_5\ Q_6), D(Q_3\ Q_4\ Q_5\ Q_6), D(Q_4\ Q_5\ Q_6), D(Q_5\ Q_6), D(Q_6)$

1^{st} term $= D(Q_1) = Q_1^2 = \left(\sqrt{3}x^{5/2}\right)^2 = 3x^5$

2nd term = D(Q1 Q2) = 2Q1 Q2 = $2\left(\sqrt{3}x^{5/2}\right)\left(\frac{1}{\sqrt{3}}x^{3/2}\right) = 2x^4$

3rd term = D(Q1 Q2 Q3) = 2Q1 Q3 + Q_2^2 = $2\left(\sqrt{3}x^{5/2}\right)\left(\frac{1}{3\sqrt{3}}x^{1/2}\right) + x^3 = x^3$

4th term = D(Q1 Q2 Q3 Q4) = 2Q1 Q4 + 2Q2 Q3 = $2\left(\sqrt{3}x^{5/2}\right)\frac{17}{9\sqrt{3}}x^{-1/2} + 2\frac{1}{\sqrt{3}}x^{3/2}\frac{1}{3\sqrt{3}}x^{1/2} = 4x^2$

5th term = D(Q1 Q2 Q3 Q4 Q5) = 2Q1 Q5 + 2Q2 Q4 + Q3 =

$$2\left(\sqrt{3}x^{5/2}\right)\frac{50}{27\sqrt{3}}x^{-3/2} + 2\frac{1}{\sqrt{3}}x^{3/2}\frac{17}{9\sqrt{3}}x^{-1/2} + \frac{1}{27}x = 5x$$

6th term = D(Q1 Q2 Q3 Q4 Q5 Q6) = 2Q1 Q6 + 2Q2 Q5 + 2Q3 Q4 =

$$2\left(\sqrt{3}x^{5/2}\right)\left(\frac{-53}{162\sqrt{3}}\right)x^{-5/2} + 2\frac{1}{\sqrt{3}}x^{3/2}\frac{50}{27\sqrt{3}}x^{-3/2} + 2\frac{1}{3\sqrt{3}}x^{1/2}\frac{17}{9\sqrt{3}}x^{-1/2} = 1$$

Enblock Terms

D(Q2 Q3 Q4 Q5 Q6) = 2Q2 Q6 + 2Q3 Q5 + Q_4^2 =

$$\frac{53}{162\sqrt{3}}x^{-5/2}\Big) + 2\frac{1}{3\sqrt{3}}x^{1/2}\left(\frac{50}{27\sqrt{3}}x^{-3/2}\right) + \frac{289}{243} \quad \frac{112}{81}\frac{1}{x}$$

D(Q3 Q4 Q5 Q6) = 2Q3 Q6 + 2Q4 Q5 =

$$\frac{1}{3\sqrt{3}}\left(\frac{53}{162\sqrt{3}}x^{-5/2}\right) + 2\frac{17}{9\sqrt{3}}x^{-1/2}\left(\frac{50}{27\sqrt{3}}x^{-3/2}\right) x^{-2}\left(\frac{-106}{1458} + \frac{1700}{729}\right) = \frac{61}{27}$$

D(Q4 Q5 Q6) = 2Q4 Q6 + Q_5^2 =

$$2\frac{17}{9\sqrt{3}}x^{-1/2}\left(\frac{53}{162\sqrt{3}}x^{-5/2}\right) + \frac{2500}{2187}x^{-3} = x^{-2}\left(\frac{-1802}{4374} + \frac{2500}{2187}\right) = \frac{3198}{4374}x^{-2} = \frac{533}{729}x^{-3}$$

D(Q5 Q6) = 2Q5 Q6 + = $\frac{50}{27\sqrt{3}}x^{-3/2}\left(\frac{-53}{162\sqrt{3}}\right) \quad \frac{5300}{3122}x = \frac{-2650}{6561}x$

D(Q6) = Q_6^2 = $\frac{2809}{78732}x$

Similarly an extension of nine terms will give the following. It is also seen that in case of perfect the Duplex of the number of quotients is equal to the number of terms in the given expansion.

Duplex of

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9

term = $Q_1^2 = \left(\sqrt{3}x^{5/2}\right)^2 = 3x^5$

2nd term = 2Q1 Q2 = $2\left(\sqrt{3}x^{5/2}\right)\left(\frac{1}{\sqrt{3}}x^{3/2}\right) = 2x^4$

3^{rd} term = $2Q_1 Q_3 + Q_2^2 = 2\left(\sqrt{3}x^{5/2}\right)\left(\frac{1}{3\sqrt{3}}x^{1/2}\right)+\left(\frac{1}{\sqrt{3}}x^{3/2}\right)^2 = \frac{2}{3}x^3+\frac{1}{3}x^3 = x^3$

4^{th} term = $2Q_1 Q_4 + 2Q_2 Q_3$

$$= 2\left(\sqrt{3}x^{5/2}\right)\left(\frac{17}{9\sqrt{3}}x^{-1/2}\right)+2\left(\frac{1}{\sqrt{3}}x^{3/2}\right)\left(\frac{1}{3\sqrt{3}}x^{1/2}\right)=\frac{34}{9}x^2+\frac{2}{9}x^2=\frac{36}{9}x^2= 4x^2$$

5^{th} term = $2Q_1 Q_5 + 2Q_2 Q_4 + Q_3^2$

$$= 2\left(\sqrt{3}x^{5/2}\right)\left(\frac{50}{27\sqrt{3}}x^{-3/2}\right)+2\left(\frac{1}{\sqrt{3}}x^{3/2}\right)\left(\frac{17}{9\sqrt{3}}x^{-1/2}\right)+\left(\frac{1}{3\sqrt{3}}x^{1/2}\right.$$

$$=\frac{100}{27}x+\frac{34}{27}x+\frac{1}{27}x=\frac{135}{27}x = 5x$$

6^{th} term = $2Q_1 Q_6 + 2Q_2 Q_5 + 2Q_3 Q_4 = 2\left(\sqrt{3}x^{5/2}\right)\left(\frac{-53}{162\sqrt{3}}x^{-5/2}\right)+2\left(\frac{1}{\sqrt{3}}x^{3/2}\right)\left(\frac{50}{27\sqrt{3}}x^{-3/2}\right)+$

$$2\left(\frac{1}{3\sqrt{3}}x^{1/2}\right)\left(\frac{17}{9\sqrt{3}}x^{-1/2}\right)=\frac{-53}{81}+\frac{100}{81}+\frac{34}{81}=\frac{81}{81}= 1$$

7^{th} term = $2Q_1 Q_7 + 2Q_2 Q_6 + 2Q_3 Q_5 + Q_4^2$

$$= 2\left(\sqrt{3}x^{5/2}\right)\left(\frac{-56}{81\sqrt{3}}x^{-7/2}\right)+2\left(\frac{1}{\sqrt{3}}x^{3/2}\right)\left(\frac{-53}{162\sqrt{3}}x^{-5/2}\right)+$$

$$2\left(\frac{1}{3\sqrt{3}}x^{1/2}\right)\left(\frac{50}{27\sqrt{3}}x^{-3/2}\right)+\left(\frac{17}{9\sqrt{3}}x^{-1/2}\right)^2=-\frac{112}{81}x^{-1}-\frac{53}{243}x^{-1}+\frac{100}{243}x^{-1}+\frac{289}{243}x^{-1}=0x^{-1}$$

8^{th} term = $2Q_1 Q_8 + 2Q_2 Q_7 + 2Q_3 Q_6 + 2Q_4 Q_5 = 2\left(\sqrt{3}x^{5/2}\right)\left(\frac{-1311}{1458\sqrt{3}}x^{-9/2}\right.$

$$2\left(\frac{1}{\sqrt{3}}x^{3/2}\right)\left(\frac{-56}{81\sqrt{3}}x^{-7/2}\right)+2\left(\frac{1}{3\sqrt{3}}x^{1/2}\right)\left(\frac{-53}{162\sqrt{3}}x^{-5/2}\right)+2\left(\frac{17}{9\sqrt{3}}x^{-1/2}\right)\left(\frac{50}{27\sqrt{3}}x^{-3/2}\right.$$

$$=-\frac{1311}{729}x^{-2}-\frac{112}{243}x^{-2}-\frac{53}{729}x^{-2}+\frac{1700}{729}x^{-2}=0x^{-2}$$

9^{th} term = $2Q_1 Q_9 + 2Q_2 Q_8 + 2Q_3 Q_7 + 2Q_4 Q_6 + Q_5^2 = 2\left(\sqrt{3}x^{5/2}\right)\left(\frac{24}{2187\sqrt{3}}x^{-11/2}\right)+$

$$2\left(\frac{1}{\sqrt{3}}x^{3/2}\right)\left(\frac{-1311}{1458\sqrt{3}}x^{-9/2}\right)+2\left(\frac{1}{3\sqrt{3}}x^{1/2}\right)\left(\frac{-56}{81\sqrt{3}}x^{-7/2}\right)+2\left(\frac{17}{9\sqrt{3}}x^{-1/2}\right)\left(\frac{-53}{162\sqrt{3}}x^{-5/2}\right)+\left(\frac{50}{27\sqrt{3}}\right.$$

$$\frac{48}{2187}x^{-3}-\frac{1311}{2187}x^{-3}-\frac{112}{729}x^{-3}-\frac{901}{2187}x^{-3}+\frac{2500}{2187}x^{-3}=\frac{-48}{2187}x^{-3}+\frac{48}{2187}x^{-3}=0x^{-3}$$

and so on

Verification by Urdhva Tiryak Multiplication Using Left to Right Method:

$Q_1\ Q_2\ Q_3\ Q_4\ Q_5\ Q_6\ Q_7\ Q_8\ Q_9$

$Q_1\ Q_2\ Q_3\ Q_4\ Q_5\ Q_6\ Q_7\ Q_8\ Q_9$

A $(Q_1^2) + (2Q_1 Q_2) + (2Q_1 Q_3 + Q_2^2) + (2Q_1 Q_4 + 2Q_2 Q_3) + (2Q_1 Q_5 + 2Q_2 Q_4 + Q_3^2) +$
$(2Q_1 Q_6 + 2Q_2 Q_5 + 2Q_3 Q_4) + (2Q_1 Q_7 + 2Q_2 Q_7 + 2Q_3 Q_5 + Q_4^2) + (2Q_1 Q_8 + 2Q_2 Q_7$

$+ 2Q_3 Q_6 + 2Q_4 Q_5) + (2Q_1 Q_9 + 2Q_2 Q_8 + 2Q_3 Q_7 + 2Q_4 Q_6 + Q_5^2)$ + B and enbloc terms (see blow)

These will give rise to the terms given in the expression as

$3x^5 + 2x^4 + x^3 + 4x^2 + 5x \quad + 1 + 0x^{-1} + 0x^{-2} + 0x^{-3}$

$Q_1 \quad Q_2 \quad Q_3 \quad Q_4 \quad Q_5 \quad Q_6 \quad Q_7 \quad Q_8 \quad Q_9$

$$\sqrt{3}x^{5/2} \quad \frac{x^{3/2}}{\sqrt{3}} \quad \frac{+x^{1/2}}{3\sqrt{3}} \quad \frac{17x^{-\frac{1}{2}}}{9\sqrt{3}} \quad \frac{50x^{-\frac{3}{2}}}{27\sqrt{3}} \quad \frac{53x^{\frac{-5}{7}}}{162\sqrt{3}} \quad \frac{-56x^{-7/2}}{81\sqrt{3}} \quad \frac{-1311}{1458\sqrt{3}}x^{-9/2} + \frac{24}{2187\sqrt{3}}x^{-\frac{11}{2}}.$$

Squaring of $Q_1 \; Q_2 \; Q_3 \; Q_4 \; Q_5 \; Q_6 \; Q_7 \; Q_8 \; Q_9$

A

DQ_1

$D(Q_1 Q_2)$

$D(Q_1 Q_2 Q_3)$

$D(Q_1 Q_2 Q_3 Q_4)$

$D(Q_1 Q_2 Q_3 Q_4 Q_5)$

$D(Q_1 Q_2 Q_3 Q_4 Q_5 Q_6)$

$D(Q_1 Q_2 Q_3 Q_4 Q_5 Q_6 Q_7)$

$D(Q_1 Q_2 Q_3 Q_4 Q_5 Q_6 Q_7 Q_8)$

$D(Q_1 Q_2 Q_3 Q_4 Q_5 Q_6 Q_7 Q_8 Q_9)$

B

$D(Q_2 \; Q_3 \; Q_4 \; Q_5 \; Q_6 \; Q_7 \; Q_8 \; Q_9)$

$D(Q_3 \; Q_4 \; Q_5 \; Q_6 \; Q_7 \; Q_8 \; Q_9)$

$D(Q_4 \; Q_5 \; Q_6 \; Q_7 \; Q_8 \; Q_9)$

$D(Q_5 \; Q_6 \; Q_7 \; Q_8 \; Q_9)$

$D(Q_6 \; Q_7 \; Q_8 \; Q_9)$

$D(Q_7 \; Q_8 \; Q_9)$

$D(Q_8 \; Q_9)$

$D(Q_9)$

$$D(Q_2 Q_3 Q_4 Q_5 Q_6) = 2\frac{x^{\frac{3}{2}}}{\sqrt{3}} \times \frac{53x^{-\frac{5}{2}}}{162}$$

$$2(Q_3 Q_5) = \frac{50x}{27\sqrt{3}} \times \frac{x^{1/2}}{3\sqrt{3}}$$

$$Q_4^2 = \frac{289x^{-1}}{81 \times 3}$$

$D(Q_2 Q_3 Q_4 Q_5 Q_6) \neq 0$

The sum of the terms of A will answer the given dividend in case perfect or imperfect squares. But in case of imperfect squares, the terms of B will also exist.

If Q_7 onwards are all zero. Then it is a perfect square.

If it were to be a perfect square then all Duplex

$D(Q_2 Q_6)$, $D(Q_3 Q_6)$, $D(Q_4 Q_6)$, $D(Q_5 Q_6)$ should vanish

$Q \ldots\ldots Q_6 \; (Q_3 \ldots\ldots Q_6)$

As applied to this problem

$D(Q_2 Q_6) = 2Q_2Q_6 + 2Q_3Q_5 + Q_4^2 \neq 0$
$D(Q_3 Q_6) = 2Q_3Q_6 + 2Q_4Q_5 \neq 0$
$D(Q_5 Q_6) = 2Q_5Q_6 \neq 0$

All Quotients beyond Q_6 should vanish if the given expression is a perfect square

1) If the given expression is sum of Duplexes of the Quotients taken in order for example in this case
$D(Q_1) + D(Q_1 Q_2) + D(Q_1 Q_2 Q_3) + D(Q_1 Q_2 Q_3 Q_4) + D(Q_1 Q_2 Q_3 Q_4 Q_5) + D(Q_1Q_2Q_3Q_4Q_5Q_6)$
2) The other enbloc duplexes vanish for example
$D(Q_2 Q_3 Q_4 Q_5 Q_6), D(Q_3 Q_4 Q_5 Q_6), D(Q_4 Q_5 Q_6), D(Q_5 Q_6), D(Q_6) = 0$ If it is a perfect square

<u>Ascending Powers of x</u>

$$2 \,|\, 1 + 5x + 4x^2 + x^3 + 2x^4 + 3x^5$$

$$\underset{Q_1}{1} + \underset{Q_2}{\frac{5}{2}x} - \underset{Q_3}{\frac{9}{8}x^2} + \underset{Q_4}{\frac{53}{16}x^3} - \underset{Q_5}{\frac{1013}{128}x^4} + \underset{Q_6}{\frac{6403}{256}x^5}$$

$$4x^2 - \frac{25}{4}x^2 = -\frac{9}{4}x^2 \times \frac{1}{2} = -\frac{9}{8}x^2 \longrightarrow Q_3$$

$$x^3 - 2\left(\frac{5x}{2}\right)\left(-\frac{9x^2}{8}\right) = x^3 + \frac{45}{8}x^3 = \frac{53}{8}x^3 \times \frac{1}{2} = \frac{53}{16}x^3 \qquad Q_4$$

$$2x^4 - 2\left(\frac{5x}{2}\right)\left(\frac{53}{16}x^3\right) - \frac{81}{64}x^4 = \left(2 - \frac{265}{16} - \frac{81}{64}\right)x^4 = \frac{128 - 1060 - 81}{64}$$

$$= \frac{1013}{64}x^4 \times \frac{1}{2} = -\frac{1013}{128}x^4 \longrightarrow Q_5$$

$$3x^5 - 2\left(\frac{5}{2}\right)x\left(\frac{-1013}{128}x^4\right) - 2\left(-\frac{9}{8}x^2\right)\left(\frac{53}{16}x^3\right) = \left(3 + \frac{5065}{128} + \frac{477}{64}\right)x^5 = \frac{6403}{128}x^5 \times \frac{1}{2} = \frac{6403}{256}x^5 \longrightarrow Q_6$$

Square root of $(1 + 5x + 4x^2 + x^3 + 2x^4 + 3x^5) = 1 + \frac{5}{2}x - \frac{9}{8}x^2 + \frac{53}{16}x^3 - \frac{1013}{128}x^4 + \frac{6403}{256}x^5$

rren Method

$\sqrt{3}x^{5/2}$	$3x^5+2x^4+x^3+4x^2+5x+1$	$\sqrt{3}x^{5/2}+\frac{x^{3/2}}{\sqrt{3}}+\frac{x^{1/2}}{3\sqrt{3}}$ $+\frac{17}{9\sqrt{3}}x^{-1/2}+\frac{50}{27\sqrt{3}}x^{-3/2}$ $-\frac{53}{162\sqrt{3}}x^{-5/2}+\ldots$
$2\sqrt{3}x^{5/2}+\frac{x^{3/2}}{\sqrt{3}}$	$\underline{3x^5}$	
$2\sqrt{3}x^{5/2}+\frac{2x^{3/2}}{\sqrt{3}}+\frac{x^{1/2}}{3\sqrt{3}}$	$2x^4+\frac{x^3}{3}$	
$2\sqrt{3}x^{5/2}+\frac{2x^{3/2}}{\sqrt{3}}+\frac{2x^{1/2}}{3\sqrt{3}}+\frac{17}{9\sqrt{3}}x^{-1/2}$	$\frac{2x^3}{3}+4x^2+5x$	
	$\frac{2x^3}{3}+\frac{2x^2}{9}+\frac{x}{27}$	
$2\sqrt{3}x^{5/2}+\frac{2x^{3/2}}{\sqrt{3}}+\frac{2x^{1/2}}{3\sqrt{3}}+$	$\frac{34x^2}{9}+\frac{134x}{27}+1$	
$\frac{34}{9\sqrt{3}}x^{-1/2}+\frac{50}{27\sqrt{3}}x^{-3/2}$	$\frac{34x^2}{9}+\frac{34x}{27}+\frac{34}{81}+\frac{289x^{-1}}{243}$	
	$\frac{100x}{27}+\frac{47}{81}-\frac{289}{243}x^{-1}$	
	$\frac{100x}{27}+\frac{100}{81}+\frac{100}{243}x^{-1}+\frac{1700}{729}x^{-2}+\frac{2500}{2187}x^{-3}$	
$2\sqrt{3}x^{5/2}+\frac{2x^{3/2}}{\sqrt{3}}+\frac{2x^{1/2}}{3\sqrt{3}}+\frac{34}{9\sqrt{3}}x^{-1/2}$	$-\frac{53}{81}-\frac{389}{243}x^{-1}-\frac{1700}{729}x^{-2}-\frac{2500}{2187}x^{-3}$	
$+\frac{100}{27\sqrt{3}}x^{-3/2}-\frac{53}{162\sqrt{3}}x^{-5/2}$	$-\frac{53}{81}-\frac{106}{486}x^{-1}-\frac{106}{1458}x^{-2}-\frac{1802}{4324}x^{-3}-\frac{2650}{13122}x^{-4}$	
	$-\frac{672}{486}x^{-1}-\frac{3294}{1458}x^{-2}-\frac{3198}{4274}x^{-3}-\frac{2650}{13122}x^{-4}$	

Section – B

Cubes, Expansions and Roots

General Introduction

A general method of cubing. The method that is conceived by Swamiji can be easily extended to expand any expression containing any number of terms to any positive power.

The algebrical principle given by Swamiji is $(a + 10b + 100c + 1000d + \ldots\ldots)^3$

This can be also visualized as $(\ldots\ldots.dcba)^3$

This expansion can be also worked out if the number is written as (abcd.....)

Where a is the higghest placement (refer Table B)

The same table can also be used for the decimals as a.bcde......... where b is in 10^{-1}, c is in 10^{-2}..... a is the integer part.

Swamiji has expllained the evaluation of cube roots by two different methods.

In both the metthods he has grouped the number into units, each contains three digits from the RHS, any left out in the final group containing less than 3 is also considered as one group and the method developed is explained.

1) The first method makes use of Argumenta onal Character (... JKL) and
2) The Second method making use of Straight Division for both the perfect and non perfect to the required decimal, are explained.

   Second method has a modified version by considering the first two groups as one unit, thus is called two digit method which gives a result in a more elegant way.

   Swamijis method is workable to the cubic equations also with a re–orientation of the concerned expressions as explained in the text. It is seen that the general method explained by Swamiji can be used for working out the roots of any degree (the integer) equation.

   The British Authors have explained the cube roots and one solution of the cubic equation by using a re–arrangement of Taylor's expansion and introducing the solution as a.bcde.. we noticed that the method developed by Swamiji is extended to equations is simpler.

   The method suggested by Swamiji leads to developing such a method to any degree equation whatsoever and also the construction of the relevant expansion tables which are considered to be novel of this work. So far, no one has given all the solutions of cubic and higher degree equations but Swamiji

Suggested a method called Argumentation for obtaining the remaining solutions making use of Sutras Adyamadyena Antyamantyena, coupled with Argumentation process.

It is surprisingly significant and interesting to note that the result can be tested by the Simple Sutram called "Gunita Samuccayah Samuccaya Gunitah". Both the procedures developed by Swamiji and later by British authors have been exemplified through working out a number of problems in cubic 4^{th} degree, 5^{th} degree, 6^{th} degree, Seventh and 8^{th} degree equations.

The elaborate working details are shown with a view to extend similar working for other problems as well.

The method of finding out the roots, Square, Cubes, for polynomials is also detailed. While doing so, an attempt is made to workout the roots both in (a) descending and (b) ascending powers of x.

If the root is an exact one, then both the a and b methods should give the same result. On the other hand a non perfect root gives two different results one for the descending order and the other for ascending order.

These are also shown in the examples. One can proceed to work out to any powers of x of ones choice.

Cubes

Method I: Yavadhunam Sutra for cubing:

The following steps are worked out for finding out the cube of a number

1) A base of the nature of 10^n is chosen so that it is the nearest value to the given number
2) The excess or deficiency of the given number over the base so chosen is determined
3) The answer of cube is worked out in three parts.

Part I consists of the given number plus / minus twice the excess or deficiency.

Part II consists of product of new excess/ deficiency and original excess / deficiency respectively.

Part III consists of cube of original excess / deficiency

In the II and III parts provisions of digits is as per one digit less than the base. (Refer Lecture Notes I on multiplication)

Example I

$(103)^3$ Base = 100 Excess = + 3.

Part I = 103 + 06 = 109

Part II = (09) (03) = 27

Part III = $(03)^3$ = 27

$(103)^3$ = 109 / 27 / 27 = 1092727

Example 2:

$(994)^3$

Base = 1000 Deficiency = – 006

Part I = 994 – 012 = 982

Part II = (018) (006) = 108

Part III = $(-006)^3 = \overline{216}$

$(994)^3 = 982 / 108 / \overline{216}$ = 982107784

Example 3: $(10009)^3$

Base = 10000 Excess = + 0009

Part I = 10009 + 00018 = 10027

Part II = (0027) (0009) = 0243

Part III = $(0009)^3$ = 0729

$(10009)^3$ = 10027 / 0243 / 0729

It is interesting to note that one need not always aim at 10^n base. Instead, one can use multiples and sub – multiples as working base (WB) of such a theoretical bases (TB). The following are few examples. The working details for such bases are that one has to modify II and I parts by multiplying with ratios $\frac{WB}{TB}$ and $\left(\frac{WB}{TB}\right)^2$ respectively

Example 1: $(396)^3$

Theoretical Base, TB = 100

Working base, WB = 400

Deficiency = $\overline{04}$

Part I = 396 + $\overline{08}$

Part II = ($\overline{12}$) ($\overline{04}$) = 48

Part III = ($\overline{04}$)^3 = $\overline{64}$

$(396)^3$ = 388 / 48 / $\overline{64}$ = 6208 / 92 / $\overline{64}$ = 620992$\overline{64}$ = 62099136

Example 2: $(2998)^3$

Theoretical Base, TB = 1000

Working base, WB = 3000

Deficiency = 00$\overline{2}$

Part I = 2998 + 00$\overline{4}$ = 2994

Part II = (00$\overline{2}$) (00$\overline{6}$) = 012

Part III = (00$\overline{2}$)^3 = 00$\overline{8}$

$(2998)^3$ = 2994 / 012 / 00$\overline{8}$

= 26946 / 036 / 00$\overline{8}$ = 26946035992

Example 3: $(5012)^3$

Theoretical Base, TB = 10,000

Working base, WB = 5000

Excess = 0012

$\frac{WB}{TB} = \frac{5000}{10,000} = \frac{1}{2}$

Part I = 5012 + 0024 = 5036

Part II = (0012) (0036) = 0432

Part III = $(0012)^3$ = 1728

$(5012)^3$ = 5036 / 0432 / 1728

$\times\frac{1}{4}$ $\times\frac{1}{2}$

= 1259 / 0216 / 1728 = 125902161728

Example 4: $(39998)^3$

Theoretical Base, TB = 10000

Working base, WB = 40000

Deficiency = $0000\bar{2}$

Part I $= 39998 + 0000\bar{4} = 39994$

Part II $= (0000\bar{6})(0000\bar{2}) = 00012$

Part III $= (0000\bar{2})^3 = 0000\bar{8}$

$(39998)^3 = 39994 / 00012 / 0000\bar{8}$

$\times 16 \quad \times 4$

$= 639904 / 0048 / 0000\bar{8} = 63990400479992$

A few more examples

Let us consider 109^3.

Original Excess of the given number over the base 100 is 9. The answer is written in three parts. The last part of the answer consists of cube of the excess. i.e. $9^3 = 729$. The provision for last two parts is dependent on the base and is one digit less than the base. The first part is calculated using the formula.

"Given Number + Twice the excess"

i.e. $109 + (2\times9) = 109 + 18 = 127$

From this a new excess 27 is derived.

Now the second (middle) of the answer part is obtained by using the formula.

" New excess × Original excess " (27×9)

is also equal to $(3 \times 9) \times 9 = 243$ "Thrice the original excess × Original excess"

i.e. 27×9 or $(3 \times 9) \times 9 = 243$ $\therefore$ $109^3 = 127/243/729$

Applying the provision principle, it is written as

$(109)^3 = 127/_2 43/_7 29 = 1295029$

1) 109^3

Current Method

```
  109      11881
× 109      × 109
  981     106929
 000      00000
109      11881
.1881    1295029
```

or

$(100+9)^3 = (a+b)^3 = 100^3 + 3 \times 100^2 \times 9$

$+ 3 \times 100 \times 9^2 + 9^3$

$a^3 + 3a^2b + 3ab^2 + b^3$

Vedic Method

$109^3 = 127/{}_{2}43/{}_{7}29$

= 1295029

First term = Given Number + Twice the excess

Middle term = New excess × original excess

(or) Three times the original excess × original excess.

Last term = cube of the original excess

Placement is only two digits for the last and middle terms as the base is 100 (3digits)

Examples:

2) 126^3

Current Method

```
   126
  ×126
   756
  252
 126
 15876
  ×126
 95876
 31752
15856
2000376
```

$(100+20+6)^3$

(100 + 20 + 26)

(a + b + c)

$a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2c + 3bc^2$

$3ab^2 + 3ac^2 + 6abc$

Vedic Method

126^3

Base = 100 excess 26

Theoretical Base = 10

Working Base = 2 × 10

= 20

$38 / {}_{10}8 / {}_{21}6$

×4 ×2

$= 152 / {}_{21}6 / {}_{21}6$

$26^3 = 17576$

$126^3 = 178/20 \;\; 28/175 \;\; 7/6 = 200376$

3) 10000013^3

Current Method

```
          10000013
        × 10000013
          30000039
         10000013
        00000000
       00000000
      00000000
     00000000
    00000000
   10000013
   100000260000169
        × 10000013
   300000780000507
  100000260000169
 000000000000000
000000000000000
000000000000000
000000000000000
000000000000000
100000260000169
1000003900005070002197
```

$(10000000+13)^3$

Vedic Method by one step!

10000013^3 = base is 10000000

10000039 / 0000507 / 0002197

Provision for the last and middle terms is 7 digits.

$13^3 = 2197$

4) 639^3

Current Method

```
   639
  ×639
  5751
 1917
3834
408321
  × 639
3674889
1224963
2449926
260917119
```

Vedic Method (Yavadhanam)

Theoretical Base is 100

Working Base is 600 excess is 39

The result is multiplied as clearly shown by 26, $\frac{WB}{TB}$ 6 of the first and middle terms.

639^3 =

717 / $_{45}$63 / $_{593}$19

× 36 / × 6 /

= 25812 / $_{273}$78 / $_{593}$19

= 260917119

In case the number consists of any digit greater than 5, then the number is converted into vinculum form, and the same procedure is applied.

5) 9993^3

Current Method

9993	99860049
9993	×9993
29979	299580147
89937	898740441
89937	898740441
89937	898740441
99860049	997901469657

Also as or $(1000 - 7)^3$

Vedic Method using V

$9993 = 100\bar{1}3$

$100\bar{3}9 / 02\bar{6}7 / \bar{1}7\bar{5}7$

$= 997901469657$

$9993 + \bar{2}6 = 99.79 \quad 100\bar{3}9$

$\bar{3}9 \times \bar{1}3$

$2\bar{6}7$

$(\bar{1}3)^3 = \bar{1}7\bar{5}7$

Cubing by Anurupyena Sutram:

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

a^3	a^2b	ab^2	b^3	are in GP
	$2a^2b$	$2ab^2$		
a^3	$3a^2b$	$3ab^2$	b^3	

Let us consider 11^3. The two digits are in the ration 1 : 1. Starting with cube of the first digit, one hast to write down Geometrical progression series of four terms (which include first term) with the Geometrical ration as 1. ie 1 1 1 1 1. The two middle terms are multiplied by twice the value of the middle terms and then add up.

Current Method

```
   11
  ×11
   11
  11
  121
  ×11
  121
 121
 1331
```

Vedic Method

$11^3 =$

```
1  1  1  1
   2  2
1  3  3  1
```

1) 17^3 $(ab)^3$

Current Method

```
  17
 ×17
 119
 17
 289
 ×17
2023
289
4913
```

$(10+7)^3$, $(20-3)^3$

$107^3 = (100+7)^3$

$100^3 + 3(100)^2 \times 7 + 3 \times 100 + 343$

$100^3 + 3(10)^2 + 3(10)49 + 343$

$a^3 + 3a^2b + 3ab^2 + b^3$

Vedic Method

The ratio between the two terms is 7:1 (b:a)

```
17³ = 1   7   49  343
         14   98
      4 9    1    3
```

$17^3 = 4913$

$a^3 \times \frac{b}{a} = a^2b = 7$

$a^2b \times \frac{b}{a} = ab^2 = 49$

$ab^2 \times \frac{b}{a} = b^3 = 343$

A few more examples are given below

2) 26^3

Current Method

```
   26
  ×26
  156
  52
  676
  ×26
 4056
1352
17576
```

$(20+6)^3$

Vedic Method

The ratio between the two terms is 3:1

26^3

```
= 8   24   72   216
          48   144
 17    5    7     6
```

$26^3 = 17576$

3) 34^3

Current Method

```
   34
  ×34
  136
 102
 1156
  ×34
 4624
3468
39304
```

$(30+4)^3$

Vedic Method (Anurupya)

The ratio between the two terms is 4:3

34^3

```
=  27   36   48   64
        72   96
   39    3    0    4
```

$34^3 = 39304$

4) 65^3

Current Method

```
     65
    ×65
    325
   390
   4225
    ×65
  21125
 25350
 274625
```

$(60+5)^3$

Vedic Method

The ratio between the two terms is 5:6

65^3

```
=  216   180   150   125
         360   300
   274     6     2     5
```

$65^3=274625$

5) 83^3

Current Method

```
    83
   ×83
   249
  664
  6889
   ×83
 20667
55112
571787
```

$(80 + 3)^3$

Vedic Method

The ratio between the two terms is 3:8

83^3

```
=  512   192   72   27
         384  144
   571     7    8    7
```

$8^3 = 512$

$512 \times \frac{3}{8} = 192$

$192 \times \frac{3}{8} = 72$

Proof is as follows.

Proof:

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$a^3 + \quad a^2b + \quad ab^2 + b^3$ The Successive terms are in GP $\left(\frac{b}{a}\right)$

$\underline{\qquad 2a^2b + 2ab^2}$

$a^3 + 3a^2b + 3ab^2 + b^3$

This method of Anurupya is applicable to cube of any number of digits provided the number is written in terms of two group units. For example consider 383. This can be written in two different sets.

1. (38)3
2. 3(83)

In applying the above method, the answer is shown in four parts. In the first part one has to write down cube of the first group. In the second part the ratio of second to first group is applied to the first part of the answer. This is continued till we complete four parts which includes first part also. Next Step is to put down below the second and third parts twice their respective values. Final addition gives the answer. While doing so one has to consider that the provision of the last three parts is dependent on the number of digits in the second group (right hand most group). And hence care is taken to see that the remaining digits of each part are carried over to the next left part and then complete the addition.

This is illustrated in the above two cases.

383^3

Current Method

```
     383
   × 383
    1149
   3064
  1149
  146689
   × 383
  440067
 1173512
 440067
56181887
```

Vedic Method

383^3

Grouping 1: 3 (83) — Ratio between the two groups is 83:3

```
383³ =  27   747   20667   571787
             1494  41334
        56   18    18      87
```

Grouping 2 : (38) 3 — Ratio is $\frac{8}{3}$

```
38³ =  27   72   192   512
            144  384
       54   8    7     2
```

```
383³ =  54872  4332  342  27
               8664  684
        56181  8     8    7
```

Ratio is $\frac{3}{38}$

$\therefore 383^3 = 56181887$

Some more examples are given below.

Examples:

1) 127^3

Current Method

```
     127
    ×127
     889
    254
   127
   16129
    ×127
  112903
  32258
 16129
 2048383
```

Vedic Method

Grouping:1 127 is grouped 1 and 27,

Ratio = $\frac{27}{1}$

$127^3 = 1\ (27)$

$127^3 = 100 + 27 = 1(27)$,

```
1   27    729   19683
    54   1458
2   04     83      83
```

127^3=2048383

Grouping2: 127 is grouped as 12 and 7

The ration between the two groups is 7:12

$(12)^3 = 1728$

$1728 \times \frac{7}{12} = 1008$

$1008 \times \frac{7}{12} = 588$

$588 \times \frac{7}{12} = 343$

are in GP

$127^3 = 120 + 7 = 12(7)$

```
1728   1008   588   343
       2016  1176
2048      3     8     3
```

127^3= 2048383

2) 3456^3

Current Method

```
       3456
      ×3456
      20736
     17280
    13824
   10368
   11943936
     × 3456
   71663616
  59719680
 47775744
35831808
41278242816
```

Vedic Method

3456^3

(i) Grouping = 3 (456) Ratio = 456 : 3 = $\frac{456}{1}$

```
27   4104   623808   94818816
     8208  1247616
41    278      242        816
```

3456^3=41278242816

Ratio 56:34 = $\frac{56}{34} = \frac{28}{17}$ =28:17

(ii) Grouping = (34) (56)

```
39304   64736   106624   175616
       129472   213248
41278      24       28       16
```

3456^3=41278242816

(iii) Grouping = (345) 6 Ratio 6:345

345^3= Ratio 45:3 = 15:1 = $\frac{15}{1}$

```
27   405   6075   91125
     810  12150
41    06     36      25
```

345^3=41063625

3456^3=

```
41063625   714150   12420   216
          1428300   24840
41278242        8       1
```

3456^3=41278242816

This method can be extendable to work out for higher powers as well.

The grouping method which is applied for squaring and cubing is also applicable to higher orders as well. The proof for fourth power on the basis of Anurupya method is given below.

Proof:

$$(a+b)^4 = a^4 + a^3b + a^2b^2 + ab^3 + b^4 \text{ the Successive terms are in GP}$$
$$\underline{\quad 3a^3b + 5a^2b^2 + 3ab^3 \quad}$$
$$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

When applied to 11^4, it can be written

Current Method

11^4

```
   11
  ×11
   11
  11
  121
  ×11
  121
 121
 1331
  ×11
 1331
1331
14641
```

Vedic Method

```
11^4 = 1  1  1  1  1
          3  5  3
       1  4  6  4  1
```

(But the terms in between the first and last are different in evaluating different powers of a number. This has to be first obtained as is clear from the proof. Consider 15^4. Which is written in five parts. The First part consists of 4[th] power of the first digit 1. And then the ratio is followed completing the five parts. The provision is one digit just as in the case of powers of two digited number, where as for a number consisting of more than two digits, the provision for all the parts excepting the first part is dependent on the number of digits in the right hand most group unit, while working out the value of any power of the given number.

Thus 15^4 = 1 5 25 125 625

2nd, 3rd, 4th, parts take additional quantities 3 times, 5 times and 3 times respectively of the original parts. Finally the summation gives the value 50625, considering the provision of only one digit in every part and carrying over the remaining to next left part.)

1) 15^4

Current Method

```
   15
  ×15
   75
  15
  225
  ×15
 1125
 225
 3375
  ×15
16875
3375
50625
```

Vedic Method

15^4

```
1   5   25   125   625
    15  125  375
5   0   6    2     5
```

$15^4=50625$

(The similar procedure can be adopted to 4th power of any number provided we partition the given number into two groups and then apply the Geometrical ratio, and additional quantities are supplied.)

2) 36^4

Current Method

```
     36
     36
    216
   108
   1296
   × 36
   7776
  3888
  46656
   × 36
 279936
139968
1679616
```

Vedic Method

36^4 =

```
81   162   324   648   1296
     486   1620  1944
167  9     6     1     6
```

$36^4=1679616$

A further extension to the 5^{th} power clearly shows the additional quantities that are to be supplied on the basis of binomial expansion. The method is self explanatory for the fifth power by an example and is further extended to the sixth power by another example.

Examples:

1) 13^5

Current Method	Vedic Method
13	$13^5 =$
×13	
39	1 3 9 27 81 243
13	12 81 243 324
169	3 7 1 2 9 3
×13	
507	
169	
2197	
×13	$13^5=371293$
6591	
2197	
28561	
× 13	
85683	
28561	
371293	

Proof:

$$
\begin{array}{l}
(a+b)^5 = a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5 \\
\qquad\qquad\quad \underline{4a^4b + 9a^3b^2 + 9a^2b^3 + 4ab^4} \\
\qquad\qquad a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{array}
$$

Sixth Power:

Example:

24^6

Current Method

```
       24
      ×24
       96
      48
      576
     × 24
     2304
    1152
    13824
     × 24
    55296
   27648
   331776
      ×24
  1327104
 663552
  7962624
     × 24
 31850496
15925248
191102976
```

Vedic Method

$24^6 =$

64	128	256	512	1024	2048
	640	3584	9728	14336	10240
191	1	0	2	9	7

$24^6 = 191102976$

Proof:

$$
\begin{aligned}
(a+b)^6 = a^6 + \; & a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6 \\
& + 5a^5b + 14a^4b^2 + 19a^3b^3 + 14a^2b^4 + 5ab^5 \\
\hline
& a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6
\end{aligned}
$$

(A general expansion is also clearly shown taking a and b as two digits and n^{th} power is worked out.)

In general,

$$(a+b)^n = a^n + \quad a^{n-1}b + \quad a^{n-2}b^2 + \ldots + \quad a^2b^{n-2} + \quad ab^{n-1} + b^n$$

$$+ (n_{c_1} - 1)a^{n-1}b + (n_{c_2} - 1)a^{n-2}b^2 + \ldots + (n_{c_{n-1}} - 1)a^2b^{n-2} + (n_{c_{n-1}} - 1)ab^{n-1} +$$

$$a^n + nc_1a^{n-1}b \quad + nc_2a^{n-2}b^2 \quad + \ldots + nc_{n-1}a^2b^{n-2} \quad + nc_nab^{n-1} \quad + b^n$$

In all these, current method working details are also shown.

Readers can try any power with the help of this for a two digited number and also for many digited numbers.

General Expansions

Swamiji's method makes use of symmetry in working out the various terms in an expansion. This can be extended to any digited expression and also to any power.

1) It is also interesting to note that from a general table containing expansion of higher powers, one can pick out the expansion terms of lower powers. From the $(d\ c\ b\ a)^6$ expansion one can read out the expansion of sixth power having less number of digits ie $(c\ b\ a)^6$, $(b\ a)^6$.
2) From the Symmetry considerations $(c\ b\ a)^3$ can be made use for obtaining the additional terms of $(c\ b\ a)^4$ by multiplying the former terms with $(c\ b\ a)$ using Urdhva – tiryak Sutram. The same method can be made use for obtaining higher powers by Urdhva tiryak method. Eg: Starting from the expansion terms of the square, we can get terms for cube, and similarly from the expansion terms of cube we can get the expansion terms of any power provided, one can take the corresponding expansion terms required and multiply the former by Urdhva tiryak.

   The terms of $(c\ b\ a)^3 = a^3, b^3, c^3, 3a^2b, 3ab^2, 3a^2c, 3ac^2, 3b^2c, 3bc^2, 6abc$

If we want the expansion of $(c\ b\ a)^5$ we get the corresponding terms by multiplying $(c\ b\ a)^2$ terms with those of $(c\ b\ a)^3$ by applying urdhva tiryak sutram.

If one wants $(c\ b\ a)^6$, duplex of $(c\ b\ a)^3$ can be used. If one wants to get $(c\ b\ a)^7$ the duplex terms of $(a\ b\ c)^3$ are to be multiplied by the $(c\ b\ a)$ by application of Urdhva tiryak and so on.

Table A

Terms of $(a\ b\ c)^3 = a^3, b^3, c^3, 3a^2b, 3ab^2, 3a^2c, 3ac^2, 3b^2c, 3bc^2, 6abc$

$(a\ b\ c)^4 =$

	a^3	b^3	c^3	$3a^2b$	$3ab^2$	$3a^2c$	$3ac^2$	$3b^2c$	$3bc^2$	$6abc$	
	a	b	c	0	0	0	0	0	0	0	
a^4	$4a^3b$	$4ab^3$	$4a^3c$	$4ac^3$	b^4	$6a^2b^2$	$4b^3c$	$4bc^3$	c^4	$6a^2bc$	
								$12b^2ac$	$6abc^2$	$6a^2c^2$	$6b^2c^2$

a^4 b^4 c^4 $4a^3b$ $4ab^3$ $4a^3c$ $4ac^3$ $4b^3c$ $4bc^3$ $12a^2bc$
$12ab^2c$ $12abc^2$ $6a^2b^2$ $6a^2c^2$ $6b^2c^2$

All these are well comparable with the general expansion terms using, symmetry. (Refer Table C)

Now it is interesting to note that the table so prepared, for any expansion containing any number of digits together with their placement is useful also for the evaluation of roots of any order. For example, if we want to get the cube root, it can be evaluated using the cubic terms of the expansion table consisting of the required terms. Similarly if one wants to have a 7^{th} root or 12^{th} root, in general, any root, one has to prepare corresponding expansion table to the required power so that, the corresponding terms which play an important role in the evaluation, are made use of.

It is further interesting to note that decimals also can be worked out using the same table. But placements are to be read as 10^{-1}, 10^{-2}, 10^{-3}, …….. It's also that the following example shows that naming of terms when reversed i.e. occupying the total different positions (in the reverse order) can also be worked out, starting from the reverse order.

Table B

1) $(a\ b\ c)^3$
2) $(c\ b\ a)^3$

1^{st} Case	$(c\ b\ a)^3$	2^{nd} Case	$(a\ b\ c)^3$
10^0	a^3		c^3
10^1	$3a^2b$		$3bc^2$
10^2	$3ab^2$, $3a^2c$		$(3b^2c)$ $(3ac^2)$
10^3	b^3, $6abc$		$(b^3, 6abc)$
10^4	$3ac^2$, $3b^2c$		$(3a^2c)$ $(3ab^2)$
10^5	$3bc^2$		$3a^2b$
10^6	c^3		a^3

From this, one can easily notice that the corresponding placement has to be taken care of in the evaluation, and working from R to L and L to R as the case may be.

The expansion terms are conceived in groups, each making use of symmetry and homogeneity. The terms each group will have the same coefficient.

1) $(h\ g\ f\ e\ d\ c\ b\ a)^3$
2) $(d\ c\ b\ a)^6$

Table C

3) $(g\ f\ e\ d\ c\ \ b\ a)^4$ 7 digited expansion to the power of 4 concern with (3)

Eg. Preparation of the expansion table making use of symmetry is as follows. That the

1) group a^4, b^4, c^4, d^4, e^4, f^4, g^4 (7 terms). This group has the coefficients as $n_{c_0} = 1$

2) Second groups consists of (By reduction by one degree and maintaining homogeneity and following symmetry) (two digit combination) Where in 4 is the power of expansion

a^3b, a^3c, a^3e, a^3f, a^3g,

b^3a, b^3c, b^3d, …..

c^3a, c^3b, c^3d, …..

d^3a, d^3b, d^3c, …..

Number of terms
$7 \times 6 = 42$

e^3a, e^3b, e^3c, ...
f^3a, f^3b, f^3c, ...
g^3a, g^3b, g^3c, ...

Thus Second Group has 42 terms. The coefficients are n_{c_3} $n-1_{c_1}$ = $4_{c_1} \times 1_{c_1} = 4$

3) Third groups consists of again reduction by one more and maintaining the homogeneity and symmetry (two digit combination) a^2b^2, a^2c^2, a^2d^2, a^2e^2, a^2f^2, a^2g^2,

b^2c^2, b^2d^2, ...
c^2d^2, c^2e^2, ...
d^2e^2, d^2f^2, ...
e^2f^2, $e\ g^2$, ...
f^2g^2

Number of terms 21

Third group has 21(terms) The coefficients are n_{c_2} $n-2_{c_2}$ = $4_{c_2} \times 2_{c_1} = 6$

4) Fourth groups consists of three digit combination with the same square term used in the third group (three digit combination) a^2bc, a^2bd, a^2be, a^2bf, a^2bg,

a^2cd, a^2ce, a^2cf, a^2cg, a^2de,
a^2df, a^2dg, a^2ef, a^2eg, a^2fg, 7 × 15 = 105 terms
b^2ac, b^2ad, ... 15 terms
c^2ab, c^2ad, ... "
d^2ab, d^2ac, ... "
e^2ab, b^2ac, ... "
f^2ab, f^2ac, ... "
g^2ab, g^2ac, ... "

Coefficients are n_{c_2} $n-2_{c_1}$ $n-3_{c_1}$ = $4_{c_2} \times 2_{c_1} \times 1_{c_1} = 12$

5) Fifth groups consists of 4 digit combination

abcd, abce, abcf, abcg, abde, abdf, abdg, abef,
abeg, abfg, acde, acdf, acdg, acef, aceg, acfg, 20 terms
adef, adeg, adfg, aefg
bcde, bcdf, bcdg, bcdg, bcef, bceg,
bcfg, bdef, bdeg, bdfg, befg. 10 terms
cdef, cdeg, cdfg, cefg 4terms

defg 1 term

35 terms

Coefficients are n_{c_1} $n-1_{c_1}$ $n-2_{c_1}$ $n-3_{c_1}$ = $4_{c_1} \times 3_{c_1} \times 2_{c_1} \times 1_{c_1} = 24$

This expansion has a total of 210 terms = $\frac{m(m+1)(m+2)(m+3)}{4!}$

Where m is the number of digits, and n is the power $\Rightarrow \frac{7\times8\times9\times10}{4\times3\times2\times1} = 210$

After this preparation a table is to be prepared for the placement of Individual terms as per $10^0, 10^1, 10^2$.....

The placements of various combinations are also given in the table.

From these tables one can get the value of expansion as the sum of all the terms with powers with the respective placements

For example $(7356)^4$, $(356)^4$ (Ref. Table C for the terms)

10^0	1296
10^1	4320×10
10^2	$(2592 + 5400)\ 10^2 = 7992$
10^3	$(6048 + 3000 + 6480)\ 10^3 = 15528$
10^4	$(625 + 1944 + 15120 + 5400)10^4 = 23089$
10^5	$(1500 + 9072 + 12600 + 3240)10^5 = (26612)\ 10^5$
10^6	$(648 + 3500 + 10584 + 1350 + 15120)10^6 = (31202)10^6$
10^7	$(540 + 6300 + 4536 + 17640)10^7 = (29016)10^7$
10^8	$(81 + 7350 + 3780 + 10584)10^8 = (21795)10^8$
10^9	$(8232 + 756 + 8820)10^9 = (7808)10^9$
10^{10}	$(860 + 2646)10^{10} = 9506$
10^{11}	$(4116)10^{11}$
10^{12}	$(2401)10^{12}$

$(7356)^4$ = Duplex of $(7356)^2$ =

D[D(7356)]

D(7) = 49 D(73) = 42 D(735) = 79

D(7356) = 114 D(356) = 61

D(56) = 60 D(6) = 36

49 | 42 | 79 | 114 | 61 | 60 | 36

= 54110736

D(54110736) = 25 | 40 | 26 | 18 | 9 | 72 | 87 | 98 | 68 | 18 | 61 | 42 | 93 | 36 | 36

= 2927971750461696

```
             1296
            43200
           799200
         15528000
        230890000
       2641200000
      31202000000
     290160000000
    2179500000000
   17808000000000
   95060000000000
  411600000000000
 2401000000000000
 2927971750461696   = (7356)^4
```

$(356)^4$	
10^0	1296
10^1	4320×10
10^2	$(2592 + 5400)10^2 = 7992$
10^3	$(3000 + 6480)10^3 = 9480$
10^4	$(625 + 1944 + 5400)10^4 = 23089$
10^5	$(1500 + 3240)10^5 = 4740$
10^6	$(648 + 1350)10^6 = 1998$ D[D(356)]
10^7	$(540)10^7$
10^8	$(81)10^8$ = 16062013696

The above examples can be verified by duplex also

Note: The table so prepared is also useful for the evaluation of the roots as well.

Seven digited expansion to the power 4 with the placements

Let us consider the expansion of $(2\ 1\ 4\ 7\ 3\ 5\ 6)^4$

g f e d c b a (Part of Table P)

Using the expansion table

$(2\ 1\ 4\ 7\ 3\ 5\ 6)^4$

g f e d c b a

10^0	a^4
10^1	$4a^3b$
10^2	$4a^3c + 6a^2b^2$
10^3	$4a^3d + 4b^3a + 12a^2bc$
10^4	$b^4 + 4a^3e + 6a^2c^2 + 12a^2bd + 12b^2ac$
10^5	$4a^3f + 4b^3c + 12a^2be + 12a^2cd + 12b^2ad + 12c^2ab$
10^6	$4a^3g + 4b^3d + 4c^3a + 6a^2d^2 + 6b^2c^2 + 12a^2bf + 12a^2ce + 12b^2ae + 24abcd$
10^7	$4b^3e + 4c^3b + 12a^2bg + 12a^2cf + 12a^2de + 12b^2af + 12b^2cd + 12c^2ad + 12d^2ab + 24abce$
10^8	$c^4 + 4b^3f + 6a^2e^2 + 6b^2d^2 + 12a^2(cg + df) + 12b^2(ag + ce) + 12c^2(ae + bd) + 12d^2ac + 24ab(cf + de)$
10^9	$4ad^3 + 4b^3g + 4c^3d + 12a^2(dg + ef) + 12b^2(cf + de) + 12c^2(af + be) + 12d^2bc + 12e^2ab + 24ab(cg + df) + 24acde$
10^{10}	$4bd^3 + 4c^3e + 6a^2f^2 + 6b^2e^2 + 6c^2d^2 + 12a^2eg + 12b^2(cg + df) + 12c^2(ag + bf) + 12d^2ae + 12e^2ac + 24ab(dg + ef) + 24acdf + 24bcde$
10^{11}	$4bd^3 + 4c^3f + 12a^2fg + 12b^2(dg + ef) + 12c^2(bg + de) + 12d^2(af + be) + 12e^2(ad + bc) + 24abeg + 24ac(dg + ef) + 24bcdf + 12f^2ab$
10^{12}	$d^4 + 4c^3g + 6a^2g^2 + 6b^2f^2 + 6c^2e^2 + 12b^2eg + 12c^2df + 12d^2(ag + bf + ce) + 12e^2bd + 12f^2ac + 24(abfg + aceg + adef + bc(dg + ef)$

To be continued till we get e^4 (10^{16}) $f^4(10^{20})$ and g^4 (10^{24}) and the corresponding terms of 10^{13} upto 10^{24} following the Symmetry and Homogeneity as per the choice.

$(2147356)^4 = 21262591725153678927421696$

To Verify this by duplex method which is as follows.

Consider evaluation of square of (2147356) by duplex and then the result is subjected to the duplex again thus $(2147356)^4$ is evaluated

D(2147356) = Sum of all the following duplexes.

D(2)	D(21)	D(214)	D(2147)	D(21473)	D(214735)	D(2147356)	D147356)	D(47356)	D(7356)	D(356)	D(56)	D(6)
4	4	17	36	42	82	107	94	127	114	61	60	36

4611137790736 = 4611137790736

D(4611137790736) = Sum of all the following Duplexes.

D(4) | D(46) | D(461) | D(4611) | D(46111) | D(461113) | D(4611137) | D(46111377) | D(461113779) | D(4611137790) | D(46111377907) | D(461113779073) | D(4611137790736) | D(611137790736) | D(11137790736) | D(1137790736) | D(137790736) | D(37790736) | D(7790736) | D(790736) | D(90736) | D(0736) | D(736) | D(36) | D(6) = 21262591725153678927421696

$(2147356)^4$ = exactly the given number form the duplex concept

Table D

$(a + b + c + d)^4$ n = 4 s = the number of elements a, b, c, d

Combination Method (Expansion in groups)

a^4	a^3b	a^2b^2	a^2bc	abcd
b^4	a^3c	a^2c^2	a^2bd	
c^4	a^3d	a^2d^2	a^2cd	
d^4	b^3a	b^2c^2	b^2ac	
	b^3c	b^2d^2	b^2ad	
	b^3d	c^2d^2	b^2cd	
	c^3a		c^2ab	
	c^3b		c^2ad	
	c^3d		c^2bd	
	d^3a		d^2ab	
	d^3b		d^2ac	
	d^3c		d^2bc	

No. of terms =

$$\frac{s(s+1)(s+2)(s+3)}{n! = 24} = \frac{4.5.6.7}{24} = 35$$

I) Coefficients of each term of the groups by combination method.

$a^4 \Rightarrow 4_{c_4}$

$a^3b \Rightarrow n_{c_1} \times (n-1)_{c_3} = 4$

$a^2b^2 \Rightarrow n_{c_2} \times (n-2)_{c_2} = 6$

$a^2bc \quad n_{c_2} \times (n-1)_{c_1} \times (n-1-1)_{c_2} = 12$

$abcd \Rightarrow n_{c_1} (n-1)_{c_1} [(n-1)-1]_{c_1} (n-3)_{c_1} = 24$

2) Coefficients by Derivative Method

a^4	a^3b	a^2b^2	a^2bc	$abcd$
1	$\frac{4}{1} = 4$	$\frac{4 \times 3}{2} = 6$	$\frac{6 \times 2}{1} = 12$	$\frac{12 \times 2}{1} = 24$

By Derivation

To get the coefficient of terms of the group a^3b type.

Derivative of a^4 is $4a^3$ and in order to maintain the homogeneity another variable (supposed to be b or c or d) is to be considered and the power of b or c or d is '1'. Therefore the coefficient 4, is to be divided by the power of b or c or d which is 1

Coefficient of a^2b^2 terms of the group

This is obtained from $4a^3b$ term, the coefficient of which is 4. As the coefficient of terms of the group a^2b^2 term is required, the differentiation of $4a^3b$ term is $12a^2b$. But the required term is a^2b^2.Hence we have to consider the derivative being multiplied by b to get the required term a^2b^2. As the power of b is '2' we have to divide 12 by 2 in order to get the coefficient corresponding to a^2b^2 Thus the coefficient is 6.

Coefficient of a^2bc

In order to get the coefficient of a^2bc term, from $6a^2b^2$ one has to reduce b through derivation, keeping $6a^2$ as it is. By this operation, one can get the coefficient of a^2bc term as 12. As the power of c is '1' division by the power of c is uneffected.

Coefficient of abcd

In order to get the coefficient of abcd term from $12a^2bc$ term. one has to reduce the a^2b by derivation, which gives the value $12 \times 2a = 24abc$ but the term now is incomplete, unless d is also considered. Therefore the coefficient of abcd is 24.

The similar argument applies to all the elements coming under the same group for the determination of the coefficient

Table E

$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

1) Binomial method

$6_{c_0} a^6 + 6_{c_1} a^5b + 6_{c_2} a^4b^2 + 6_{c_3} a^3b^3 + 6_{c_4} a^2b^4 + 6_{c_5} ab^5 + 6_{c_6} b^6$

$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

2) Derivative Method

$$a^6 + \frac{6a^5b}{1} + \frac{6\times5a^4b^2}{2} + \frac{15\times4a^3b^3}{3} + \frac{20\times3a^2b^4}{4} + \frac{15\times2ab^5}{5} + \frac{6b^6}{6}$$

3) By Combinations Method

a^6	$\Rightarrow$	$6_{c_6} = 1$
a^5b	$\Rightarrow$	$6_{c_1}(6-1)_{c_5} = b \times 1 = 6 = 6_{c_5}\ 6-5_{c_1}$
a^4b^2	$\Rightarrow$	$6_{c_2}(6-2)_{c_4} \Rightarrow 15 \times 1 = 15 = 6_{c_4}\ 6-4_{c_2}$
a^3b^3	$\Rightarrow$	$6_{c_3}(6-3)_{c_3} \Rightarrow 20 \times 1 = 20 = 6_{c_3}\ 6-3_{c_3}$
a^2b^4	$\Rightarrow$	$6_{c_4}(6-4)_{c_2} \Rightarrow 15 \times 1 = 15 = 6_{c_2}\ 6-2_{c_4}$
ab^5	$\Rightarrow$	$6_{c_1}(6-5)_{c_1} \Rightarrow 6 \times 1 = 6 = 6_{c_5}\ 6-5_{c_1}$
b^6	$\Rightarrow$	$6_{c_6} = 1$

4) Derivative Method $(a+b+c+d+e)^6$

Term	a^6	a^5b	a^4b^2	a^4bc	a^3b^3	a^3b^2c
Coeff		$\frac{6}{1} = 6$	$\frac{6\times5}{2} = 15$	$\frac{15\times2}{1} = 30$	$\frac{15\times4}{3} = 20$	$\frac{20\times3}{1} = 60$
Term	a^3bcd	a^2b^3c	$a^2b^2c^2$	a^2b^2cd	a^2bcde	
Coeff	$\frac{60\times2}{1} = 120$	$\frac{15\times4}{1} = 60$	$\frac{60\times3}{2} = 90$	$\frac{90\times2}{1} = 180$	$\frac{180\times2}{1} = 360$	

5) Combinations Method

Term	a^6	a^5b	a^4b^2
Coeff	$6_{a_6} = 1$	$6_{c_1} \times 6-1_{c_5} = 6$	$6_{c_2} \times 6-2_{c_4} = 15$
Term	a^4bc	a^3b^3	a^3b^2c
Coeff	$6_{c_1} \times 6-1_{c_1} \times 6-2_{c_4} = 30$	$6_{c_3} \times 6-3_{c_3} = 20$	$6_{c_1} \times 6-1_{c_2} = 60$
Term	$a^2b^2c^2$	a^2b^2cd	a^2bcde
Coeff	$6_{c_2} \times 6-2_{c_2} \times 6-4_{c_2} = 90$	$6_{c_1} \times 6-1_{c_1} \times 6-1-1_{c_2} \times 6-1-3_{c_2} = 150$	$6_{c_1} \times 6-1_{c_1} \times 6-1-1_{c_1} \times 6-4_{c}$

Table F

A method for writing down the expansion of $(a+b+c+d+e+\_\_\_\_\_)^n$, n being any positive integer on the basis of symmetry and maintaining homogeneity the following steps are to observed.

Step 1: To write down the n^{th} power of individual terms for example a^n, b^n, c^n ...... etc. These will form a group.

Step 2: To consider products of two terms, wherein one of the terms is raised to the power of (n – 1) and the second raised to the power of 'one'. A complete exhaustion of such combinations is to be considered. For example $a^{n-1}b$, $a^{n-1}c$, $a^{n-1}d$, $a^{n-1}e$, $b^{n-1}c$, ..., $c^{n+}d$, ... Along with each term a reverse combination such as ab^{n-1}, ac^{n-1},..

ad^{n-1}, ... follows indicating a pair of terms as with $a^{n-1}b$, ab^{n-1}, ... and so on. These will form a group

Step 3: (i) To consider a reduction in the power of 'a' by '1' than in the step 2 ie, a^{n-2}. In this case one gets the following two types of combinations of a^{n-2} with other terms in groups as

(i) $a^{n-2}b^2$, $a^{n-2}c^2$, $a^{n-2}d^2$..........

(ii) $a^{n-2}bc$, $a^{n-2}bd$, $a^{n-2}bc$ $a^{n-2}cd$, $a^{n-2}ce$ $a^{n-2}de$, acb^{n-2}, abc^{n-2}, $a^{n-3}b^3$, a^3b^{n-3}, etc

These will form two different groups.

Similarly we have to write down the series starting from b^{n-2} and taking the products of square of other terms which will form one series. This procedure is continued with other terms both to have type i and ii type as well.

This procedure has to be continued and completed with each term separately. Thus a series of terms starting with b, another starting with c, d, e .... and so on each giving rise to two sets of terms different from one another. Terms starting from 'b' will form series in similarity with the terms starting from 'c' to form series: These form the elements of groups i and ii of step 3.

Step 4: Starting from a^{n-3} to write down all possible combinations with other terms (maintaining homogeneity) For example combinations such as

I (i) $a^{n-3}b^3$, $a^{n-3}c^3$, $a^{n-3}d^3$, ... group

(II) $a^{n-3}b^2c$, $a^{n-3}b^2d$, $a^{n-3}b^2e$, ... group

$a^{n-3}bc^2$, $a^{n-3}bd^2$, $a^{n-3}be^2$, ... (These are the symmetrical reverse combinations)

(III) $a^{n-3}bcd$, $a^{n-3}bce$, ... groups

$a^{n-3}cde$, $a^{n-3}cef$, ... groups

Similar terms have to be written starting from $b^{n-3}$and completing the combinations with other terms under group I as

I $b^{n-3}c^3$, $b^{n-3}d^3$ ....

II $b^{n-3}a^2c$ ($b^{n-3}ac^2$), $b^{n-3}a^2d$, ($b^{n-3}ad^2$) .... $b^{n-3}c^2d$ ($b^{n-3}cd^2$), $b^{n-3}c^2e$, ($b^{n-3}ce^2$) .... and so on until with the combination with the last term.

III $b^{n-3}acd$, $b^{n-3}ace$, $b^{n-3}acf$ ....

$b^{n-3}cde$, $b^{n-3}cef$ ....

$b^{n-3}def$ ....

This process is to be confirmed starting from c^{n-3}, d^{n-3}, e^{n-3}, ... etc and so on. This reduction in the power by one each time stepwise and to continue to write down the various products maintaining the homogeneity is to be completed. The final series of terms will be the last series in the expression raised to power n.

How to write down the coefficients of each term belonging to various groups. From symmetry considerations, the result consists of symmetrical groups such as.

Group I a^n, b^n, c^n, ... etc

Group II $a^{n-1}b$, $b^{n-1}c$, ... etc

Group III	$a^{n-2}b^2, b^{n-2}c^2$, ... etc
Group IV	$a^{n-2}bc, b^{n-2}bd, b^{n-2}be$ .... etc
Group V	$a^{n-3}b^3$ .... etc
Group VI	$a^{n-3}b^2c$ .... etc
Group VII	$a^{n-3}bcd$ .... etc and so on

In all these cases the coefficient of terms will be same in each group and may be different from group to group. But all the terms belonging to the same group will have the same coefficient. Two formulae could be suggested to arrive at the coefficients of each term belonging to different groups.

Table G

To find out the coefficient of the terms belonging to the three

1) The higher power of single numbers as coefficient
2) Derivation of the first gives $3a^2b$
3) The derivative of $3a^2b$ in 6abc making homogeneous

$(a + b + c + d + e)^6$

I Group a^6, b^6, c^6, d^6, e^6

Coefficient $6_{c_6} = 1$

II Group a^5b, a^5c, a^5d, a^5e

ab^5, ac^5, ad^5, ae^5

$b^5c, b^5d, b^5e, c^5d, c^5e, d^5e$

$bc^5, bd^5, be^5, cd^5, ce^5, de^5$

Coefficient $6_{c_1}\ 5_{c_5} = 6$

III Group $a^4b^2, a^4c^2, a^4d^2, a^4e^2, b^4c^2, b^4d^2, b^4e^2$

$a^2b^4, a^2c^4, a^2d^4, a^2e^4, b^2c^4, b^2d^4, b^2e^4$

c^4d^2, c^4e^2, d^4e^2

$c^2d^4, c^2e^4, d^2e^4,$

Coefficient $6_{c_4}\ 2_{c_2} = 15$

IV Group $a^4bc, a^4bd, a^4be, a^4cd, a^4ce, a^4de$

$ab^4c, ab^4d, ab^4e, ac^4d, ac^4e, ad^4e$

$abc^4, abd^4, abe^4, acd^4, ace^4, ade^4$

$b^4cd, b^4ce, b^4de, c^4de$

$bc^4d, bc^4e, bd^4e, cd^4e$

$bcd^4, bce^4, bde^4, cde^4$

Coefficient $6_{c_4}\ 2_{c_1}\ 1_{c_1} = 15$

V Group $a^3b^3, a^3c^3, a^3d^3, a^3e^3, b^3c^3, b^3d^3, b^3e^3, c^3d^3, c^3e^3, d^3e^3,$

Coefficient $6_{c_3}\ 3_{c_3} = 20$

VI Group a^3b^2c, a^3b^2d, a^3b^2e, a^3c^2d, a^3c^2e, a^3d^2e,
a^3bc^2, a^3bd^2, a^3be^2, a^3cd^2, a^3ce^2, a^3de^2,
a^3a^2c, b^3a^2d, b^3a^2e, b^3c^2d, b^3c^2e, b^3d^2e
b^3ac^2, b^3ad^2, b^3ae^2, b^3cd^2, b^3ce^2, b^3de^2
c^3a^2b, c^3a^2d, c^3a^2e, c^3b^2d, c^3b^2e, c^3d^2e
c^3ab^2, c^3ad^2, c^3ae^2, c^3bd^2, c^3be^2, c^3de^2
d^3a^2b, d^3a^2c, d^3a^2e, d^3b^2c, d^3b^2e, d^3c^2e
d^3ab^2, d^3ac^2, d^3ae^2, d^3bc^2, d^3be^2, d^3ce^2
e^3a^2b, e^3a^2c, e^3a^2d, e^3b^2c, e^3b^2d, e^3c^2d
e^3ab^2, e^3ac^2, e^3ad^2, e^3bc^2, e^3bd^2, e^3cd^2
Coefficient $6_{c_3}\ 3_{c_2}\ 1_{c_1} = 60$

VII Group a^3bcd, a^3bce, a^3bde, a^3cde,
ab^3cd, ab^3ce, ab^3de, ac^3de
abc^3d, abc^3e, abd^3e, acd^3e
$abcd^3$, $abce^3$, $abde^3$, $acde^3$
$bcde^3$, bcd^3e
bc^3de, $bcde^3$
Coefficient $6_{c_1}\ 3_{c_1} 2_{c_1}\ 1_{c_1} = 120$

VIII Group $a^2b^2c^2$, $a^2b^2d^2$, $a^2b^2e^2$, $a^2c^2d^2$, $a^2c^2e^2$, $a^2d^2e^2$, $b^2c^2d^2$, $b^2c^2e^2$, $b^2d^2e^2$, $c^2d^2e^2$
Coefficient $6_{c_2} 4_{c_2}\ 2_{c_2} = 90$

IX Group a^2b^2cd, a^2b^2ce, a^2b^2de
abc^2d^2, abc^2e^2, abd^2e^2
a^2c^2bd, a^2c^2be, a^2c^2de
acb^2d^2, acb^2e^2, acd^2e^2
adb^2c^2, adb^2e^2, adc^2e^2
a^2e^2bc, a^2e^2bd, a^2e^2cd
aeb^2c^2, aeb^2d^2, aec^2d^2
b^2c^2de, bcd^2e^2, b^2d^2ce
bdc^2e, c^2d^2be, cdb^2e^2
Coefficient $6_{c_1} 5_{c_2} 3_{c_2} 1_{c_1} = 180$

X Group a^2bcde, ab^2cde
abc^2de, $abcd^2e$
$abcde^2$
Coefficient $6_{c_1} 5_{c_1} 4_{c_1} 3_{c_1} 2_{c_2} = 360$

The number of combinations in the expansion $(a + b + c + \ldots\ldots)^m$ where m is positive integer is $\frac{n(n+1)(n+2)....m\ terms}{m!}$ where n stands for number of terms a, b, c in the given expression.

<u>For Example</u>:

$(a + b + c)^3$ will have m = 3 n = 3

$\therefore$ Number of combinations = $\frac{3.4.5}{3!} = 10$

$(a + b + c + d + e)^6$ will have m = 6, n = 5

$\therefore$ Number of combinations = $\frac{5.6.7.8.9.10}{6!} = 210$

A few examples

i) $(a + b)^2$ will have three combinations Group I a^2, b^2 with coefficient $2_{c_2} = 1$

Group II ab with the coefficient $2_{c_1}\ 1_{c_1} = 2$

ii) $(a + b + c)^2$ will have $\frac{3.4}{2!} = 6$ combinations

I Group a^2, b^2, c^2 Coefficient $2_{c_2} = 1$

II Group ab, ac , bc with the coefficient $2_{c_1} 1_{c_1} = 2$

iii) $(a + b + c + d)^3$ will have $\frac{4.5.6}{3!} = 20$ combinations

I Group a^3, b^3, c^3, d^3 with the Coefficient $3_{c_3} = 1$

II Group a^2b, a^2c, a^2d, ab^2, ac^2, ad^2, b^2c, b^2d, c^2d, bc^2, bd^2, cd^2 with the coefficient $3_{c_2}\ 1_{c_1} = 3$

III Group abc, abd, acd, bcd with the coefficient $3_{c_1}\ 2_{c_1}\ 1_{c_1} = 6$

$(a + b + c + d)^3$ = sum of all the combinations $a^3 + b^3 + c^3 + d^3 + 3(a^2b + ab^2 + a^2c + ac^2 + a^2d + ad^2 + b^2c + bc^2 + b^2d + bd^2 + c^2d + cd^2) + 6(abc + abd + acd + bcd)$

Table H

Expansions use of terms in roots

$(2\ 5\ 3)^5$ using $(cba)^5$ expansion. A is in 10^0, b is 10^1, c is in 10^2's place

$10^0 \longrightarrow a^5 = 3^5 = 243$

$10^1 \longrightarrow 5a^4b = 2025$

$10^2 \longrightarrow 10a^3b^2 + 5ca^4 = 6750 + 810 = 7560$

$10^3 \longrightarrow 10a^2b^3 + 20a^3bc = 11250 + 5400 = 16650$

$10^4 \longrightarrow 5ab^4 + 10c^2a^3 + 30a^2b^2c = 9375 + 1080 + 13500 = 23955$

$10^5 \longrightarrow b^5 + 20b^3ca + 30c^2a^2b = 3125 + 1500 + 5400 = 23525$

$10^6 \longrightarrow 5b^4c + 10c^3a^2 + 30b^2c^2a = 6250 + 720 + 9000 = 15970$

$10^7 \longrightarrow 10b^3c^2 + 20c^3ab = 5000 + 2400 = 7400$

$10^8 \longrightarrow 10b^2c^3 + 5c^4 = 2000 + 240 = 2240$

$10^9 \longrightarrow 5bc^4 = 400$

$10^{10} \longrightarrow c^5 = 32$

```
 320000000000
 400000000000
 224000000000
  74000000000
  15970000000
   2352500000
    239550000
     16650000
       756000
        20250
          243
-------------
1036579476493
```

Note: $(253)^2$ = D(253) 4|20 | 37 | 30 | 9

$(253)^2 = 64009$ —— (A)

D(64009) = 36 | 48 | 16 | 0 | 108 | 72 |0 |0 | 811 = 4097152081 —— (B)

(4097152081 × 253)

using urdhva tiryak $\begin{pmatrix} 4097152081 \\ 000000253 \end{pmatrix} = 1036579476493$

$(253)^5 = (253)B$. B = D[D(253)] This can be carried out using urdhva tiryak method. This is also seen equal to evaluating by the duplex.

8 | 20 | 30 | 59 | 64 | 36 | 32 | 25 | 22 | 72 | 29 | 3 = 1036579476493

Note 1: Fifth power can be evaluated by using duplex concept also. This is shown below $(253)^5$ can be written as $[(253)^2]^2$ 253. $(253)^2$ can be expanded...... using Duplex.

Note 2: $(253^2)^2$ is obtained by considering the duplex of (253^2)

It is interesting to note that the same table is used for obtaining the fifth power and fifth root. Similarly one can attempt for sixth root and so on with the corresponding terms in the expansion.

Table I

$(a + b + c)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 + 6a^5c + 15a^4c^2 + 20a^3c^3 + 15a^2c^4 + 6ac^5 + c^6 + 6b^5c + 15b^4c^2 + 20b^3c^3 + 15b^2c^4 + 6bc^5 + 30a^4bc + 30b^4ac + 30c^4ab + 60a^3b^2c + 60a^3bc^2 + 60b^3a^2c + 60b^3c^2a + 60c^3a^2b + 60c^3b^2a + 180a^2b^2c^2$ (28 terms)

One can also get the fourth digit contribution to the table and similarly any number of digits provided one takes care of the additional terms consequent on the additional digit.

10^0	a^6								
10^1	$6a^5b$								
10^2	$6a^5c$	$15a^4b^2$							
10^3	$6a^5d$	$20a^3b^3$	$30a^4bc$						
10^4		$15a^2b^4$	$15a^4c^2$	$60a^3b^2c$	$30a^4bd$				
10^5		$6ab^5$	$60a^3c^2b$	$60b^3a^2c$	$30a^4cd$	$60a^3b^2d$			
10^6		b^6	$20a^3c^3$	$30b^4ac$	$15a^4d^2$	$60b^3a^2d$	$120a^3bcd$		
10^7		$6b^5c$	$60b^3c^2a$	$60c^3a^2b$	$180a^2b^2c^2$	$60a^3bd^2$	$60a^3c^2d$	$360a^2b^2cd$	
10^8		$15a^2c^4$	$15b^4c^2$	$60c^3b^2a$	$30ab^4d$	$60a^3cd^2$	$180a^2b^2d^2$	$120ab^3cd$	$360a^2c^2bd$
10^9		$20b^3c^3$	$30c^4ab$	$20a^3d^3$	$6b^5d$	$60b^3ad^2$	$60c^3a^2d$	$360a^2d^2bc$	$360adb^2c^2$
10^{10}		$6ac^5$	$15b^2c^4$	$15b^4d^2$	$30b^4cd$	$60b^3c^2d$	$180a^2c^2d^2$	$120abc^3d$	$360acb^2d^2$
10^{11}		$6bc^5$	$30ac^4d$		$60d^3a^2b$	$60d^3a^2c$	$60b^3cd^2$	$60c^3b^2d$	$360abc^2d^2$
10^{12}		c^6	a^2d^4	$20b^3d^3$	$60d^3ab^2$	$60c^3ad^2$	$60d^3bc^2$	$180b^2c^2d^2$	$120abcd^3$
10^{13}					$30bc^4d$	$30abd^4$	$60d^3ac^2$	$60c^3bd^2$	
10^{14}				b^2d^4	c^5d	$30acd^4$		$60d^3b^2c$	
10^{15}					c^4d^2	$30bcd^4$			
10^{16}				bd^5	c^3d^3				
10^{17}					c^2d^4				
10^{18}	d^6				cd^5				

Table J

The total number of terms in the expansion $(a + b + c \ldots)^n$. (. . 1)(. . 2) ...upto m terms

the last term being $(n + m - 1)$

n = No. of elements in the expression for expansion, m = power

$(a + b + \ldots\ldots)^2 = \frac{n(n+1)}{2!}$ terms

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b + c)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 3ac^2 + c^3 + 3b^2c + 3bc^2 + 6abc$

$(a + b + c + d)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 3ac^2 + c^3 + 3a^2d + 3ad^2 + d^3 + 3b^2c +$
$3bc^2 + 3b^2d + 3bd^2 + 3c^2d + 3cd^2 + 6abc + 6abd + 6acd + 6bcd$

$(a + b)^2 = \frac{n(n+1)}{2!}$ where n is the number of elements and this can be extended to any number of elements

For two elements $n = 2 \Rightarrow \frac{2(3)}{-} = 3$ terms in the expansion

For three elements $(a + b + c)^2 \Rightarrow \frac{3(4)}{2}$ = 6 terms in the expansion

For four elements $(a + b + c + d)^2 \Rightarrow \frac{4(5)}{2}$ = 10 terms in the expansion and so on.

From this one can have an idea about the number of terms in the expansion.

$(a + b)^3 = \frac{n(n+1)(n+2)}{3!}$ where n is the number of elements. This can be extended to any number of elements

For two elements $(a + b)^3 = \frac{2(3)(4)}{6}$ = 4 terms in the expansion

For three elements $(a + b + c)^3 = \frac{3(4)(5)}{3!}$ = 10 terms in the expansion

For four elements $(a + b + c + d)^3 = \frac{4(5)(6)}{3!}$ = 20 terms in the expansion

$(a + b)^4 = \frac{n(n+1)(n+2)(n+3)}{4!}$

If m = 4 for two elements, 5 terms 3 elements 15 terms for 4 elements 35 terms

Similarly

$(a + b)^5 = \frac{n(n+1)(n+2)(n+3)(n+4)}{5!}$ n is the number of elements

For two elements, 6 terms

for three elements, 21 terms, 4 elements 56 terms and so on in the expansion

Similarly

$(a + b)^6 = \frac{n(n+1)(n+2)(n+3)(n+4)(n+5)}{6!}$ where n is the number of elements

for two elements 7 terms, 3 elements 28 terms and so on.

The procedure we adopted here for finding out the various terms in each expansion is different from the usual binomial expansion.

The method makes use of the following salient points

1) The expansion consists of homogeneous terms which represent combinations of its elements. For example if it is a square as $(a + b)^2$ all the terms in the expansion should be combinations of the elements maintaining homogeneity such as a^2, b^2, ab. If it is a sixth degree expansion then the terms will satisfy that homogeneity.
2) Exhaustive products of the elements through all possible combinations is aimed at maintaining the homogeneity as well.
3) To place each term under 10^α place where α is 0,1,2,..... One has to designate the units place (10^0), 10's place, 10^1 ....., 100's place 10^2 and so on

For example

..... + c + b + a

a is in units place-ie 10^0
b is in 10's place ie 10^1
c is in 100's place ie 10^2
Product of ba is in 10's place
Product of ca is in 100's place
Similarly any combination of abc.. is to be identified in their respective places for example c^4 is to be found in 10^8 place, a^3bc in 100's place, ac^2 in 10^4's place and so on.

4) After obtaining a particular expansion for example $(c + b + a)^3$, if one can add another element 'd' also in the expansion by making suitable additions of the element 'd' in combinations with a, b, c, d into the table already obtained for $(c + b + a)^3$ (Table K)
If d were to be introduced in thousands place then in the table K, terms marked in the circles are to be added which is equal to $(d + c + b + a)^3$ Refer Table L
From the same table K intended for $(c + b + a)^3$, one can read and also $(b + a)^3$

5) Terms in the expansion are to be eliminated if they don't satisfy the rule concerned with that of homogeneity of the term in relation to the power of the expansion
This expansion table is useful for finding out not only n^{th} order expansion but also n^{th} root of a given number which is demonstrated in the problems (section).

The usefulness of such tables explained by Swamiji is really significant and novel, wherein a most general expansion and finding out the roots of the numbers, can be obtained is possible

How to write down the expansion without using Binomial expansion is demonstrated below.

This is obtained

1) From symmetry consideration
2) Homogeneity
3) Number of terms in the expansion from the formulae
4) Placement of each term against a particular position in the final expansion.
5) Grouping different terms along with the determination of the coefficients of each term coming under different groups (terms belonging to one group have the same coefficient).

It is illustrated as

$(c + b + a)^6$......(Table I)

The readers those who are interested can try different combinations of elements.

$(.....l + k + j + ... + d + c + b + a)^n$

It is seen that the coefficients worked out by this method are same as the coefficients in the binomial expansion of two elements.

Table K

Binomial expansion

1) $(a + b)^3 = 3_{C_0} a^3 + 3_{C_1} a^2b + 3_{C_2} ab^2 + b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

2) By considering a as one element and (b + c) as the second element.

$(a + b + c)^3 = (a + (b + c))^3$

$= a^3 + (b + c)^3 + 3a^2(b + c) + 3a\,(b + c)^2$

$= a^3 + (b + c)^3 + 3a^2b + 3a^2c + 3a(b^2 + 2bc + c^2)$

$= a^3 + (b + c)^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2$

$= a^3 + b^3 + c^3 + 3ab^2c + 3bc^2 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2$

3) Symmetry consideration Maintaining homogeneity

10^0	a^3			
10^1		$3a^2b$		
10^2		$3a^2c$	$3ab^2$	
10^3	b^3			$6abc$
10^4		$3b^2c$	$3ac^2$	
10^5				
10^6	c^3		$3bc^2$	
10^7				

Note: Needless to say that Swamiji's method is more elegant and easier even for substitution and finding out the sum.

Symmetry considerations in writing down the expansion

$(c + b + a)^3$

a is in 10^0 place

b is 10^1 place

c is 10^2 place

in the cubic expansion

Step 1: a^3 will be in 10^0 place

Step 2: Reduce the power of a^3 by 1 and keep the coefficient equal to 3 the power in the first terms ie $3a^2$ to make this term homogeneous it will be $3a^2b$ and is placed in 10^1 place

Step 3: Reduce the power of 'a' by one more so that the term contains 'a'. In order to maintain the homogeneity it will be ab^2

The coefficient of the term ab^2 is obtained $\frac{3 \times 2}{2}$ (3 is the coefficient of a^2b term, 2 is the power of 'a' in the term) divided by '2' is the power of b in the term ab^2. Thus the third term in the expansion is $3ab^2$

Step 4: Further reduction of one more along with the homogeneity concept leads to b^3 the coefficient of b^3 is $\frac{3\times 1}{3}$ (3 is the coefficient of term containing a in the third term) dividing by '3' representing the power of b.

Step 5: The second combination will be between a and c. the coefficients are exactly same as obtained in the combinations of a and b and in the same order

$3a^2c$ is to be placed 10^2 place

$3ac^2$ is to be placed 10^4 place

c^3 is to be placed 10^6 place

Step 6: The third combination is between b and c and the terms are

$3b^2c$ is to be placed in10^4 place

$3bc^2$ is to be placed in 10^5 place

Step 7: Last combination will be abc the coefficient of which is 3 × 2 (3 is the coefficient of a^2 term, 2 is the power of a) is to be placed in 10^3 place. This method of finding the coefficient is also similar to the derivative method. Now the following is the consolidated table. Showing the terms in the $(c + b + a)^3$ expansion.

Table L

10^0	a^3			
10^1		$3a^2b$		
10^2		$3a^2c$	$3ab^2$	
10^3	b^3	$3a^2d$		$6abc$
10^4		$3b^2c$	$3ac^2$	$6abd$
10^5		$3b^2d$	$3bc^2$	$6acd$
10^6	c^3		$3ad^2$	$6bcd$
10^7		$3c^2d$	$3bd^2$	
10^8		$3cd^2$		
10^9	d^3			
10^{10}				

Table gives the necessary terms required for working out the cube of a number $a + 10^1b + 10^2c + 10^3d$. One can extend this table to any digit in any position of the number even more than d.

Same tables are useful for finding out the roots of numbers.

For the determination of the coefficients of the expanded terms is attempted on the basis of symmetry grouping of the terms and general formale applicable for each group is developed and is explained

Let us consider the number 253 (cba) = $a + 10^1b + 10^2c$ (Refer Table K)

$$cba = 3 + 50 + 200 = 253$$

10^0 $a^3 = 27$

10^1 $3a^2b = 135$

10^2		$3a^2c = 54$	$3ab^2 = 225$	
10^3	$b^3 = 125$			$6abc = 180$
10^4		$3b^2c = 150$	$3ac^2 = 36$	
10^5			$3bc^2 = 60$	
10^6	$c^3 = 8$			
10^7				
10^8				

$27 + 135(10^1) + 279(10^2) + 305(10^3) + 186(10^4) + 60(10^5) + 8(10^6) = 16194277$

$(253)^3 = 16194277$

Example

$(23.23)^3 = 12535.67227$ for cubic expansion terms (Refer Table K)

a = 23 which can be taken as in 10^0 for the purpose of expansion in decimal numbers b is in 10^{-1} and c is in 10^{-2} here b = 2, c = 3 $(a.\ b\ c)^3$

$a^3 =$	12167	10^0
$3a^2b =$	3174	10^{-1}
$3a^2c =$	4761	10^{-2}
$3ab^2 =$	276	10^{-2}
$b^3 =$	8	10^{-3}
$6abc =$	828	10^{-3}
$3b^2c =$	36	10^{-4}
$3ac^2 =$	621	10^{-4}
$3bc^2 =$	54	10^{-5}
$c^3 =$	27	10^{-6}

```
12167.000000
   317.4
   47.61
      2.76
       .008
       .828
       .0036
       .0621
       .00054
       .000027
12535.672267
```

D(23.23) = 539.96329

By Urdhva – Tiryak

```
  539.6329   Or   5396329
  00023.23           2323
12535.672267  12535.672267
```

6 decimal points to be counted from RHS

For verification, the working of the cube root of 12535.672267, Refer Cube roots Eg 7 using Swamiji's Method.

Cubing Using Series Multiplication

Series multiplication (Refer Lecture Notes – I on Multiplication). This can also be expanded using the symmetries in a general way as shown below.

Series Multiplication (numbers)

$(\ldots.c + b + a)^3$

a	b	c	d
a	b	c	d
a	b	c	d

Units	a^3	
Tens	$3a^2b$	10^1
Hundreds	$3b^2a + 3a^2c$	10^2
Thousands	$3a^2d + 6abc + b^3$	10^3
Ten Thousands	$3b^2c + 3c^2a + 6abd$	10^4
Lakh	$3bc^2 + 3b^2d + 6acd$	10^5
Ten Lakhs	$6bcd + c^3 + 3ad^2$	10^6
One Crores	$3c^2d + 3bd^2$	10^7
Ten Crores	$3cd^2$	10^8
Hundred Crores	d^3	10^9

General expansion of $(a + b + c + d + e + f + g)^3$ with the coefficients

$a^3 + 3a^2 (b + c + d + e + f + g) + 3a(b + c + d + e + f + g)^2 + b^3 + 3b^2 (c + d + e + f + g) + 3b (c + d + e + f + +g)^2 + c^3 + 3c^2 (d + e + f + g) + 3c (d + e + f + g)^2 + d^3 + 3d^2 (e + f + g) + 3d (e + f + g)^2 + e^3 + 3e^2 (f + g) + 3e (f + g)^2 + f^3 + 3f^2g + 3fg^2 + g^3$

Cubing of Polynomials

Eg $(2x + 3y + z + 2)^3$

By given expression.

$(2x + 3y + z + 2)^2 = 4x^2 + 12xy + 4xz + 9y^2 + 8x + 6yz + 12y + 4z + 4 + z^2$

$(2x + 3y + z + 2)^3$ = Duplex $\times$ given expression by Urdhva Tiryak

$(4x^2 + 12xy + 4xz + 9y^2 + 8x + 6yz + 12y + 4z + 4 + z^2)(2x + 3y + z + 2) =$

$54y^2x + 24x^2 + 27y^3 + 6xz^2 + 36xyz + 72xy + 27y^2z + 54y^2 + 24x^2z + 9yz^2 + 24x + 36yz + 6z^2 + 36y + 12z + z^3 + 8 + 12x^2z + 36x^2y + 8x^3$

Series multiplication for $(2x + 32x + 3z + 2)^3$

2x	3y	z	2
2x	3y	z	2
2x	3y	z	2

By using Series multiplication procedure

$(2x)^3 + 3(2x)(3y)^2 + 3(2x)(z)^2 + 3(2x)(2)^2 + 3(3y)(z)^2 + 3(3y)(2)^2 + 3(3y)(2x)^2 + 3(z)(2x)^2 + 3(z)(3y)^2 + 3(z)(2)^2 + 3(2)(2x)^2 + 3(2)(3y)^2 + 3(2)(z)^2 + (3y)^3 + z^3 + 2^3 + 6(2x)(3y)(z) + 6(3y)(z)(2) + 6(2x)(3y)(2) + 6(2x)(z)(2) = 8x^3 + 36x^2y + 12x^2z + 24x^2 + 54xy^2 + 6xz^2 + 24x + 36xyz + 72xy + 24xz + 27y^3 + 27y^2z + 54y^2 + 9yz^2 + 36y + 36yz + z^3 + 12z + 6z^2 + 8$

By using the general expansion $8x^3 + 12x^2 (3y + z + 2) + 6x (3y + z + 2)^2 + 27y^3 + 27y^2 (z + 2) + 9y (z + 2)^2 + z^3 + 12z + 6z^2 + 8 = 8x^3 + 36x^2y + 12x^2z + 24x^2 + 54xy^2 + 6xz^2 + 24x + 36xyz + 72xy + 24xz + 27y^3 + 27y^2z + 54y^2 + 9yz^2 + 36y + 36yz + z^3 + 12z + 6z^2 + 8$

In the series multiplication the coefficient is automatically obtainable

Cube Roots And Higher Roots (Numbers And Polynomials in One Variable)

Cube Roots

On the basis of Vedic Sutras determination of cube roots is explained (1) with the help of the method of straight division for both exact and non exact cubes adopted by Vedic Mathematics and (2) an indirect application of Lopana Sthapanabhyam Sutram, in addition to the application of well–known first principles which is also based throughout by inspection and argumentation, (for the equation of cube roots of exact cubes). The methods explained for cube roots are most general in the sense that they can by applied to any digited numbers. This method is also paving way to the roots of higher powers. A few examples for the determination of roots of 4^{th} and 5^{th} powers are also attempted.

By Argumentation – for cube roots of exact cubes.

We start with well-known basic principles

Cube roots of exact Cubes

(1) Cubes of 1,2, ... 9 are given below

$1^3 = \underline{1}$

$2^3 = \underline{8}$

They all have their own district endings and there no possibility of overlapping as is found in Squares.

$3^3 = 2\underline{7}$

$4^3 = 6\underline{4}$

$5^3 = 12\underline{5}$

$6^3 = 21\underline{6}$

$7^3 = 34\underline{3}$

$8^3 = 51\underline{2}$

$9^3 = 72\underline{9}$

(2) The last digit of the cube root of an exact cube is as given below.

If (1)	Cube ends in 1,	Cube root ends in 1
(2)	Cube ends in 2,	Cube root ends in 8
(3)	Cube ends in 3,	Cube root ends in 7
(4)	Cube ends in 4,	Cube root ends in 4
(5)	Cube ends in 5,	Cube root ends in 5
(6)	Cube ends in 6,	Cube root ends in 6
(7)	Cube ends in 7,	Cube root ends in 3
(8)	Cube ends in 8,	Cube root ends in 2

1, 4, 5, 6, 9, 0 repeat the cubes in the Cube endings. That means Last digit of the cube and the last digit of cube root are same.

2, 3, 7, 8 have an inter – play of complements from 10. This is to be understood as: if cube ends in 2 cube roots ends in 8 and vice versa

(3) The given number is grouped into 3–digit groups from right to left. While doing so if the Left most group contains one or two digits as the case may be this is to be counted as one group. The left most is counted as first group from which the cube root starts (1) The number of digits (n) in the cube root is the same as the number of groups. The number of digits, the first digit and the last digit of the cube root of an exact cube are the data with which we start working to extract the cube root. For preliminary work on the above factors few examples are given below This grouping is necessary to frame the left hand most group, From this we can determine the first digit (F) in the cube root. Examples are given below.

(i) 79507

Grouping: (79) (507)

n = 2 i.e. there are two digits in the cube root

Since the cube ends in 7, cube root should end in 3. This is designated by L $\therefore$ L = 3 consider the Cube root of left most group (79) trying the cubes of 1–9 which fits and should be <79 nearly the considered group. (79). Therefore in this case first digit (F) of the cube root is 4, F = 4. The data on n, L, F are thus obtained.

A few more examples for obtaining n, L, F

(ii) 22188041

Groupings : 22 188 041

n = 3, L = 1, F = 2

(iii) 487443403

Groupings: 487 443 403

n = 3, L = 7, F = 7

(iv) 80677568161

Groupings: 80 677 568 161

n = 4, L = 1, F = 4

The Algebrical principle ultilised is given below.

Algebrical Principle

Any number JKL can be written in algebrical form as
L + 10K + 100J + 1000H etc.,
Consider a three – digits number JKL.
Algebrical form is L + 10K + 100J.

This means that the studies of J,K and L are JKL stand (Hundreds, Tens, and Units) respectively, consider its cube i.e., $(L + 10K + 100J)^3$. The cubing of JKL. A Series multiplication of JKL

JKL
JKL

(1) Units Place is $\underline{L^3}$
(2) Tens Place is $\underline{3L^2K}$
(3) Hundreds place is $\underline{3LK^2 + 3L^2J}$
(4) Thousands place is $\underline{K^3 + 6LKJ}$
(5) Ten thousands place is $3LJ^2 + 3K^2J$
(6) Lakhs place is $3KJ^2$
(7) Millions place is J^3

The method adopted by Swamiji is to consider the given cube for extraction of the cube root as starting from Units place digit moving towards left. For this purpose the letters designated are L 'in Units ' place, K in ten place , J in hundreds' place and H in thousands' place and so on.

The analytical sorting of the above parts in the algebrical expansion into their respective places results in eliminating letter after letter and determining the previous digit in the given cube. The amounts to working of the elimination of the digit in the units place, tens' place and in the hundreds' place of the given cube which are algebrical expansion parts $\underline{L^3}$, $\underline{3L^2K}$, and $\underline{3L^2J + 3LK^2}$ whose values are to be first obtained.

(b) (1) From Units place of the given cube we subtract the value of L^3 and that eliminates the last digit in the cube. It fixes the digit in the tens place.

(2) From such fixed tens place digit, the value of K can be determined. We subtract the value of $3L^2K$ and thus eliminate the penultimate digit tens place. This procedure when extended to the digit in the hundreds place the value of J can be determined. In case where J is to be the first digit of the

cube root, J can be thus independently determined without recoursing to the above procedure.

(3) From the hundreds place, we Subtract the value of $3L^2J + 3LK^2$ and eliminate the pre penultimate digit (Hundreds Place) and so on.

(c) The values of L, K, J in the Cube root are determinable from the exact expression and the values of the expansion parts of the given cube concerned with units, tens and hundreds.

If it is more than 3–digits such as L, K, J, H also in the Cube root, then one has to go into the expression for thousands and so on.

Knowing the value of L, which can be determined clearly from any given cube, one can evaluate the tens' digit value K and similarly by substituting the values of L and K, the value of J can be determined. The method of such determinations are clearly explained in the following examples.

The Lopana sthapanabhyam sutram is applied by eliminating successively the digits in units, Tens, and so on in the cube and simultaneously establishing the penultimate digits.

From the hundreds place, we Subtract the value of $3L^2J + 3LK^2$ and eliminate the pre penultimate digit (Hundreds Place) and soon.

Argumentation

(1) 79507

Grouping: 79 507

n = 2, F = 4, L = 3

Step1: $L^3 = 27$

Subtracting L^3 from the Units place, which eliminates the last digit from the given cube

$$\begin{array}{r} 79507 \\ -\underline{\quad 27} \\ 7948 \end{array}$$

Step2: $3L^2K = 27K$ (as derived from 1st Step)

27K should end in 8 for the elimination of the tens place digit

By argumentation, this is achieved by multiplying 7 by 4 only

$\therefore$ K = 4 $\therefore$ $\sqrt[3]{79507} = 43$

(2) 22188041

Grouping: 22 188 041

n = 3 , F = 2, L = 1

Step1: $L^3 = 1$

Subtracting L^3 from the units place and thus eliminating the last digit from the given cube

22188041

2218804

Step2: $3L^2K = 3K$

3K ends in 4 as shown in step 1

By argumentation this is achieved by this multiply 3 by 8 $\therefore$ K = 8

Subtracting $3L^2K$ from tens places and thus eliminates the penultimate digit.

$3L^2K = 24$

2218804
24
221878

Step 3: $3L^2J + 3LK^2 = 3J + 192$ ends in 8 as shown in step 2

By argumentation this is achieved by making 3J end in 6 i.e. J should be 2

$\therefore$ 3J ends in 6 $\therefore$ J = 2 $\therefore$ $\sqrt[3]{22188041} = 281$

(3) 487443403

Groupings : 487 443 403

n = 3, F = 7 L = 7

Step 1: $L^3 = 343$

Subtracting L^3, from the given cube

487443403
343
48744306

Step 2: $3L^2K = 147K$ ends in 6 (step 1)

By argumentation it is achieved by multiply 7 by 8 $\therefore$ K = 8

$3L^2K = 1176$

Subtracting $3L^2K$ from the result of Step 1

48744306
1176
4874313

Step 3: $3L^2J + 3LK^2 = 147J + 1344$ ends in 3 (Step 2)

$\therefore$ 147J ends in 9

By argumentation 7 has to be multiplied by 7

$\therefore J = 7$ $\therefore \sqrt[3]{487443403} = 787$

(4) 178453547

Grouping : 178 453 547

n = 3, F = 5, L = 3

Step 1 : $L^3 = 27$

Subtracting L^3, from the given cube

178453547
27
17845352

Step 2: $3L^2K = 27K$ ends in 2 (Step 1)

By argumentation it is achieved by multiplying 7 by 6 $\therefore$ K = 6

$3L^2K = 162$

subtracting $3L^2K$, from the result of step 1

17845352
162
1784519

Step 3: $3L^2J + 3LK^2 = 27J + 324$ ends in 9 (obtained in Step 2)

$\therefore$ 27J ends in 5

By argumentation it is achieved by multiplying 7 by 5

$\therefore J = 5$ $\sqrt[3]{178453547} = 563$

(5) 1089547389

Grouping : 1 089 547 389

n = 4, F = 1, L = 9

Step 1 : $L^3 = 729$

Subtracting L^3, from the given cube

```
1089547389
       729
108954666
```

Step 2: $3L^2K = 243K$ ends in 6

By argumentation this is achieved by multiplying 3 by 2 $\therefore$ K = 2

$3L^2K = 486$

subtracting $3L^2K$, from the result of Step 1

```
108954666
      486
10895418
```

Step 3: $3L^2J + 3LK^2 = 243J + 108$ ends in 8 obtained in Step 2

$\therefore$ 243J ends in 0

By argumentation this is achieved by multiplying 3 by 0

$\therefore$ J = 0

$3L^2J + 3LK^2 = 108$

subtracting $3L^2J + 3LK^2$ from the result of Step 2

```
10895418
     108
1089531
```

Step 4: $3L^2H + 6LKJ + K^3 = 243H + 0 + 8$ ends in 1 obtained in step 3

$\therefore$ 243H ends in 3

By argumentation this is achieved by multiplying 3 by 1

$\therefore$ H = 1 $\therefore$ $\sqrt[3]{1089547389} = 1029$

(6) 80677568161

Grouping : 80 677 568 161

n = 4, F = 4, L = 1

Step 1 : $L^3 = 1$

Subtracting L^3, from the given cube

80677568161

8067756816

Step 2 : $3L^2K = 3K$ ends in 6 obtained in Step 1

By argumentation this is achieved by multiplying 3 by 2 $\therefore$ K = 2

$3L^2K = 6$

subtracting $3L^2K$, from the result of Step 1

$$\begin{array}{r} 8067756816 \\ \underline{\quad 6} \\ 806775681 \end{array}$$

Step 3 : $3LK^2 + 3L^2J = 3J + 12$ ends in 1 obtained in step 2

3J ends in 9 By argumentation this is achieved by multiplying 3 by 3

J = 3

$3L^2J + 3LK^2 = 21$

subtracting $3L^2J + 3LK^2$, from the result of step 2

$$\begin{array}{r} 806775681 \\ \underline{\quad 21} \\ 80677566 \end{array}$$

Step 4 : $3L^2H + 6LKJ + K^3 = 3H + 36 + 8$

3H + 44 ends in 6 obtained in step 3

3H ends in 2 By argumentation this is achieved by multiplying 3 by 4

H = 4 $\therefore \sqrt[3]{80677568161} = 4321$

(7) 183085943000687

Grouping : 183 085 943 000 687

n = 5, F = 5, L = 3

Step 1: $L^3 = 27$

Subtracting L^3 from the given cube

$$\begin{array}{r} 183085943000687 \\ \underline{\quad 27} \\ 18308594300066 \end{array}$$

Step 2 : $3L^2K = 27K$ ends in 6 obtained in step 1

By argumentation this is achieved by multiplying 7 by 8

K = 8

$3L^2K = 216$

subtracting $3L^2K$, from result of step 1

$$\begin{array}{r} 18308594300066 \\ \underline{\quad 216} \\ 1830859429985 \end{array}$$

Step 3 : $3L^2J + 3LK^2 = 27J + 576$ ends in 5 obtained in step 2

27J end in 9 By argumentation this is achieved by multiplying 7 by 6

$\therefore J = 7$

$3L^2J + 3LK^2 = 189 + 576 = 765$

subtracting $3L^2J + 3LK^2$, from the result of step 3

```
  1830859429985
            765
  183085942922
```

Step 4 : $3L^2H + 6LKJ + K^3 = 27H + 1008 + 512$

$27H + 1520$ ends in 2

$27H$ ends in 2

$\therefore H = 6$

$3L^2H + 6LKJ + K^3 = 162 + 1520 = 1682$

subtracting $3L^2H + 6LKJ + K^3$,

```
  183085942922
          1682
  18308594124
```

Step 5: $3L^2E + 3LJ^2 + 3JK^2 + 6LKH$

$= 27E + 441 + 1344 + 864$

$= 27E + 2649$ ends in 4 obtained in step 4

$27E$ ends in 5 By argumentation this is achieved by multiplying 7 by 5

$E = 5$

$\therefore \sqrt[3]{183085943000687} = 56783$

(8) 13278548255430401

Grouping : 13 278 548 255 430 401

$n = 6, F = 2, L = 1$

Step 1: $L^3 = 1$

Subtracting L^3, from the given cube

```
  13278548255430401
                  1
  1327854825543040
```

Step 2: $3L^2K = 3K$ ends in 0 obtained in step 1

By argumentation this is achieved by multiplying 3 by 0 $\therefore K = 0$ $3L^2K = 0$

subtracting $3L^2K$ from the result of step 1

```
  1327854825543040
                 0
  132785482534304
```

Step 3: $3L^2J + 3LK^2 = 3J + 0$ ends in 4 obtained in step 2

$\therefore$ 3J ends in 4 By argumentation this is achieved by multiplying 3 by 8

$J = 8$

$3L^2J + 3LK^2 = 24$

subtracting $3L^2J + 3LK^2$

```
 132785482554304
             24
 13278548255428
```

Step 4: $3L^2H + 6LKJ + K^3 = 3H + 0 + 0$ ends in 8 obtained in step 3

$\therefore$ 3H ends in 8 By argumentation this is achieved by multiplying 3 by 6

$\therefore H = 6$

subtracting $3L^2H + 6LKJ + K^3$, from the result of step 3

```
 13278548255428
             18
 1327854825541
```

Step 5 : $3L^2E + 3LJ^2 + 3JK^2 + 6LKH$

$= 3E + 192 + 0 + 0$ ends in 1 obtained in step 4

$\therefore$ 3E ends in 9 By argumentation this is achieved by multiplying 3 by 3

$\therefore E = 3$

$3L^2E + 3LJ^2 + 3JK^2 + 6LKH = 201$

subtracting $3L^2E + 3LJ^2 + 3JK^2 + 6LKH$, from the result of step 4

```
 1327854825541
           201
 132785482534
```

Step 6: $3L^2D + 3KJ^2 + 3K^2H + 6LKE + 6LJH$

$= 3D + 0 + 0 + 0 + 288$ ends in 4 obtained in step 5

$\therefore$ 3D ends in 6 By argumentation this is achieved by multiplying 3 by 2

$D = 2$

$\therefore \sqrt[3]{13278548255430401} = 236801$

If the cube is even:

(1) 262144

Method 1: Grouping : 262 144

$n = 2,\ F = 6,\ L = 4$

Step 1: $L^3 = 64$

Subtracting L^3, from the given cube

```
 262144
     64
 26208
```

Step 2: $3L^2K = 48K$ ends in 8 obtained in step 1

By argumentation this is achieved by multiplying 8 by 1 or 8 by 6 $\therefore$ K = 1 or 6

But we know that first digit is 6

$\therefore \sqrt[3]{262144} = 64$

Method 2 :

$$\begin{array}{r|r} 8 & 262144 \\ \hline 8 & 32768 \\ \hline 8 & 4096 \\ \hline 8 & 512 \\ \hline 8 & 64 \\ \hline 8 & 8 \end{array}$$

$\sqrt[3]{1} = 1$ cube roots of unity, 1, w, w^2

$\therefore \sqrt[3]{262144} = 8 \times 8 \times 1 = 64$

2) 2299968

Method1: Grouping: 2299 968

N = 3, F = 1, L = 2

Step1: $L^3 = 8$

Subtracting L^3, from the given cube

$$\begin{array}{r} 2299968 \\ \underline{\quad 8} \\ 229996 \end{array}$$

Step2: $3L^2K = 12K$ ends in 6 obtained in step 1

By argumentation this is achieved by multiplying 2 by 3 or 2 by 8

$\therefore$ K = 3 or 8

(i) Suppose K = 3

$3L^2K = 36$

Subtracting $3L^2K$ from the result of step 1

$$\begin{array}{r} 229996 \\ \underline{\quad 36} \\ 22996 \end{array}$$

(ii) Suppose K = 8

$3L^2K = 96$ subtraction from the result of step 1

229996
  96
22990

Step3: $3L^2J + 3LK^2$

(i) Suppose K = 3

$3L^2J + 3LK^2 = 12J + 54$ ends in 6 obtained in step 2 (i)

12J ends in 2 By argumentation this is achieved by multiplying 2 by 1 or 2 by 6

J = 1 or 6

(ii) Suppose K = 8

$3L^2J + 3LK^2 = 12J + 384$ ends in 6 obtained in step 2 (ii)

12J ends in to By argumentation this is achieved by multiplying 2 by 1 or 2 by 6

$\therefore$ J = 3 , 8

Method 2:

8) 2299968
8) 287496
 35937

Grouping : 35 937

n = 2, F = 3, L = 3

Step 1: $L^3 = 27$

Subtracting L^3, from the given cube

35937
  27
3591

Step 2: $3L^2K = 27K$ ends in 1 obtained in step 1

K = 3 By argumentation this is achieved by multiplying 7 by 3

$\therefore \sqrt[3]{35937} = 33$

Cube root of 2299968 in 33 x 4 = 132

(3) 958585256

8) 958585256
 119823157

Grouping : 119 823 157

n = 3, F = 4, L = 3

Step 1: $L^3 = 27$

Subtracting L^3, from the given cube

119823157
27
11982313

Step 2: $3L^2K = 27K$ ends in 3 obtained in step 1
By argumentation this is achieved by multiplying 7 by 9
K = 9
$3L^2K = 243$
subtracting $3L^2K$, from the result of step 1
11982313
243
1198207

Step 3: $3L^2J + 3LK^2 = 27J + 729$ ends in 7 obtained in step 2
∴ 27J ends in 8 By argumentation this is achieved by multiplying 7 by 4
J = 4
∴ cube root of 119823157 is 493
cube root of 958585256 in 2 x 493 = 986

(4) 2461468662008
24614686620083
307683582751
Grouping : 307 683 582 751
n = 4, F = 6, L = 1

Subtracting L^3, from the given cube
307683582751

30768358275

Step 2: $3L^2K = 3K$ ends in 5 obtained in step 1
By argumentation this is achieved by multiplying 3 by 5
K = 5
$3L^2K = 15$
subtracting $3L^2K$, from the result of step 1
30768358275
15
3076835826

Step 3 : $3L^2J + 3LK^2 = 3J + 75$ ends in 6 obtained in step 2
∴ 3J ends in 1 By argumentation this is achieved by multiplying 3 by 7
J = 7

$3L^2J + 3LK^2 = 96$

subtracting $3L^2J + 3LK^2$, from the result of step 2

3076835026
96
307683573

Step 4: $3L^2H + 6LKJ + K^3 = 3H + 210 + 125$

$= 3H + 335$ ends in 3 obtained in step 3

3H ends in 8 By argumentation this is achieved by multiplying 3 by 6

H = 6

∴ cube root of 307683582751 in 6751

cube root of 2461468662008 in 2 x 6751 = 13502

(5) 429208068033352434936

8) 429208068033352434936

53651008504190543 67

Grouping : 5 365 100 850 419 054 367

n = 7, F = 1, L = 3

Step 1: $L^3 = 27$

Subtracting L^3, from the given cube

5365100850419054367
27
536510085041905434

Step 2: $3L^2K = 27K$ ends in 4 obtained in step 1

K = 2 By argumentation this is achieved by multiplying 7 by 2

$3L^2K = 54$

Subtracting $3L^2K$ from the result of step 1

536510085041905434
54
53651008504190538

Step 3: $3L^2J + 3LK^2 = 27J + 36$ ends in 8 obtained in step 2

27J ends in 2 By argumentation this is achieved by multiplying 7 by 6

J = 6 $3L^2J + 3LK^2 = 198$

Subtracting $3L^2J + 3LK^2$ from the result of step 2

53651008504190538
198
5365100850419034

Step 4: $3L^2H + 6LKJ + K^3 = 27H + 216 + 8$

= 27H + 224 ends in 4 obtained in step 3

27H ends in 0 By argumentation this is achieved by multiplying 7 by 0

H = 0

$3L^2H + 6LKJ + K^3 = 224$

Subtracting $3L^2H + 6LKJ + K^3$ from the result of step 3

5365100850419034
224
5365100850418 81

<u>Step 5:</u> $3L^2E + 3LJ^2 + 3JK^2 + 6LKH$

$= 27E + 324 + 72 + 0$

= 27E + 396 ends in 1 obtained in step 4

27E ends in 5 By argumentation this is achieved by multiplying 7 by 5

E = 5

$3L^2E + 3LJ^2 + 3JK^2 + 6LKH = 531$ from the result of step 4

Subtracting 531

536510085041881
531
53651008504135

<u>Step 6:</u> $3L^2D + 3KJ^2 + 3K^2H + 6LKE + 6LKH$

$= 27D + 216 + 0 + 180 + 0$

= 27D + 396 ends in 5 obtained in step 5

27D ends in 9 By argumentation this is achieved by multiplying 7 by 7

D = 7

$3L^2D + 3KJ^2 + 3K^2H + 6LKE + 6LJH = 585$

Subtracting 585 from the result of step 5

53651008504135
585
5365100850355

<u>Step 7:</u> $3L^2C + 3LH^2 + 3K^2E + J^3 + 6LKD + 6LJE + 6KJH$

$= 27C + 0 + 60 + 216 + 252 + 540 + 0$

= 27C + 1068 ends in 5 obtained in step 6

27C ends in 7 By argumentation this is achieved by multiplying 7 by 1

C = 1

∴ Cube root of 536510085041905 4367 is 1750623

Cube root of 4292080680335243 4936

is $2 \times 1750623 = 3501246$

A few more examples

1) Find the cube root of N = 8157016197

(8) (157) (016) (197) = N

The number is grouped into 4 $\therefore$ n = 4

Therefore the cube root contains 4 digits as HJKL

From the first group F one has to determine the nearest cube root of the group value. F = H = 2

As the given number ends in 7 L = 3 (Refer Table U)

L = 3 n = 4

		8 157 016 197
$L^3 = 27$	$-L^3$	$\underline{-27}$
$3L^2K = 27K$ ending in 7	$\therefore$ K = 1	01617
	$-27K$	$\underline{-27}$
		0159
$3L^2J + 3LK^2$ ending in 9 = $3L^2J + 9$		$\underline{-9}$
27J ending in 0	J = 0	0150
	$-27J$	$\underline{-0}$
$3L^2H + 6LKJ + + K^3$ ending in 5		157015
27H + 1		
27H ending in 4	$\therefore$ H = 2	157014

$\therefore$ 2013 is the cube root of N.

2) Find the cube root of N = 81464295375

(81) (464) (295) (375)

The number is grouped into 4 $\therefore$ n = 4

Therefore the cube root contains 4 digits as HJKL

From the first group F, one has to determine the nearest cube root of the group value. F = H = 4

As the given number ends in 5, L = 5 (Refer Table U)

H = 4 n = 4 L = 5

		81 464 295 375
L = 5	125	$\underline{-125}$
$3L^2K$ ending in 5		295 25
75K ends in 5		
$\therefore$ K = 1, 3, 5	Let K = 3	$\underline{-225}$
		2930
	$3L^2J + 3LK^2 = 75J + 135$	$\underline{-135}$

75J ending in 5

$\therefore$ J = 1, 3, 5 Let J = 3

$3L^2H + 6LKJ + K^3$

75H + 270 + 27 = 75H + 297

75H ends in 0

$\therefore$ H = 0, 2, 4, 6, 8 H = 4 is already fixed

$\therefore$ 4335 is the cube root of N.

2795
– 225
464 257
– 297
463960

Another method

5	81464295375
5	16292859075
5	3258571815
	651714363

Now find the CR of 651 714 363 = N. This is grouped into 3 as 651 714 363

J = 8 Being the nearest cube root of first group N = 3 L = 7 as the number ends in 3.

		651 714 363
	$-L^3$	– 343
$3L^2K$ = 147K ends in 2		71402
	K = 6	– 882
		7052
	$3L^2J + 3LK^2$ = 1475 + 756	– 756
147J ends in 6		6296

$\Rightarrow$ J = 8

$\therefore$ (867) is the Cube Root of 651714363

Multiply (867) by 5 = (867 × 5) = 4335

Note: In all such cases where there is ambiguity one can remove it by dividing the given number suitably, so that the ambiguity can be removed.

3) Find the cube root of N = 21001731479

(21) (001) (731) (479)

The number is grouped into 4 $\therefore$ n = 4

Therefore the cube root contains 4 digits as HJKL

From the first group F one has to determine the nearest cube root of the group value. $\therefore$ F = H = 2

As the given number ends in 9 L = 9 (Refer Table U)

	21	001	731	479
$-L^3$				– 729
	21	001	730	75

$3L^2K = 243K$ ends in 5

21 001 730 75
− 12 15
21 001 7186
− 675
21 001 6511
− 17 01
21 001 48 1

$\therefore$ K = 5

$3L^2J + 3LK^2 = 3x81J+3x9x25=243J+675$

243J ends in 1

J = 7

$\therefore$ CR = HJKL = 2759

Verification

$3L^2H + 6LKJ + K^3$

$= 3 \times 81H + 6 \times 9 \times 5 \times 7 + 125$

$= 243H + 1890 + 125 = 243H + 2015$

243H + 2015

21001481
− 2015
21999466

243H ends in 6 $\therefore$ H = 2

2759 is the cube Root of N.

Method I: (using first principles)

The given number is made into n groups where each group contains 3 digits. n denotes also the number of digits in the cube root which is ......J,K,L. L, the last digit is found by using the first principle of cube roots from the table U.

A general method to find the Cube Roots (perfect cube) of any digited number (Here 4 digits are given but can be extended to any digited number).

4) Find the cube root of N = 9261,

(9) (261)

The number is grouped into 2 $\therefore$ n = 2

Therefore the cube root contains 2 digits as KL

From the first group F, one has to determine the nearest cube root of the group value.

F = K = 2

As the given number ends in 1

L = 1 (Refer Table U)

9 2 6 1
$-L^3$ − 1
9 2 6 0

K = 2 (It is the nearest cube root of the first group)

To verify the value of K

Consider $3L^2K = 3K$ ends with 6

$\therefore$ K = 2

$\therefore$ The cube root of the given number is 21

5) Find the cube root of N = 1442897

(1) (442) (897)

The number is grouped into 3 $\therefore$ n = 3

Therefore the cube root contains 3 digits as JKL

From the first group F one has to determine the nearest cube root of the group value.

F = J = 1

As the given number ends in 7, L = 3 (Refer Table U)

L = 3		1 442 897
	$-L^3$	– 27
Now $3L^2K = 27K$ ends in 7		1 442 870
$\therefore$ K = 1		
	$-3L^2K$	– 27
To verify the value of J		14426
$3L^2J + 3LK^2 = 27J + 9$		– 9
i.e.27 J ends in 7 $\therefore$ J = 1		1441 7

$\therefore$ Cube root of the given number is 113

6) Find the Cube Root of N = 76 928 302 277

(76) (928) (302) (277)

The number is grouped into 4 $\therefore$ n = 4

Therefore the cube root contains 4 digits as HJKL

From the first group F one has to determine the nearest cube root of the group value.

F = H = 4

As the given number ends in 7, L = 3 (Refer Table U)

		76 928 302 277
	$-L^3$	– 27
$3L^2K = 27K$ ends with 5		76 928 302 25
$\therefore$ K = 5		
	$-3L^2K$	– 135
$3L^2J + 3LK^2 = 27J + 225$		769283009
	$-3LK^2$	– 225
27J ends with 4		769282784
$\therefore$ J = 2		
	$-3L^2J$	– 54
Verification of H		76928273
$3L^2H + 6LKJ + K^3$		
= 27H + 180 + 125	= 27H + 305	– 305
$3L^2H$ = 27 H ends in 8		76928968

$\therefore$ H = 4 $\therefore$ Cube root of N is 4253

Verification by Urdhva Tiryak D = Duplex

$(4253)^2$ D (4 2 5 3) 1 8 0 8 8 0 0 9

4 2 5 3

7 6 9 2 8 3 0 2 2 7 7 = N

3 2 8 7 6 7 6 2 4 2

7) Find the Cube root of N = 1 830 623 053 337

(1) (830) (623) (053) (337)

The number is grouped into 5 $\therefore$ n = 5

Therefore the cube root contains 5 digits as IHJKL

From the first group F one has to determine the nearest cube root of the group value.

F = I = 1

As the given number ends in 7, L = 3 (Refer Table U)

Consider grouping of 3 digits from the right side of the given number. The first group (from LHS) even if it is incomplete it should be taken as a group

There are five groups. Therefore number of digits in cube root are five. n = 5

		1 830 623 053 337
L = 3 Since number ends in 7(Ref. Tab.) – L^3		-- 27
$3L^2K = 27K$ ends in 1	$\therefore$ K = 3	623 053 31
	$-3L^2K$	– 81
$3L^2J + 3Lk^2 = 27J + 81$		623 0525
		– 81
As 27J ends in 4 $\therefore$ J = 2		623 0444
	$-3L^2J$	– 54
$3L^2H + 6LKJ + K^3 = 27H + 108 + 27$		623 039
	$-(6LKJ + K^3)$	– 135
Since 27H ends in 4		622 904
$\therefore$ H = 2		
To Verify the value of I	$-3L^2H$	– 54
$3L^2I + 6LKH + 3LJ^2 + 3K^2J = 27I + 3L^2I +$		622 85
$108 + 36 + 54 = 27I + 198$		
		– 198
27I ends in 7		620 87

$\therefore$ I = 1 $\therefore$ Cube Root of the given number is 1 2 2 3 3

8) Find the cube root of N = 2299968

(2) (299) (968)

The given number is grouped into three as 2 299 968 ∴ n = 3

∴ Therefore cube root contains 3 digits as JKL

From the first group F, one has to determine the nearest cube root of the group value

F = J = 1

As the given number ends in 8, L = 2 (Refer Table U)

		2299968
	$-L^3$	-8
$3L^2K = 12K$ ends in 6		996
∴ K = 3 or 8		
If K = 3	$3L^2K$	-36
$3L^2J + 3LK^2$ ends in 6		86
∴ 12J + 54		-54
12J ends in 2		932

J = 1 or 6

J = 6 is not valid since the first groups cube root value is 1 ∴ 132 is the cube root of N

if K = 8

		2299968
	$-L^3$	-8
$3L^2K = 12K$		996
If K = 8, 12K		-96
$3L^2J + 3LK^2$		990
12J + 384		-384
12J ends in 6		616

⇒ J = 8 is invalid

∴ 132 is the cube root of given number

9) To find out the method by which one can locate the non – perfect cube nature.

Find the cube root of 2298968

(2) (298) (968)

The number is grouped in 3 ∴ n = 3

Therefore the cube root contains 3 digits as JKL

From the first group F one has to determine the nearest cube root of the group value.

F = J = 1

As the given number ends in 8, L = 2 (Refer Table U)

		2298968
	$-L^3$	-8
$3L^2K = 12K$ ends in 6 ∴ K = 3, 8		896
If K = 3	$3L^2K$	-36
$3L^2J + 3LK^2$ ends in 6		86
12J + 54		-54
12J ends in 2		932
J = 1 or 6 (J = 6 is invalid) ∴ J = 1,	$-12J$	-12
To show that H = 0		92
$3L^2H+6LKJ+K^3 = 12H + 36 + 27 = 12H+63$		-63
12H ends in 9 is invalid		29

∴ given number is not perfect cube

Test $(132)^3 = 2299968 \neq N = 2298968$

For the decimal working refer Taylor's Series Method (Eg 2)

10) Find the cube root of 1860867

(1) (860) (867)

The number is grouped into 3 ∴ n = 3.

Therefore the cube root contains 3 digits as JKL from the first group F one has to determine the nearest cube root of the group value.

F = J = 1

As the number ends in 7, L = 3. (Refer Table U)

∴ L = 3

		1860867
	$-L^3$	-27
$3L^2K = 27K$ ends in 4 K = 2		84
	$-3L^2K$	-54
$3L^2J + 3LK^2$		603
27J + 36	$-3LK^2$	-36
27J ends in 7 J = 1		67
To show that H = 0		-27
$6L^2H+6LJK+K^3 = 27H+36 + 8 = 27H + 44$		64
		-44
27H ends in 0		20

∴ H = 0

∴ 123 is he Cube root of given number

11) Find the cube root of N = 2905841483

(2) (905) (841) (483)

The number is grouped into 4 ∴ n = 4

∴ Therefore cube root contains 4 digits as HJKL

From the first group F one has to determine the nearest cube root of the group value.

F = H = 1

As the number ends in 3, L = 7 (Refer Table U)

	2 905 841 483
	– 343
$3L^2K$ = 147K is ending in 4 ∴ K=2	2905841 14
– 147K	– 294
$3L^2J + 3LK^2$ ending in 2	29058382
= 3 × 49J + 3 × 7 × 4 = 147J + 84	– 84
$3L^2J$ is ending in 8	29058298
= 147J (ends in 8) ∴ J = 4	
	– 588
$3L^2H + K^3 + 6LKJ$	771
= 147H + 344	– 344
$3L^2H$ = 147H ending in 7	427
∴ H = 1	

1427 is the cube Root of N.

12) Find the cube root of N = 277295358761

(277) (295) (358) (761)

The number is grouped into 4 ∴ n = 4

Therefore cube root contains 4 digits as HJKL

From the first group F one has to determine the nearest cube root of the group value

F = H = 6

As the number ends in 1, L = 1 (Refer Table U)

	277 295 358 761
(L) 1	– 1
(K) $3L^2K$ ending in 6	27729535876
∴ K = 2	– 6
(J) $3L^2J + 3LK^2$ ending in 7	2772953587
	– 12
$3L^2J$ ending in 5	2772953575
∴ J = 5	– 68
Verification of H	356
(H) $3L^2H + 6LKJ + K^3$ ending in 6	
$3L^2H + 60 + 8 = 3L^2H + 68$	– 68
3H ending in 8	288
∴ H = 6	

∴ (6521) is the cube root of N

13) Find the cube root of N = 8, 303, 765, 625

The number is grouped into 4 $\therefore$ n = 4

Therefore the cube root contains 4 digits as HJKL

From the first group F one has to determine the nearest cube root of the group value

F = H = 2

As the number ends in 5 L = 5 (Refer Table U)

		8 303 765 625
	$-L^3$	-125
$3L^2K = 75K$ ends in 0		76550
$\therefore$ K = 0, or 2		
If K = 0	$-3L^2K$	0
$3L^2J + 3LK^2 = 75J$ ends in 5		83037655
$\therefore$ J = 5, 1		
If J = 5, $3L^2J = 375$	$-3L^2J$	-375
$3L^2H + 6LKJ + K^3$ 75H ends in 8		728

Let us consider J = 1

		7655
$3L^2J = 75$	$-3L^2J$	-75
$3L^2H + 6LKJ + K^3$		7680
$3L^2H = 75H$ ends in 8		
is not valid		

Hence Let us consider K = 2

		76550
$3L^2K = 150$	$-3L^2K$	-150
		7640
$3L^2J + 3LK^2$ ending in zero		-60
$3L^2J + 60$		
$3L^2J$ ends in = 0		758
J = 0, 5		
If J = 0	$-3L^2J$	-0
		758
$3L^2H + 6LKJ + K^3$		-8
$3L^2H + 8$		
$3L^2H$ ends in 0		750
75H ends in 0		

$\therefore$ H is 0, 2

H = 2 is valid

$\therefore$ 2025 is the cube root of N.

14) Find the cube root of N = 12732581112551

(12) (732) (581) (112) (551)

The number is grouped into 5 ∴ n = 5

Therefore the cube root contains 5 digits as GHJKL

From the first group F, one has to determine the nearest cube root of the group value

F = G = 2

As the given number ends in 1, L = 1 (Refer Table U)

		12 732 581 112 551
	$-L^3$	-1
$3L^2K$ is ending in 5		1273258111255
= 3K ending in 5	∴ K = 5	
	$-3K$	-15
$3L^2J + 3LK^2$ ending in 4		127325811124
3J + 75		-75
$3L^2J$ ending in 9	∴ J = 3	811049
		-9
$3L^2H + 6LKJ + K^3$		2581104
3H + 215		
		-215
3H ending in 9		2580889
∴ H = 3		
$3L^2G + 3LJ^2 + 3K^2J + 6LKH$		...258088
	3G + 27 + 225 + 90	-342
3G + 342		
3G ends in 6	∴ G = 2	257746

Thus G is confirmed from the value of Cube root of first group.

∴ (23351) is the cube root of N.

15) Find the Cube root of N = 3463512697

(3) (463) (512) (697)

The number is grouped into 4 ∴ n = 4

Therefore the cube root contains 4 digits as HJKL

From the first group F, one has to determine the nearest cube root of the group value

F = H = 1

As the number ends in 7, L = 3 (Refer Table U)

		3463512697
	$-L^3$	-27
$3L^2K = 27K$ ending in 7		1267
$\therefore K = 1$	$-3L^2K$	-27
$3L^2J + 3LK^2$ ends in 4		1214
$27J + 9$	$-3LK^2$	-9
$27J$ ends in 5		35115
$J = 5$	$3L^2J$	-135
$3L^2H + 6LKJ + K^3$ ends in 8		3498
$3L^2H + 90 + 1$	$(-6LKJ - K^3)$	-91
$= 3L^2H + 91$		
$3L^2H$ is ending in 7		3407
$27H$ ends in 7 $\Rightarrow H = 1$		

$\therefore$ CR of (3463512697) is 1513

16) Find the cube root of N = 248 858189
(248) (858) (189)
The number is grouped into 3 $\therefore n = 3$
Therefore the cube root contains 3 digits as JKL
From the first group F one has to determine the nearest cube root of the group value
F = J = 6
As the number ends in 9, L = 9 (Refer Table U)

L = 9		2488 58189
	$-L^3$	-729
$3L^2K = 243K$ ends in 6	K = 2	24885746
	$-3L^2K$	-486
$3L^2J + 3LK^2 = 243J + 108$ ends in 6		2488526
		-108
$\Rightarrow 243J$ ends in 8 $\therefore J = 6$		2488418

$\therefore$ 629 is the cube root of N

17) Find the cube root of N = 105823817
(105) (823) (817)
The number is grouped into 3 $\therefore n = 3$
Therefore the cube root contains 3 digits as JKL
From the first group F one has to determine the nearest cube root of the group value
F = J = 6
As the number ends in 7, L = 3 (Refer Table U)

L = 3		105 823 817
	$-L^3$	-27
$3L^2K=3(3)^2K=27K$ ends in 9 $3L^2K=189$		10,582,379
$\therefore K = 7$		
If K = 7, $3L^2K = 189$	$-3L^2K$	-189
		1,058,219
$3L^2J + 3LK^2 = 27J + 441$	$-3LK^2$	-441
$27J$ ends in 8 $\therefore J = 4$		1057778

$\therefore$ 473 is the cube root of N

18) Find the cube root of N = 192100033

(192) (100) (033)

The number is grouped into 3 $\therefore$ n = 3

Therefore the cube root contains 3 digits as JKL

From the first group F one has to determine the nearest cube root of the group value

F = J = 5

As the number ends in 3, L = 7 (Refer Table U)

		192 100 033
	$-L^3$	– 343
$3L^2K$ = 147K ends in 9	$\therefore$ K = 7	19209969
	$-3L^2K$	– 1029
$3L^2J + 3LK^2$		1920894
= 147J + 1029	$-3LK^2$	– 1029
147J ends in 5	$\therefore$ J = 5	1919865

$\therefore$ 577 is the cube root of N.

19) Find the cube root of N = 642735647

(642) (735) (647)

The number is grouped into 3 $\therefore$ n = 3

Therefore the cube root contains 3 digits as JKL

From the first group F, one has to determine the nearest cube root of the group value

F = J = 8

As the number ends in 7, L = 3 (Refer Table U)

		642 735 647
	$-L^3$	– 27
$3L^2K$ = 27K is ending in 2 $\Rightarrow$ K = 6		642 735 62
	$-3L^2K$	– 162
		642 7340
Confirmation of J		
$3L^2J + 3LK^2$ ending in 0		440
27J + 324	$-3LK^2$	– 324
27 J ending in 6 J = 8		116

$\therefore$ 863 is the cube root of N

20) Find the cube root of N = (2) (307) (660) (544) (340) (523)

The number is grouped in the 6 $\therefore$ n = 6

Therefore the cube root contains 6 digits as IGHJKL

From the first group F, one has to determine the nearest cube root of the group value.

F = I = 1

As the given number ends in 3, L = 7 (Refer Table U)

L) $L^3 = 343$

K) $3L^2K = 147$ K ends in 8

∴ K = 4

J) $3L^2J + 3LK^2$

= 147J + 336 ends in 3

147J ends in 7

∴ J = 1

H) $3L^2H + 6LKJ + K^3$

= 147H + 6 × 7 × 4 × 1 + 64

147H + 232 ends in 6

147H ends in 4

∴ H = 2

G) $3L^2G + 6LKH + 3LJ^2 + 3K^2J$ ends in 6

147G + 6×7×4×2 + 3×7×1 + 3×16×1

i.e. 147G + 336 + 21 + 48 = 147G + 405

147G ends in 1 ∴ G = 3

	2 307 660 544 340 523	
$-L^3$	– 343	
	……., 34018	
	– 588	→ – 147K
	3343	
	336	
	3007	
	147	→ – 147J
	544286	
	– 232	
	544054	
	– 294	→ – 147K
	54376	
	– 405	
	53971	
	– 441	→ – 147G
	5353	
	– 696	
	4657	

Verification

$3L^2I + 6LKG + 6LJH + 3K^2H + 3KJ^2$

147I + 6.7.4.3 + 6.7.1.2 + 3.16.2 + 3.4.1

147I + 504 + + 84 + 96 + 12 = 147I + 696

147I ends in 7 ∴ I = 1

132147 is the cube root of N

21) Find the cube root of N = 194537321140807

(194) (537) (321) (140) (807)

The number is grouped into 5 ∴ n = 5

Therefore the cube root contains 5 digits as GHJKL

From the first group F one has to determine the nearest cube root of the group value.

F = G = 5

As the given number ends in 7, L = 3 (Refer Table U)

$L^3 = 27$		140807
	$-L^3$	–27
$3L^2K = 27K$ ends in 8		14078
∴ K = 4		– 108
	– 27K	1397
$3L^2J + 3LK^2$		– 144

= 27J + 144 1253

27J ends in 3 − 243

∴ J = 9 − 27J 321101

$3L^2H + 6LKJ + K^3$ − 712

$3L^2H + 648 + 64 = 3L^2H + 712$ 389

27H ends in 9 ∴ H = 7 − 189

− 27H 32120

$3L^2G + 3LI^2 + 3K^2J + LKH$ − 1665

27G + 729 + 432 + 504 20455

= 27G + 1665

27G ends in 5

∴ G = 5

57943 is the cube root of N.

22) Find the cube root of 12450388553011648

(12) (450) (388) (553) (011) (648)

The number is grouped into 6 ∴ n = 6

Therefore the cube root contains 6 digits as IGHJKL

From the first group F, one has to determine the nearest cube root of the group value

F = I = 2

As the given number ends in 8, L = 2 (Refer Table U) $L^3 = 8$

011 648

$-L^3$ − 8 ⟶ $-L^3$ (L = 2)

$3L^2K$ = 12K ends in 4 ∴ K = 7 or 2 55301164

Consider K = 7

$-3L^2K$ = − 84 ⟶ 12K (K = 7)

$3L^2J + 3LK^2 = 3.4J + 3.2.49$ 5530108

12J + 294

− 294

12J ends in 4 ∴ J = 2 or 7 5529814

Consider J = 7

− 84 ⟶ − 12J (J = 7)

$3L^2H + 6LKJ + K^3$ 552973

12H + 588 + 343 = 12H + 931 − 931

12H ends in 2 ∴ H = 1 or 6 552042

Consider H = 1

$3L^2G + 6LKH + + 3LJ^2 + 3K^2J$ − 12 ⟶ − 12H(H = 1)

12G + 6.2.7 + 3.2.49 + 3.49.7 55203

12G + 84 + 294 + 1029 ⇒ 12G + 1407

− 1407

12 G ends in 6 53796

∴ G = 3 or 8 − 36 ⟶ 12G (G = 3)

Consider G = 3

5376

$3L^2I + 6LKG + 6LJH + 3K^2H + 3KJ^2$ − 1512

12I+252+84+147+1029=12I+1512

12I ends in 4 3864

$\Rightarrow I = 2$ or 7

But $I = F = 2$ is already fixed $\therefore$ (231772) is the cube root of N.

$\therefore$ Cube of (231772) is exactly the given number.

23) Find the cube root of N = 14377 925503802791853

(14) (377) (925) (503) (802) (791) (853)

The number is grouped into 7 $\therefore n = 7$

Therefore the cube root of N contains 7 digits as MIGHJKL

From the first group F, one has to determine the nearest cube root of the group value F

M = 2

As the given number ends in 3, $\therefore$ L = 7 (Refer Table U)

	14 377 925 503802791853
$-L^3$	-343
$3L^2K = 147K$ ends in $1 \Rightarrow K = 3$	79151
$-147K$	-441
$3L^2J + 3LK^2 = 147J + 189$	8027871
	-189
$147J$ ends in $2 \Rightarrow J = 6$	8027682
$-147J$	-882
$3L^4H + 6LKJ + K^3 = 147H + 756 + 27$	802680
$= 147H + 783$	-783
$147H$ ends in $7 \Rightarrow H = 1$	801897
$-147H$	-147
$3L^2G + 6LKH + 3LJ^2 + 3K^2J = 147G + 126 + 756 + 162$	80175
$= 147G + 1044$	-1044
$147G$ ends in $1 \Rightarrow G = 3$	50379131
$-147G$	-441
$3L^2I + 6LKG + 6LJH + 3K^2H + 3KJ^2 = 147I + 378 + 252 + 27 + 324$	5037869
$= 147I + 981$	-981
$147I$ ends in $8 \Rightarrow I = 4$	5036888
$-147I$	-588
$3L^2M + 6LKI + 6LJG + 3LH^2 + 3K^2G + 6KJH + J^3 = 147M + 504 + 756 + 21 + 81 + 108 + 216 = 147M + 1686$	503630
	-1686
$147M$ ends in $4 \Rightarrow M = 2$	501944

Cube root of N = 2431637

The same Argumentation Method can be applicable for finding out the Square Root also.

1) Find the square root of N = 141376

(14) (13) (76)

The number is grouped into 3

Therefore the square root contains 3 digits as JKL

From the first group F one has to determine the nearest square root of the group value. F = J = 3

As the given number ends in 6 L = 4 or 6 (Refer Table U)

Let us consider L = 6

		141376
	$-L^2$	-36
2LK = 12K ends in 4		14134
∴ K = 2 or 7 If K = 2		
	$-2LK$	-24
$2LJ + K^2 = 12J + 4$		1411
		-4
12J ends in 7 (is not valid)		1407

As J ≠ 3 as determined in the beginning

∴ Let us consider the other value for K as 7

L = 6 K = 7

		141376
	$-L^2$	-36
		14134
2LK = 84		-84
$2LJ + K^2 = 12J + 49$		1405
	$-K^2$	-49
12J ends in 6		1356

∴ J = 3 or 8

J K L

The square root is 3 7 6. One can continue to show the exactness of the root as follows.

L has two values 4, 6.

If L = 4

Then		141376
	$-L^2$	-16
2 LK = 8K ends in 6		14136
K = 7 or 2, if K=7		
	$-2LK$	-56
$2LJ + K^2 = 8J + 49$		1408
		-49
8J ends in 9		1359

This is not valid. Hence consider another value for K = 2

If L = 4, K = 2

Then	14136
− 2LK	− 16
$2LJ + K^2 = 8J + 4$	1412
$- K^2$	− 4
8J ends in 8	1408

J = 1 which is not correct as J is 3.

∴ 376 is the correct square root of the given number.

Method II

Let us divide N = 141 376 by 4 successively thrice

$\frac{141376}{4} = \frac{35344}{4} = \frac{8836}{4} = 2209$ Since 22 09 is divided into two groups the Square Root has two digits say KL. K = 4 (∵ 4 is the nearest square root of 22)

The number ends in 9, ∴ L = 3 or 7. (Refer Table U) Let us consider L = 3

		K	L
		(22)	(09)
	$- L^2$		− 9
2LK = 6K ends in 0 ∴ K = 0 or 5		22	0
This is not valid			
Now Let L = 7		22	09
	$- L^2$		− 49
2LK = 14K ends in 6 ∴ K = 4 or 9		21	6

Square root = 47

Hence square root = KL × 2 × 2 × 2 = 47 × 8 = 376

General Theory (Straight Division)

The theory for the cube roots determination is explained by Swamiji in the following manner. Depending on the number of digits in the cube root, the cube is written as expansion of the linear combination of digits occupying units, tens, hundreds, thousands etc as the case may be.

For example 27^3 can be written as

$(a + b)^3$ with a = 20, (i.e. a = 2 in tens place)
b = 7

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

a^3 is in thousands
$3a^2b$ is in hundreds
$3ab^2$ is in tens
b^3 is in units

For the cube root determination of $(a + b)^3$, the following is the procedure.

Considering $a^3 + 3a^2b + 3ab^2 + b^3$,

One proceeds to work with digit by digit.

Step (1) : Put down the cube root of the term a^3 which is a. This is the first quotient

Step (2): To consider $3a^2$ as the divisor throughout the work. These can be shown

	a^3	$3a^2b$	$3ab^2$	b^3
$3a^2$		0		
	a			

The next dividend is $0 + 3a^2b = 3a^2b$

Dividing this by $3a^2$, we get b as the next quotient and 0 as the remainder

	a^3	$3a^2b$	$3ab^2$	b^3
$3a^2$		0	0	
	a	b.		

As we are considering a two-digited number in the cuberoot, the decimal starts after the two quotients i.e. after b.

Step (3):The dividend now is $0 + 3ab^2 = 3ab^2$

From this one has to subtract $3ab^2$

(Refer the expansion of $(a + b)^3$) (Table B)

New dividend $= 3ab^2 - 3ab^2 = 0$

Dividing by $3a^2$ we get 0 as quotient and 0 is remainder

	a^3	$3a^2b$	$3ab^2$	b^3
$3a^2$		0	0	0
	a	b.	0.	

Step 4: Now the dividend in $0 + b^3 = b^3$

From which one has to subtract b^3

(Refer the expansion of $(a + b)^3$) (Table B)

We get as the new dividend

Divide 0 by $3a^2$ we get 0 as

Quotient and 0 as remainder

	a^3	$3a^2b$	$3ab^2$	b^3
$3a^2$		0	0	0
	a	b.	0.	0

We have worked out cube root of a perfect cube $(a + b)^3$ as $(a + b)$. It is to be clearly seen that after the decimal point, the quotient should be zeroes.

<u>General Method</u> (Straight Division)

(1) 14706125 (Reduction Method)

Following similar programming as in exact cubes (ref. argumentation) the given numbers is grouped as : 14 706 125

<u>Step 1:</u> Consider the Cube root of left most group as the first quotient digit in the answer by trying the cubes of 1,2, ... 9 which fits nearly (≤) the considered group (14). Therefore in this problem first digit of the cube root is 2, with first remainder we represent 6. This quotient is a. ∴ a = 2. The first remainder (R_1) 6 is placed between the left most group and the working is carried out digit by digit. After obtaining the first quotient the operation is carried out with next digit of the given cube.

```
 | 14 :      706    125
 |           /
 |    :     6
-+-------------------------
 |  2 :
    a
```

This gives the first gross dividend 67.

<u>Step 2:</u> Divisor is framed as thrice the square of the first quotient digit, (a) as 12.

```
12   14        706     125
               /
```

<u>Step 3:</u> First gross dividend 67 is divided by the divisor, which gives second quotient digit b as 5 and second remainder R_2 as 7. Second gross dividend is 70.

```
12) 67 (5(b)
    60
     7 R2
```

```
12 | 14 :  7   0   6   1   2   5
   |      /   /
   |  :6  7
---+-------------------------------
   |  2 :  5
      a    b
      a = 2     b = 5
```

Step 4: From this second gross dividend subtract $3ab^2$ to obtain new dividend

$ND = 70 - 3 \times 2 \times 5^2$

$= 70 - 150 = -80$

To avoid negative new dividend, we reduce the quotient b by 1. This b becomes 4 (modified)

12) 67 (4 (b(m)) m ⇒ modified
48
19 R_2(m)

	14 :		7	0	6	1	2	5
12	:	6	19	10				
	2 :	4 (modified)						

a = 2, b = 4

Thus the second gross divided in modified to 190.

ND = Gross dividend $- 3ab^2$

$= 190 - 3 \times 2 \times 4^2$

$= 190 - 96$

$= 94$

Step 5: Divide this new dividend 94 by the divisor 12, which results in the third quotient , c,

digit as 7 and remainder as 10.

12) 94 (7(c)
84
10 R_3

	14 :		7	0	6	1	2	5
12		6	19	10				
		R_1	R_2	R_3				
			7.					

a = 2, b = 4, c = 7

Third gross dividend in 106.

From this subtract $6abc + b^3$ to obtain new dividend.

$ND = 106 - (6 \times 2 \times 4 \times 7 + 4^3)$

$= 106 - 400 = -294$

To avoid negative new dividend, we reduce the quotient c by 1. Thus C become 6 (modified).

12) 94 (6 (C_4(m))
72
22 R_3 (m)

	14 :	7	0	6	1	2	5
12	:	6 (R_1)	19 (R_2)	22 (R_3)			
	2 :	4	6.				

a = 2, b = 4, c = 6

Now gross dividend is 226

New dividend $= 226 - (6 \times 2 \times 4 \times 6 + 4^3) \rightarrow [6abc+b^3]$

$= 226 - 352 = -126$

∴ we further reduce the quotient 6 by 1 i.e. C = 5

12) 94 (5C [modified]
60
34 R_3 (modified)

	14 :	7	0	6	1	2	5
12	:	6	19	34 (R_3)			
	2 :	4	5.				

a = 2, b = 4, c = 5

Gross dividend = 346

New dividend $= 346 - (6 \times 2 \times 4 \times 5 + 4^3)$ $(6abc+b^3)$

$= 346 - 304$

$= 42$

Step 6 : Divide this new dividend 42 by 12

Then d = 3 $R_4 = 6$

12) 42 (3(d)
36
6 R_4

Gross dividend = 61

	14 :	7	0	6	1	2	5
12	:	6	19	34	6 (R_4)		
	2 :	4	5.	3.			
	a	b	c	d			

From the gross dividend we have to subtract $3ac^2 + 3b^2c$.

$$3ac^2 + 3b^2c = 3 \times 2 \times 5^2 + 3 \times 4^2 \times 5 = 390$$

ND = 61 – 390 = – 329 (–ve value)

∴ we reduce quotient d by 1, modified to 2

12) 42 (2 (d(m))
24
18 (R_4(m))

∴ Gross dividend = 181

	14 :	7	0	6	1	2	5
12	:	6	19	34	18		
	2 :	4	5.	2			

ND = 181 – 390 = – 209 (–ve value)

∴ we reduce quotient d further by 1 is d = 1 modified

12) 42 (1 d(m)
12
30 R_4(m)

Gross dividend in 301

	14 :	7	0	6	1	2	5
12	:	6	19	34	30		
	2 :	4	5.	1			

ND = 301 – 390 = – 89 (–ve value)

We reduce quotient d further by i.e. d = 0 modified

12) 42 (0 d(m)
0
42 (R_4 (m)

Gross dividend in 420

	14 :	7	0	6	1	2	5
12	:	6	19	34	42 (R_4)		
	2 :	4	5.	0			

ND = 421 – 390 = 31

Step 7 : Divide new dividend 31 by 12.

```
12 ) 31 ( 2 (e)
     24
      7 (R5)
```

	14 :	7	0	6	1	2	5
12	:	6	19	34	42	7 R_5	
	2 :	4	5.	0	2		
	a	b	c	d	e		

Gross dividend = 72

To obtain new dividend, we have to subtract $3bc^2$

$3bc^2 = 3 \times 4 \times 5^2 = 300$

ND = 72 – 300 = – 228 (–ve value)

$\therefore$ we reduce e by 1 then e = 1 (modified)

```
12 ) 31 ( 1 e(m)
     12
     19 R5(m)
```

	14 :	7	0	6	1	2	5
12	:	6	19	34	42	19	
	2 :	4	5.	0	1		

Gross dividend = 192

ND = 192 – 300 = –108 (–ve value)

$\therefore$ we reduce e further by 1 then e = 0

```
12 ) 31 ( 0 e(m)
      0
     31 R5(m)
```

	14 :	7	0	6	1	2	5
12	:	6	19	34	42	31	
	2 :	4	5.	0	0		
	a	b	c	d	e		

Gross dividend = 312

ND = 312 – 300 = 12

Step 8 : Divide ND 12 by 12

```
12 ) 12 ( 1
     12
      0 R6
```

	14 :	7	9	6	1	2	5
12	:	6	19	34	42	31	0
	2 :	4	5.	0	0	1	
	a	b	c	d	e	f	

Gross dividend = 5

To obtain new dividend, we have to subtract c^3 from gross dividend.

$c^3 = 125$

ND = 5 − 125 = −120 (−ve value)

∴ we reduce f by 1 i.e. f = 0 (m)

12) 12 (0 f(m)
    0
    12 R_6 (m)

	14 :	7	9	6	1	2	5	0
12	:	6	19	34	42	31	12 R_6	0
	2 :	4	5.	0	0	0	0	

Gross dividend = 125

ND = 125 − 125 = 0

Step 9 : If we divide this new dividend 0 by 12, we get quotient g as 0 and remainder as 0, resulting next gross dividend as 0.

∴ 14706125 is exact cube
and its cube root in 245.

Swamiji's General Method of evaluating Cube Roots (CR) for numbers

1) The given number is grouped from the RHS end. Each group contains a maximum number of 3 digits to the extent possible. In doing so if the first group ends with 1 or 2 digits it is still considered to be a group.
2) The cube root then consists of number of digits equivalent to number of groups.

For Eg: 1| 860 867. This umber has a CR containing 3 digits (c b a) where a is in units place, b is in 10's place, c is in 100's place.

If one writes the number from the position of the division s

$(c + b + a)^3$ is equal $\Rightarrow a \times 10^0 + b \times 10^1 + c \times 10^2$

$a^3(10^0)$	$3a^2b(10^1)$	$3ab^2(10^2)$	
$b^3(10^3)$	$3a^2c(10^2)$	$3ac^2(10^4)$	
$c^3(10^6)$	$3b^2c(10^4)$	$3bc^2(10^5)$	$6abc(10^3)$

In the bracket, the placement of the term is given. The straight division method is applied in finding out the CR. It is illustrated as below.

Consider the number 557441768 (This is grouped into 3)

557 441 768 whose CR is to be determined.

Following are the steps. (Straight Division Method)

1) Nearest CR of the first group 557 is to be considered by Vilokanam. It can be shown as 8, ($8^3 = 512$) $\therefore$ c = 8. This is less than 557 by 45. This acts as a remainder and to be tagged with 4 as 45 × 10 + 4 = 454 which is the first Intermediate dividend ID

$\overline{55754867} = 823.0492592$

$3c^2 = 192$	557	4	4	1	7	6	8
CD		45	70	32			
	8	2	3				
	c	b	a				

$3c^2 = 192$ is the common divisor (CD). The first ID 454 is to be divided by CD(192) to get the value of b with the remainder 70, b = 2

This 70 will form next ID as 704 from which one has to subtract $3cb^2 = 96$. The new dividend ND = 704 − 96 = 608. This when divided by CD will give the value for a, with the remainder ie 32 a = 3. The next ID is 321. One can stop at this as one will have only 3 digits in the cube root of this number. A confirmation is attempted by cubing the Value 823 and comparing it with the given number. The cube of 823 comes out exactly as 557 441 768. If it is not so, then further extension of this procedure leads to finding out the decimal points, which is the case of imperfect cube.

Let us consider the number. 557 541 867 whose cube root is to be determined. Following the above procedure.

1) Nearest cube root of 1st group 557 is 8 this is c, Cube Root

	557	5	4	1	8	6	7	0
$3C^2CD=192$	512	45	71	42	125	46	$\overline{68}$	161
			$3cb^2$	$3bc^2$	b^3+6abc	$3ca^2+3b^2a$	$3ba^2$	a^3
	8	2	3.	0	5	0	$\overline{7}$	11
	c	b	a.	a^1	b^1	c^1		
				d	e	f	g	

The cube of 823 ≠ given number.

Hence one has to extend the working to get the decimal after 823.

The steps are as follows

ID is 421 – (8 + 288) = 125

∴ ND = 125; 125 ÷ 192 = the 1^{st} decimal as 0 with the remainder 125 leading to 1258 as the next ID

∴ 1258 – (216 + 36 + 0) = 1006 ; 1006 ÷ 192

$3ac^2 + 3bc^2 + 6abd\,e$ = 5 with the remainder 46 leading to 466 as the next ID

$466 - (3ba^2 + 6cbe + 6cad) = \overline{68}$

$\overline{68}$ when divided by 192 gives f = 0 with the remainder as $\overline{68}$ leading to next ID as $\overline{68}\ 7 = \overline{673}$

$466 - (54 + 480) = \overline{68}$; $\overline{68} \div 19211$

$\overline{673} - (a^3 + 3cd^2 + 3b^2e + 6cbf + 6ace + 6bad)$

673 – (27 + 0 + 60 + 720) = 1480

1480 ÷ 192 = 7 , Remainder = 136

<u>Upto 4 decimals</u> The cube root of the given number is 823.0493

Swamiji's method can be applied to find the square root, cube root, or higher order roots as well. In each case one has to prepare a table for the expansion of the terms of the second degree, third degree or the fourth degree or higher degree respectively as the case may be. This expansion can be made use of not only for integers but while preparing the table, one can put it in the form of (a +b+ c + ...) where the position of 'a' is the highest and confirmed in descending order as b next c, the next to it and so on..

The same expansion terms are applicable in the ascending order also where a happens to be in the units place and b in the Ten's place and so on. But care has to be taken to consider a proper placement.. Expansion of Swamijis method is also useful for the expansion in decimals. But only thing is that one has to count the decimal placement as 10^{-1}, 10^{-2} .... b, c etc This aspect is illustrated in the example.

CR s are evaluated by two different methods as well by swamiji for perfect cubes
LKJ..... method

This method is laborious for bigger numbers in the sense that one may have to work out sometimes many probabilities before one arrives at the correct value and hence one can say that the most general method as described by Swamiji ie Straight Division Method is considered novel in the sense that one can adopt the same principle to find out the higher order roots as well.

For Eg: higher order roots to be given. Ex for 5^{th} root

Besides, we can even say that Swamijis method of finding the roots is general in the sense that it can be also applicable to imperfect cubes, perfect higher order roots, where one can attempt to work out to any required decimal point of ones choice

A comparison between Swamiji's method and Taylor's method as explained by British authors is attempted and is exemplified in the following working details.

The straight division method that is applied by Swamiji to find the Square roots, Cube roots, Higher order roots can be easily extended to finding out the different roots of a polynomial as well. Here it is of interest to note that one can attempt to find out the roots in the ascending or descending order by adopting the same working principle. A number of examples are given in this section for finding out the different roots of polynomials.

1. Find the cubic root of 131.8

$x^3 - 131.8 = 0$

$CD = 3a^2$	131.	8	0	0	0	0
	125	6	68	5	50	$\overline{40}$
	5.	0	9	0	$\bar{9}$	$\bar{5}$
	a.	b	c	d	e	f

Upto f (5 decimals) = 5.0909 5 = 5.08905

2. Find the Cube Root of 39.5

$x^3 - 39.5 = 0 \Rightarrow x^3 = 39.5$

Solution x = a.bcde ...
From Vilokanam a = 3

$=3a^2=27$	39	5	0	0	0	0
	27	12	17	26	7	$\bar{2}$
			$\overline{144}$	$\overline{64}$	$\overline{504}$	$\overline{336}+1152$ =816
			$3ab^2$	$b^3+3a.2bc$	$3b^2c+3a.2bd+3a.c^2$	$3b^2d+3bc^2+3a.2be$ $+3a.2cd$
	3.	4	0	7	$\overline{16}$	29
	a	b	c	d	e	f

Subtraction Terms

upto 5 decimals x = 3.4 07 $\overline{16}$ 29 = 3.406$\bar{4}$9 = 3.40569

Taylor's Expansion Method

We consider the cube root of 39.5 which can be written in the form of cubic equation as $E = x^3 - 39.5 = 0$. This is same as finding out Cube root of 39.5

The working details are as follows.

$E = x^3 - 39.5 = 0 \qquad f(x) = x^3 = 39.5$

The solution of this can be represented as x = a. bcdef......

which means in the expanded decimal form

$x = a + 10^{-1}b + 10^{-2}c + 10^{-3}d + 10^{-4}e + \ldots\ldots$ where a, b, c, d, e are decimal digits in 1, ..., 9 or 0.

In this procedure (table) the first row deals with the number, whose cube root is to be determined written in the form including the decimals. For example 39.5 is written as 39.50000... to as many decimals as is required to be evaluated.

Step1: The first step is to workout the nearest cube root (CR) of 39 which is 3 (4 onwards will have the cube as higher value than 39). This nearest CR of 39 is 'a'. So CR is in the form of a.bcde.... where b, c, d, e, f,. are 1st, 2nd, ... decimal digits.

Finding out the CR tentamounts to finding the cube root of 39.5

This is explained by British Authors using Taylor's Series expansion and the recurring relations of Taylor's series.

$f(x) = f[a + b \times 10^{-1} + c \times 10^{-2} + \ldots..]$ on rearranging

$10^{-1}b + 10^{-2}c + 10^{-3}d \ldots\ldots$

$$= -f(a) - \frac{1}{2}f''(a)\,[10^{-1}b + 10^{-2}c + 10^{-3}d\ldots]^2 - \frac{1}{6}\frac{f'''(a)[10^{-1}b + 10^{-2}c + 10^{-3}d + \ldots\ldots]^3}{f'(a)}$$

Now the entire problem is worked out as a function of multipliers of differentials $f', f'', f''', \ldots$

The expression $x = a + b \times 10^{-1} + c \times 10^{-2} \ldots$ is expanded to have

1) Square terms.
2) Cubic terms as follows. The square term is $[10^{-1}b + 10^{-2}c + 10^{-3}d\ldots\ldots]^2$ ie b^2, 2bc, c^2 + 2bd, 2be + 2cd..... depending on the decimal requirement. This consists of square terms, Product of two digits, which give rise to contributions to 2nd third, fourth, fifth decimals and so on. Similarly $[10^{-1}b + 10^{-2}c\ldots\ldots]^3$ gives rise to contributions to various decimals depending on the products. For example in this expansion b^3, $3b^2c$, $3bc^2 + 3b^2d + c^3 + \ldots$ Contributions to third, fourth, fifth, sixth decimals and so on. The

contributions in the square terms can be obtained using Swamijis duplex concept ie 'Dwandvayoga' which was adopted by British Authors in the working details in combination with Taylor's Series.

A Systematic evaluation of decimal point is worked out by considering the various contributions from the square, cube terms higher order terms also. These are formulated in Table upto 12^{th} order equation and for 12 decimal points along with the corresponding differentials

Taylor's Expansion (TE)

$10^{-1}b + 10^{-2}c + 10^{-3}d + 10^{-4}e + \ldots$

$$= -f(a) - \frac{1}{2} f''(a)\,[\,10^{-1}b + 10^{-2}c + 10^{-3}d + \ldots]^2 - \frac{1}{6}\;\frac{f'''(a)[10^{-1}b + 10^{-2}c + \ldots]^3}{f'(a)}$$

The contribution to various decimal points pertaining to the square terms in the expansion are to be multiplied by the second differential coefficient $\frac{1}{2} f''(x)$ at x = 1.

These are to be placed under their respective decimal points. For example b^2 in the 2^{nd} decimal, 2bc in the 3^{rd} decimal, $2bd + c^2$ in the 4^{th} decimal, 2bc + 22cd in the 5^{th} decimal and so on as contributions of duplex terms. (Refer Table M) Similarly the contributions from the cubic expansion in TE are to be multiplied by the third differential $\frac{1}{6} f'''(x)$ at x = a and the values of b^3, $3b^2c$, $3bc^2$, $3b^2d$ etc..... are to be placed under 3^{rd}, 4^{th}. 5^{th} decimal contributions and are to be placed under their corresponding decimal points (Refer Table M) The method makes use of the multiplication of second, third differentials with the respective contributions arriving out of the square cubic expansions of the TE (duplex and triplex terms) respectively.

The same procedure can be adopted to the evaluation of the roots of any degree equation provided the corresponding differentials as multipliers can be worked out and together with the expansion of decimal contributions to that degree

The duplex terms which need to be multiplied by the Second differential $\left(\frac{1}{2} f'''(a)\right)$ are read from the expression.

$[10^{-1}b + 10^{-2}c + 10^{-3}d\ldots\ldots]^2$ b, c, d are the 1^{st}, 2^{nd}, 3^{rd} decimal points

				Decimal Contribution
Duplex (b)	=	b^2	—	2nd
Duplex (bc)	=	2bc	—	3rd
Duplex (bcd)	=	$2bd + c^2$	—	4th
Duplex (cd)	=	2cd	—	5th
Duplex (d)	=	d^2	—	6th

This table is to be worked out for any range of decimals to find out the respective contributions. These will be forming the Second Horizontal setup (Table)

The Triplex terms are evaluated using the duplex terms b^2, 2bc, $c^2 + 2bd$, 2be + 2cd, d^2. Th.s needs to be multiplied by bcd……

b^2	2bc	$2bd + c^2$	2cd	d^2
b	c	d	0	0
b^3	$3b^2c$	$3b^2d + 3b^2c$	$6bcd + c^3$	$3bd^2 + 3c^2d$

Triplex terms contributions to decimal points

b^3	—	3rd
$3b^2c$	—	4th
$3b^2d + 3bc^2$	—	5th
$c^3 + 6bcd$	—	6th
$3b^2d + 3c^2d$	-	7th position

These depend on the required decimal points to start with ie b, c, d and also the required decimal in the final answer (value of x). These are all to be multiplied $\frac{1}{6}f''(a)$ and considered for subtraction under respective decimals.

These will form the Third Horizontal set up. (Table)

The first Horizontal setup consists of the following details.

Step 1: To write down the problem in the firm of a.bcd….

(The decimal points to be extended as per requirements.)

If the given problem has one or no decimal solution, then the working can be extended to any number of decimals. Then accordingly we have to write down the decimal to the required extent.

For example: 39.5 is the given number required for evaluation of cube root say to 5 decimal points. The given number is written in the 1st horizontal line is 39.50000. The cube root is in the form a. bcdef.

Evaluation of 'a' is considered by finding out the nearest CR of 39 which is 3. $\therefore$ a = 3.

The values of $f'(x)$, $f''(x)$, $f'''(x)$ at $x = a$ of $f(x) = x^3$ are to be calculated.

The values are shown in the table.

To proceed to evaluation of b, c, d, e, f

We have to frame the intermediate dividends (ID) and new dividends (ND) from the first decimal point on wards.

1) $x^3 = 39.5$

   The value of x^3 at x = a = 3 is 27. The difference between 39 and 27 is 12 which is to be coupled with the 1st decimal point of the given number to enable the formation of 1st ID. This is obtained as $(12 \times 10) + 5 = 125$ (shown in the Table) This is also new dividend (ND) in relation to the 1st decimal point.

2) A common divisor (CD) is derived from the first derivative $f'(x)$ at x=a ie at $x=3 \Rightarrow 27$) $\therefore$ CD = 27

3) The evaluation, of successive decimal points, starting from 'b' through determination of IDs NDs, and by dividing the latter by CD is as follows.

4) The 1st decimal point b:

   The 1st ID which is also the ND is divided by CD to get b and corresponding remainder.

   $125 \div 27 \Rightarrow 4 = b$ $17 = R$

   R is multiplied by 10 and coupled with the next decimal dividend to form the next ID as 170

To Evaluate 'c':

The ID is 170 from this , one has to subtract $b^2 \frac{1}{2} f''(a) = 16 \times 9 = 144$

ND = 170 – 144 = 26

ND is divided by CD giving the value c = 0; R = 26

R is to be multiplied by 10 and coupled to the next decimal dividend to get next ID as 260

To Evaluate 'd':

The ID is 260 from which one has to subtract the duplex term (2bc) and triplex term (b^3) contributions as $\left(2.\left(\frac{1}{2}\cdot{}'(.)\right)+.'\left(\frac{1}{6}\cdot{}'''(.)\right)\right)$ ie 260 – (0 + 64) = 196 (ND)

ND when divided by CD gives the value for d as 7 and 7 as the R.

R is multiplied by '10' and coupled with the next decimal dividend to get the next ID as 70

To Evaluate 'e':

The ID is 70. From this one has to subtract the duplex terms ($2bd + c^2$) and triplex term ($3b^2c$) contributions as $(2bd + c^2)\left(\frac{1}{2}f''(a)\right)+(3b^2c)\left(\frac{1}{6}f'''(a)\right)$

ND = 70 [504 + 0] = – 434

ND ÷ CD ⇒ – 434 ÷ 27 ⇒ $\overline{16}$ = e and R = $\overline{2}$

R is to be multiplied by 10 and coupled with the next decimal dividend to get next ID is $\overline{20}$

To Evaluate 'f':

ND is obtained by subtracting the duplex terms (2be + 2cd) and triplex terms ($3b^2d + 3bc^2$) Contributions as $(2be + 2cd)\left(\frac{1}{2}f''(a)\right) + (3b^2d + 3bc^2)\left(\frac{1}{6}f''(a)\right)$

ND = – 20 – [– 1152 + 336] = 796

ND ÷ CD gives the value of 'f' as 29 as 13

The cube roots is a. bcdef

= 3.4 0 7 $\overline{16}$ 29

= 3. 4 0 5 6 9

Table

		a	b	c	d	e	f
f′= f′(a)CD	f′(a)=27	39	5	0	0	0	0
			12	17	26	7	– 2
	$\frac{1}{2}f''(a)=9$			$\overline{144}$	0	$\overline{504}$	1152
				b^2	2be	2bd + c^2	2be+2cd
	$\frac{1}{6}f'''(a)=1$				$\overline{64}$	0	$\overline{336}$
					b^3	$3b^2c$	$3b^2d+3bc^2$
		3	4	0	7	$\overline{16}$	29
		a	b	c	d	e	f

* Refer Table M for Duplex and Triplex Terms used in the Taylor's Series.

Step 1: 39 – 27 = 12, ID = 125 is also ND

Step 2: 125 ÷ 27 ⇒ 4 = b R = 17 ID = 170

Step 3: $170 - b^2 \frac{1}{2} f''(a) = 170 - 144 = 26$ 26 ÷ 27 ⇒ c = 0 R = 26 ID = 260

Step 4: $260 - (2bc)\frac{1}{2} f''(a) - \frac{1}{6} f'''(a)(b^3)$ 260 – 0 – 64 ⇒ 196 ND

196 ÷ 27 ⇒ 7 = d, 7 = R ID = 70

Step 5: $70 - (2bd + c^2)\frac{1}{2} f''(a) - \frac{1}{6} f'''(a)3b^2c$ 70 – 504 – 0 = – 434 ND

– 434 ÷ 27 ⇒ – 16 = e, R = – 2 ID = – 20

Step 6: $-20 - (2be + 2cd)\frac{1}{2} f''(a) - \frac{1}{6} f'''(a)(3b^2d + 3bc^2)$

– 20 + 1152 – 336 = 796 ND

796 ÷ 27 ⇒ 29 = f R = 13

CR of 39.5 upto fifth decimal is $3.40\ 7\ \overline{16}\ 29 = 3.40569$

This method can be extended for the determination of the roots of any power of x. The following details are to be first worked out.

1) Required decimals should be clearly noted.
2)
   - i) The duplex terms in case Square roots
   - ii) Duplex, triplex terms in case of cube roots
   - iii) Duplex, triplex quadruplex terms in case of 4[th] root.
   - iv) Duplex, triplex quadruplex terms and quintet terms in case fifth roots, and so on ……

For still higher order roots, additional terms such as Sextect, Heptet etc terms as contributing to the decimals are also to be considered depending on the power of the equation.

These are clearly shown in the Table M for 12[th] order equation upto 12[th] decimal points along with the corresponding derivative multipliers.

Let x = a, b, c, d, e, ………..

$x = a + 10^{-1}b + 10^{-2}c + 10^{-3}d + \dots\dots\dots$

$f(a + 10^{-1}c + 10^{-2}d + \dots\dots)$

$$\frac{\left(-f(a) - \frac{1}{2} f''(a)(10^{-1}b + 10^{-2}c + \dots)^2 - \frac{1}{6} f''(a)(10^{-1}b + 10^{-2}c + 10^{-3}d\dots)^2\right)}{f'(a)}$$

1. Find the Cube Root 131.8

$f'(a) = 75$	131	6^{8}	68^{0}	5^{0}	50^{0}	$\overline{40}^{\,0}$
$\frac{1}{2}f''(a) = 15$			$\bar{0}$	$\bar{0}$	$\overline{1215}$	$\bar{0}$
$\frac{1}{6}f'''(a) = 1$				$\bar{0}$	$\bar{0}$	$\bar{0}$
	5	0	9	0	$\bar{9}$	$\bar{5}$

Step 1 : Nearest cube root of 131 in 5 $\therefore a = 5$

Remainder $= 131 - 5^3 = 6$

Step 2: New dividend = 68

75) 68 (0
0
68

Step 3: Intermediate dividend = 680

New dividend=$680 - \frac{1}{2}f''(a)b^2$

75) 680 (9
675
5

$\therefore c = 9$

Step 4: Intermediate dividend = 50

New dividend =

$$50 - \frac{1}{2}f''(a)b^2\,2bc - \frac{1}{6}f''(a)b^3$$

= 50

75) 50 (0
0
50

$\therefore d = 0$

Step 5: Intermediate dividend = 500

New dividend =

$$500 - \frac{1}{2}f''(a)(2db + c^2)$$

$$-\frac{1}{6}f'''(3b^3c)$$

$= 500 - 15 \times 81 = - 715$

75) $\overline{715}$ ($\bar{9}$
$\overline{675}$
$\overline{40}$

$\therefore e = \bar{9}$

Step 6: Intermediate dividend= -400

New dividend $=500 - \frac{1}{2}f''(a)$

$$(2be + 2cd) - \frac{1}{6}f'''(3bc^2 + 3b^2d$$

$= - 400$

75) $\overline{400}$ ($\bar{5}$
$\overline{375}$
$\bar{5}$ $\therefore f = \bar{5}$

$\therefore$ cube root $= 5.090\bar{9}\,\bar{5}$
$= 5.08905$

2. Find the Cube Root of 2298968

$$x^3 - 2298968 = 0 \Rightarrow x^3 = 2298968$$

$a = 131$

$f' = 3a^2 = 51483$

$\frac{1}{2} f'' = \frac{1}{2} 6a = 3a = 393$

$\frac{1}{6} f''' = 1$

	0	0	0	0	0	0	0
51483	50877	45423	10533	48009	41130	44082	
393		$\overline{31833}$	$\overline{56592}$	$\overline{25152}$	$\overline{56592}$		
1			$\overline{729}$	$\overline{1944}$	$\overline{1728}$		
131	9	8	0	8	6		
a	b	c	D	e	f	g	h

131.98086

The CR of 2298968 upto 5 decimals is 131.98086

Finding out the Roots using different methods

1. **Find the Cube Root of** 9999976000191999488

Swamiji's Method (Straight Division Method)

$CD=3a^2=243$	999	9	7	6	0	0	0	1	9	1
	729	270	36	97						
			279							
			$3ab^2$	b^3+6abc						
	9	11	0							
	a	b	c							
		10								

Reduction is necessary Hence an attempt is made with two digit method.

$CD=3a^2$ =29403	999976	0	0	0	1	9	1	9	9	9	4	88
	970299	29677	2740	3343								
			32143	32746	13990	6737	4109					
			$\overline{24057}$	$\overline{48843}$	$\overline{74358}$	$\overline{63180}$	$\overline{40338}$					
	99	10	10	9	2.	0	0					
	a	b	c	d	e	f	g					
		9	9									

Duplex Method

CR 999992

999992 Duplex$_2$

$(CR)^2$ 999984000064

999984000064

000000999992 By Urdhva tiryak

999976000191909488

Argumentation (IGHJKL Method)

Here L = a, K = b J = c I = f G = e H = d Refer Cubic expansion Table N for the subtraction terms.

Let CR be f e d c b a

F = 9

N = 6

L = 2

999976000191999488

8

99948

$3a^2b = 3 \times 4 \times b = 12b$ 48

(b=4)

12b ends in 8

$\therefore$ b = 4 or 9 9990

Let b = 4 $\Rightarrow$ 12b=48

12c ends in 4=7 c = 2 or 7 96

Let c = 2

$b^3+3a^2d \neq 6abc$ 9894

=64+12d+96 24

9870

=12d+160 160

12 d ends in 7 hence $b \neq 4$ 827

We consider the other value of b b = 9

a = 2 999488

a^3 8

b = 9 99948

12b 108

$3ac^2+3ab^2$ 9984

12c + 486 486

Let c = 4 9498

12c 48

12c ends in 8 ⇒c=4or9(c=4)		191945
Let c = 4		1161
$3a^2d + 6abc + b^3$		190784
= 12d + 432 + 729	12d	24
12d+1161 12 d ends in 4		
∴ d = 2 or 7		19076
Let d = 2		
$3a^2e + 3ac^2+3b^2c+6abd$		1284
12e+96+972+216		17792
	12e	12
12e ends in 2 ⇒ e = 1, 6		17780
Let e = 1		
$3a^2f+3b^2d+3bc^2+6abe+6acd$		1122
12f+486+432+108+96		0656
12f+1122	12f	36
12f ends in 6 ∴ f = 3, 8		6000062

Let f = 3

CR = 312492

Let us consider c = 9

a =2 b = 9

	1919984
$3a^2c+3ab^2=12c+486$	486
12c ends in 8	1919498
⇒ c = 4 or 9	
Let c = 9	108
$b^3+3a^2d+6abc$	191939
12d+729+972	1701
12d ends in 8	190238
⇒ d = 4 or 9	48
Let d = 4	
$3a^2e+3ac^2+3b^2c+6abd$	19019
12e+486+2187+432	3105
12e ends in 4	15914
e = 2 or 7	
Let e = 2	24
$3a^2f+3b^2d+3bc^2+6abe+6acd$	15890
12f+972+2187+216+432	3807
12f ends in 3 is <u>ruled out</u>	2083

Hence d = 9

	190238
	<u>108</u>
$3a^2e+3ac^2+3b^2c+6abd$	19013
12e+486+2187+972	<u>3645</u>
12e ends in 8	15368
e = or 9	<u>48</u>
Let e = 4	
$3a^2f+3b^2d+3bc^2+6abe+6acd$	1532
12f+2187+432+972	<u>5778</u>
12f ends in 4	5744

⇒ f = 2 or 7 is ruled out as f = 9 is the nearest CR of 1st group

Hence e = 9	15368
	<u>108</u>
$3a^2f+3b^2d+3bc^2+6abe+6acd$	1526
12f+2187+2187+972+972	<u>6318</u>
12f ends in 8	5208

⇒ f = 2 or 9

∴ f = 9 as already fixed as the nearest CR of first group

∴ CR of given number is 999992

Taylor's Method in grouping

243	999	9	7	6	0	0
	729	270	36	$\overline{227}$	23	$\overline{97}$
27			$\overline{3267}$	6534	$\overline{10395}$	21978
1				$\overline{1331}$	3993	$\overline{8349}$
	9	11	$\overline{11}$	12	$\overline{25}$	52
	a	b	c	d	e	f

243	999	9	7	6	0	0	0	0	0
	729	270	522	853	1246	1525	1642	1597	176
27			$\overline{2187}$	$\overline{4374}$	$\overline{6561}$	$\overline{8748}$	$\overline{7533}$	$\overline{5346}$	
1				$\overline{729}$	$\overline{2187}$	$\overline{4374}$	$\overline{7290}$	$\overline{9234}$	
	9	11	12	14	15	8	6	5	
	a	b	c	d	e	f	g	h	
		10	11	13	14	7	5	4	
			10	12	13	6	4	3	
		(9)	(9)	11	12	5	3	2	
				10	11	4	2	1	
				(9)	10	3	1	(0)	
					(9)	(2)	(0)		

Cube Root of N is 999992

Difference between Swamiji method and British Authors work.
In both methods one should prepare general $(a + b + c)^3$ expansion table
This is applicable in decimal evaluation also when a is first number b, c, d are the $a + b10^{-1}\ c10^{-2}\ d10^{-3}$ has be reckoned

2. Find the cube root of 37349.123

Swamiji

1) To determine the nearest cube root of the number preceeding the decimal. If the number has more than three. The number has 5 degits preceeding the decimal. This can be grouped into 2 groups as (37) (349) 1[st] group 2[nd] group Finally to consider the number whole to determine the cube root with the help of expansion table consider nearest cube root of the first group which is 37; as 3
2) Remainder ie $.37 - B^3 = 10$
3) 1[st] Intermediate dividend $(10 \times 10) +$ 3 103
4) ID = 103 this is divided by $3a^2 = 27$
$103 \div 27 = 3$. R = 22
5) $ID_2 = (22 \times 10) + 4 = 224$
Subtract $3ab^2 = 8$ from this is ND 224 – 81 = 143 Divide ND by 27
c = 5, R = 8
ID = 89
$(b^3 + 6abc)$ (original)
27 + 270 = 297
As the results Reduction can be attempted .
∴ c = 4 R = 8
It needs reduction at every stage. Hence two digit method is to be attempted

$CD=3a^2$	37349	1	1	3
3267		1412	1053	2414
			$\overline{1584}$	$\overline{1648}$
	33.	4	2	6
	a.	b	c	d

Taylor

1) A should be 3 as the nearest cube of 37

$x^3 = 37349.123$
a = 33
f(a) = 35937

	1	2	3
(CD) $f'(a) = 3a^2$ = 3267	1412	1053	2414
$\frac{1}{2}f''(a) = 99$		$\overline{1584}$	$\overline{1584}$
$\frac{1}{6}f'''(a) = 1$			$\overline{64}$
33.	4	2	6
a.	b	c	d

A comparison of these two clearly shows that Swamiji's result is just the same as that obtained from Taylor's series.

1) A detailed comparison shows that while multipliers of differentials are to be considered in the Taylor's giving rise to a number of terms for subtraction, the Swamiji's method makes use of a single subtraction as per the terms derived in the cube expansion

Both the methods are extendable to any root but with this difference maintained in point 2 .

In both the methods general terms in the expansion Table concerned with the required power upto the required range of decimals are to be concerned. In the Taylor's method in addition to the above on has to get all the differentials in consonance with the given power. Eg if cube root is required then 3 differentials are necessary 10^{th} root 10 differentials are to be considered

3. Finding the cube root of 971 with the comparison

Swmaji's Method

		0	0	0	0	0	0	0	0
CD=	971	242	233	143	215	206	116	80	206
$3a^2=243$									
						449	359	323	
							602	566	
			$\overline{2187}$	$\overline{729}$	$\overline{972}$	$\overline{1944}$	$\overline{4725}$	$\overline{3510}$	
	a	b	c	d	e	f	g	h	
	9	9	0	2	(4)	(10)	(5)	8	
					3	(9)	(4)		
						(8)	(3)		

Nearest integer cube root is 9 ∴ a = 9

$\sqrt[3]{971} = 9.902383$ (8)

$971 - a^3 = 242$ gives the first dividend

(971 − 729) as 2420

$3a^2$ is the common divisor CD

Taylor's Method

		0	0	0	0	0	0	0	0
	971	242	233	143	215	206	116	80	206
						449	359	323	
f'	243						602	566	
$\frac{f''}{2}$	27		$\overline{2187}$	0	$\overline{972}$	$\overline{1458}$	$\overline{3996}$	$\overline{1458}$	
$\frac{f'''}{6}$	1			$\overline{729}$	0	$\overline{486}$	$\overline{729}$	$\overline{2052}$	
	a	b	c	d	e	f	(g)	h	
	9	9	0	2	(4)	(10)	5	8	
					(3)	(9)	(4)		
						(8)	(3)		

$\sqrt[3]{971} = 9.902383\ 8$ $f(a) = 29$

RHS = 242 giving the firs dividend as 2420

Swamiji

1) 1^{st} step (group the given number into 3 digits each)
(971) is one group

The nearest cube root is a = 9

a = 9; The common divisor = $3a^2 = 243$ (c1)

2) 2^{nd} step (Subtracting the nearest cube value from the given Number
i.e. 971 – 729 = 242

The dividend is 2420 (ND)

3) Dividing the ND by CD we get 9 as the quotient for b, the remainder being 233

4) The Dividend (ID) is 2330, on subtracting from this the value of $3ab^2$, we get new dividend as 2330 – 2187 = 143

5) Dividing 143 by CD we get 0 as c and the remainder is 143. Thus the Dividend ID is 1430

6) To get the ND we have to subtract (b^3 + 6abc) form the above ID 1430

∴ 1430 – (729) = 701 ND

ND = 701

Dividing ND by CD, we get d = 2 and the remainder 215

Taylor's Series

1) Nearest CR of the given number 971 is a = 9
The common divisor CD is got from $f'(a) = 3a^2 = f'(a) = 243$

2) The same is procedure as Swamijis to get the first dividend i.e. 2420 (ND)

3) ND Dividend is divided by the common Divisor to get the value of b and the remainder, giving the next dividend
b = 9 R = 233

4) The next ID dividend is 2330 on subtracting the value of
$\frac{1}{2}f''(a)b^2 \; 3ab^2 = 2187$
From this dividend we get ND as 143 (2330 – 2187)
143 ÷ CD give c = 0, R = 143

Note

$3a = \frac{1}{2}f''(a) = 3a$ hence both have same subtraction term.

5) To get the next ID ND we have to subtract
$\frac{1}{2}f''(a)(2bc) + \frac{1}{6}f'''(a)b^3$
i.e. 3a x 0 + 1 x 729
1430 – 729 = 701
The values of d = 2 and the remainder 215 are obtained by dividing 701 by CD

Next ID is 2150.

$ND = 2150 - (3ac^2 + 6abd + 3bc^2) = 1178$

$ND \div 3a^2$ gives $e = 4$, 206 (R)

Next ID is 2060

$ND = 2060 - (6abc + 3bc^2 + 3b^2d + 6acd)$ – ve

Hence reduction is carried to from 4 to 3 e

$\therefore$ ND = 4490 – 1944 = 2546

2546 ÷ 243 = 10 as f, 116 (R)

These two methods give the same result, the procedures being different.

Note: New Dividend is 2150

729 is contributed by two terms

$$\frac{1}{2}f''(a)2bc + \frac{1}{6}f'''(a)b^3 = 729$$

$$ND = 2150 - \frac{1}{2}f''(2bd + c^2) - \frac{1}{6}f'''(3b^2c)$$

ND is same as that of Swamiji but has two different parts giving finally the same result as 1178

1178 ÷ CD gives the next dividend being 2060

e = 4, 206 (R)

$2060 - (2bc + 2cd)\,\frac{1}{2}f''(a) - \frac{1}{6}f'''(3b^2d + 3bc^2)$ gives a – ve value hence a reduction in the value of from e 4 to 3, leading to the value of f = 10, with the remainder = 116. The next ID is 1160

4. N = 179856214027

Let us consider a bigger number for finding the Cube Root

$\sqrt[3]{179856214027}$ (179) (856) (214) (027) N is grouped into 4.

From 179	8	5	6	2	1	4	0	2	7
$3a^2$=CD=75	54	23	70	70	60	16	58	64	6
		98	145	14	135	91	133	139	
				220		160	208	214	
					210	241	283	289	
					285	316	358	364	
								439	
5	7	5	6	10	5	6			
a	b	c	d	e	f	g	h	i	
	6	(4)	5	9	4	5	7	11	
			(4)	8	3	4	6		
				(7)	2	3	5		
						2	4		
							3		

Following Swamiji's Method

1) $a^3 = 125$ 179 – 125 = 54
   1st Dividend is 548
2) CD = $3a^2$
3) 548 ÷ 75 b = 7, 23R
4) Dividend is 235
   ND = 235 – $3ab^2$ (735) – ve
   Hence reduction of b to 6 with the ID dividend as 985
5) ND = 985 – $3ab^2$ = 445

445 ÷ 75 = c = 5, 70R
Dividend is 706
ND = 706 – (b^3 + (abc) – (216 + 900) – ve result
Hence c is reduced to 4 with Dividend 1456
ND = 1456 – (216 + 720) = 520
520 ÷ 75 = d = 6, 70R
Dividend is 702
ND = 702 – ($3ac^2$ + 6abd + $3b^2c$) gives – ve value
Hence reduction of d to 5 leading to the d 327 as 1452
1452 – (240 + 900 + 432) – ve value
Hence further reduction d to 4 leading to Dividend as 2202
2202 – (240 + 720 + 432) = 810
810 ÷ 75 = e = 10 R = 60
next ID = 601
ND = 601 – 7 – (6abe + $3bc^2$ + $3b^2d$ + 6acd) gives – ve result
Hence reduction of e to 9 with dividend as 1351
1351 – (1620 + 288 + 432 + 480) – ve result
Hence reduction of e to 8 with the Dividend as 2101
ND = 2101 – (1440 + 288 + 432 + 480) Still – ve
Hence further reduction of e to 7 giving rise to a Dividend as 2851
∴ ND = 2851 – (1260 + 288 + 432 + 480) = 391
391 ÷ 75 gives f = 5, R = 16
Dividend = 164
ND = 164 – ($3ad^2$ + 6abf + $3b^2e$ + 6bcd + 6ace + c^3)
– ve result given to Division with reduction of f to 4 from 5 and the next ID is 914
ND = 914 –is – ve and needs further reduction of f to 3 with Dive = 1664
ND = 1664 – ( ) – ve hence reduction of f to 2 with Division as 2414
ND = 2414 – (240 + 360 + 756 + 576 + 840) – ve hence f is reduced to 1 with Division as 3164

ND = 3164 – (240 + 180 + 756 + 576 + 840 + 64)

508 ÷ 75 g = 6   R = 58

Next ID = 580

ND = 580 – ($6abg + 3bd^2 + 3b^2f + 6bce + 3c^2d + 6acf + 6ade$)

1080 + 288 + 108 + 1008 + 192 + 120 + 840

a = 5 b = 6 c = 4 d = 4 e = 7 f = 1 g = 2 h = 8

580 ——— g = 6
1330 ——— g = 5
2080 ——— g = 4
2830 ——— g = 3
3580 ——— g = 2

For the values of g = 6,,4,3; ND gives –ve values

Hence g = 2

Dividend = 3580

ND = (3580 – 2916) = 664

= 664 ÷ 75

h = 8 (64)

ND = 662 – ($3ae^2 + 6abh + 3b^2g + 6bef + 6bde + 3cd^2 + 3c^2e + 6acg + 6adf$)

= 642 – (735 + 1260 + 216 + 144 + 1008 + 192 + 336 + 240 + 120) – ve result

h is reduced to 3

$3ae^2$ $6abh$ $3b^2g$ $6bef$ $6bde$ $3cd^2$ $3c^2e$ $6acg$ $6adf$

ND = 4392 – (735 + 440 + 216 + 144 + 1008 + 192 + 336 + 240 + 120)

Where h = 8,7,6,5,4 the ND gives –ve value. Hence h = 3

ND = 4392 – 3531 = 861 gives i = 11 and R = 36

This method leads to laborious in the sense that at every step it appears to have reduction process hence considering two digit (two groups being considered one unit).

$3a^2$=9408	179856	2	1	4	0	2	7
		4240	4770	7381	2518	3484	4746
			($3ab^2$)	(b^3+6abc)	($3b^2c+3ac^2$ $+6abd$)	($3b^2d+3bc^2$ $+6abe+6acd$)	($c^3+3ad^2+3b^2e$ $+6abf+6ace$ $+6bcd$)
			$\overline{2688}$	(24+5376) = $\overline{5440}$	(192+2688 +9408) = $\overline{12288}$	336+192 +1344+9408 =$\overline{11280}$	(64+8232+48 +2688+1344 +672)=$\overline{13048}$
	56	4	4	7	1	2	3
	a.	b	c	d	e	f	g

Upto 4 decimals the CR of given number is 5644.7123

5. **Finding out the cube root of 16194277**

by using JKL method.

16 194 277

given number has divided into three groups. Hence CR contains three digits. JKL

J = 2 beings the nearest CR of first group

Number ends in 7 ∴ L = 3

		16 194 277
		− 27
$3L^2k$ = 27K ends in 5		16 19425
∴ K = 5	− 27K	− 135
$3L^2J + 3LK^2$		1929
	27J + 225	− 225
27J ends in 4		1704

∴ J = 2 or 7

But J = 2 as already considered

∴ Cube root of (16194277) is 253

Single Digit Method (Swamijis Straight Division method)

	16	1	9	4	2	7	7
CD=$3a^2$		8	9	9	17	13	
= 12			21	21	29		
		2	6	$\bar{9}$			
		a	b				
		2	5	5	2	1	
					a	c	
				4	1	0	
				(3)	0		

CD = $3a^2$	16	1	9	4	2	7	7
	or						
= 12	1.5.6.2.5	8	21	9			
	2	6	8				0
	a	b	5				
		5	c				

Requires reduction and leads to a complicated working

Two Digit Method

CD = $3a^2$	16194	2	7	7
= 1875	15625	569	67	2
			$\overline{675}$	$\overline{27}$
	25	3	0	0
	a	b		

CR of N = 253

These methods of working are applicable in general for determination of n^{th} root and for determination of n^{th} power of a number.

This method is applicable for imperfect n^{th} roots cubes and n^{th} power of a number having decimals.

Example of imperfect cube

6. Find the Cube root of 16195277 Two Digit Method (using Swamiji's Method)

CD = $3a^2$	16195	2	7	7	0	0	0	0	0	0
= 1875	15625	570	77	102	1000	625	250	1465	1471	310
	$\overline{570}$		$\overline{675}$	$\overline{27}$	0	$\overline{2250}$	$\overline{1035}$	$\overline{54}$	$\overline{5025}$	
	A = 25	3	0	0	5	2	0	7	5	
		b	c	d	e	f	g	h	i	

Cube Root of N upto seven decimals = 253.0052075

7. Find the Cube Root of 12535.672267 (using Swamiji's Method)

CD = $3a^2$	12535	6	7	2	2	6	7
= 1587	12167	368	512	90	66	5	
			$\overline{276}$	$\overline{836}$	$\overline{657}$	$\overline{657}$	
	a = 23	2	3	0	0		
		b	c	d	e	...	

$\therefore$ CR of (12535.672267) is 23.23

8. Cube root of 61.9

Swamiji's Method

	61	9	0	0	0
27		34	25	$\overline{20}$	16
			$\overline{1296}$	6480	
	3.	12	$\overline{38}$	232	

This is not desirable for further work as it requires lot of reduction process as shown below

$x^3 = 61.9$

	61	9	25	0	0	0
27	27	34	106	196	286	421
	3.	12	12	15	22	27
	a	11	11			
		10	10			
		(9)	9			
		b	8			
			7			
			6	329		
			(5)			
			c	5	7	
				d	e	

This application of reduction is complicated and Laborious

Applied ∴ higher nearest cube root value of 61 ie 4

one can try the higher value of nearest cube root of 61, i.e. 4.

Let $x^3 = 61.9$

a = 4 be the nearest CR of 61.9 $CD = 3a^2 = 48$, $3a = 12$

Swamiji's Method

	1	2	3	4	5	6	7	8	9	10	11	12
61	9	0	0	0	0	0	0	0	0	0	0	0
)=48 $\overline{64}$	$\bar{3}=\bar{2}\bar{1}$	$\bar{2}\bar{1}$	$\overline{18}$	$\overline{36}$	$\overline{24}$	0	$\overline{44}$	$\overline{32}$	$\bar{8}$	$\bar{5}$	$\overline{17}$	$\bar{8}$
		0	0	0+0 $+\overline{192}$ $=\overline{192}$	0+0+0 $+\overline{288}$ $=\overline{288}$	64+0 +0+0 $+\overline{1056}$ $+\overline{108}$ $=\overline{1100}$	0+0+144 $+0+0+\overline{1056}$ $+\overline{792}$ $=\overline{1704}$	0+528 +108 +0 $+0+\overline{2112}$ $+\overline{792}$ $+\overline{1452}$ $=\overline{3720}$	27+0+0 +528+792 $+0+\overline{4224}$ $+\overline{1584}$ $+\overline{2904}$ $=\overline{7365}$	0+1056 +1452+ 297+792 $+\overline{8064}$ $+\overline{3168}$ $+\overline{5808}+$ $\overline{1452}$ $=\overline{14895}$	0+0 +2112 +297 +1089 +2904 +1584+ $\overline{14880}+$ $\overline{6048}+$ $\overline{11616}$ +5808 $=\overline{30366}$	1331+0 +4032 +1452 +594 +3168 +5808 +2178 $+\overline{29856}$ $+\overline{11160}$ $+\overline{22176}$ $+\overline{11616}$ $+\overline{5808}$ $=\overline{62053}$
4.	0	$\bar{4}$	$\bar{3}$	$\bar{1}\bar{1}$	$\bar{1}\bar{1}$	$\overline{22}$	$\overline{44}$	$\overline{84}$	$\overline{155}$	$\overline{311}$	$\overline{636}$	$\overline{1294}$
a	b	c	d	e	f	g	h	i	j	k	l	m

Note: The subtraction terms are given in the next page under the corresponding columns

1	2	3	4	5	6	7	8	9	10	11	12
	$3a^2b$	b^3+	$3b^2c+$	$3b^2d$	c^3+3b^2e	$3b^2f$	$3b^2g+3c^2e$	d^3+3b^2h	$3b^2i+3c^2g$	$3b^2j+3bf^2$	e^3+3b^2h
		$3a.2bc$	$3a.2bd$	$+3bc^2$	$+6bcd$	$+3bd^2$	$+3cd^2+6bcf$	$+3be^2$	$+3ce^2+3d^2e$	$+3c^2h$	$+3c^2i+3cf^2$
			$+3a.c^2$	$+3a.2be$	$+3a.2bf$	$+3c^2d$	$+6bde$	$+3c^2f$	$+6bch$	$+3d^2f$	$+3d^2g$
				$+3a.2cd$	$+3a.2ce$	$+6bce$	$+3a.2bh$	$+6bcg$	$+6bdg$	$3de^2+6bci$	$+6bcj$
					$+3a.d^2$	$+3a.2bg$	$+3a.2cg$	$+6bdf$	$+6cdf$	$+6bdh$	$+6bdi$
						$+3a.2cf$	$+3a.2df$	$+6cde$	$+6bef$	$+6beg$	$+6cdh$
						$+3a.2de$	$+3a.e^2$	$+3a.2bi$	$+3a.2bj$	$+6cef$	$+6beh$
								$+3a.2ch$	$+3a.2ci$	$+6cdg$	$+6ceg$
								$+3a.2dg$	$+3a.2dh$	$+3a.2bk$	$+6def$
								$+3a.2ef$	$+3a.2eg$	$+3a.2cj$	$+6bfg$
									$+3a.f^2$	$+3a.2di+$	$+3a.2bl$
										$3a.2eh$	$+3a.2ck$
										$+3a.2fg$	$+3a.2dj$
											$+3a.2ei$
											$+3a.2fh$
											$+3a.g^2$

Upto m (12 decimals) $x = 4.0\ \bar{4}\ \bar{3}\ \overline{11}\ \overline{11}\ \overline{22}\ \overline{44}\ \overline{84}\ \overline{155}\ \overline{311}\ \overline{636}\ \overline{1294} = 4.0\ \bar{4}\ \bar{4}\ \bar{2}\ \bar{3}\ \bar{7}\ \bar{4}\ \bar{3}\ \bar{3}\ \bar{7}\ \bar{5}\ \bar{4} = 3.955762566246$

$x^3 = 61.9$ $a = 4$ $x = a.bcdefghijklm$

Taylor's Method

	a	b	c	d	e	f	g	h	i	j	k	l	m	n
$f'(a)=48$	61	$\bar{3}$ 9	$\bar{2}\bar{1}$ 0	$\bar{1}\bar{8}$ 0	$\bar{3}\bar{6}$ 0	$\bar{2}\bar{4}$ 0	0 0	$\bar{4}\bar{4}$ 0	$\bar{3}\bar{2}$ 0	$\bar{8}$ 0	$\bar{5}$ 0	$\overline{17}$ 0	$\bar{8}$ 0	$\overline{21}$ 0
$\frac{f''(a)}{2}=12$			0	0	$\overline{192}$	$\overline{288}$	$\overline{1164}$	$\overline{1848}$	$\overline{4356}$	$\overline{8712}$	$\overline{18492}$	$\overline{38352}$	$\overline{80616}$	
$\frac{f'''(a)}{6}=1$				0	0	0	64	144	636	1347	3597	7986	18563	
	4	0	$\bar{4}$	$\bar{3}$	$\overline{11}$	$\overline{11}$	$\overline{22}$	$\overline{44}$	$\overline{84}$	$\overline{155}$	$\overline{311}$	$\overline{636}$	$\overline{1294}$	
	a	b	c	d	e	f	g	h	i	j	k	l	m	

Step 1 : $61.9 - 64 = \bar{3}9$

Step 2 : (b) $\bar{3}9 = \overline{21}$

$48)\ \bar{2}\bar{1}\ (0\ (b) \quad \therefore b = 0$

$\underline{0\,0}$

$\underline{\bar{2}\bar{1}}$

Step 3 : $12(b^2) = 12(0) = 0$

(c) $48)\ \bar{2}\bar{1}0\ (\bar{4}\ (c) \quad \therefore c = \bar{4}$

$\underline{\bar{1}\bar{9}\bar{2}}$

$\underline{\bar{1}\bar{8}2} = \bar{1}\bar{8}$

Step 4: $12(2bc) = (2 \times 0 \times \bar{4})12 = 0$

(d) $1(b^3) = 1(0) = 0$

$48)\ \bar{1}\bar{8}0\ (\bar{3}\ (d) \quad \therefore d = \bar{3}$

$\bar{1}\bar{4}\bar{4}$

$\bar{4}4 = \bar{3}\bar{6}$

Step 5: $12(2bd + c^2) = 12(0 + 16) = 192$

(e) and

$1(3b^2c) = 0$

$\bar{3}\bar{6}0 + \bar{1}\bar{9}\bar{2} = \bar{5}\bar{5}\bar{2}$

$48)\ \bar{5}\bar{5}\bar{2}\ (\bar{1}\bar{1}(e)\ \therefore e = \bar{1}\bar{1}$

$\bar{5}\bar{2}\bar{8}$

$\bar{3}6 = \bar{2}\bar{4}$

Step 6: $12(2be + 2cd)$

(f) $= 12(0 + 2 \times \bar{4} \times \bar{3}) = 288$

$1(3b^2d + +3bc^2) = 1(0 + 0) = 0$

$\bar{2}\bar{4}0 + \bar{2}\bar{8}\bar{8} = \bar{5}\bar{2}\bar{8}$

$48)\ \bar{5}\bar{2}\bar{8}\ (\bar{1}\bar{1}(f)\ \therefore f = \bar{1}\bar{1}$

$\bar{5}\bar{2}\bar{8}$

0

Step 7: $12(2bf + 2ce + d^2)$

(g) $= 12[0 + (2 \times \bar{4} \times \bar{1}\bar{1}) + 9]$

$= 12(88 + 9) = 1164$

and

$1(3b^2e + 6bcd + c^3) = (0 + 0 + \bar{6}\bar{4}) = \bar{6}\bar{4}$

$\bar{1}\bar{1}\bar{6}\bar{4} + 64 = \bar{1}\bar{1}00$

$48)\ \bar{1}\bar{1}00\ (\bar{2}\bar{2}\ (g)\ \therefore g = \bar{2}\bar{2}$

$\bar{1}0\bar{5}\bar{6}$

$\bar{4}\bar{4}$

Step 8: $12(2bg + 2cf + 2de)$

(h) $= 12[0 + (2 \times \bar{4} \times \bar{1}\bar{1})$

$+ (2 \times \bar{3} \times \bar{1}\bar{1})]$

$= 12(88 + 66) = 12(154) = 1848$

and

$1(3b^2f + 6bce + 3bd^2 + +3c^2d)$

$[0 + 0 + 0 + (3 \times 16 \times \bar{3})] = \overline{144}$

$\therefore \bar{4}\bar{4}0 + \overline{1848} + 144 = \overline{2144}$

$48)\ \overline{2144}\ (\bar{4}\bar{4}\ (h)\ \therefore h = \bar{4}\bar{4}$

$\bar{2}\bar{1}\bar{1}\bar{2}$

$\bar{3}\bar{2}$

Step 9: $12\,(2bh + 2cg + 2df + e^2)$

(i) $= 12[0 + (2 \times \bar{4} \times \bar{2}\,\bar{2})$

$+ (2 \times \bar{3} \times \bar{1}\,\bar{1}) + 12\bar{1}\,]$

$= 12\,(176 + 66 + 121) = 12 \times 363$

$= 4356$

$\therefore$ part 1 $= \bar{4}\,\bar{3}\,\bar{5}\,\bar{6}$

$1(3b^2g + 6bcf + 6bde + 3c^2e +$

$3cd^2)$

$= [0 + 0 + 0 + (3 \times 16 \times \bar{1}\,\bar{1}) +$

$(3 \times \bar{4} \times 9)]$

$\overline{528} + \bar{1}0\bar{8} = \overline{636}$

$\therefore$ part 2 $= 636$

$320 + 4356 + 636 = 4040$

$48\ )\ 4040\ (\ 8\ 4\ (i)\ \therefore i = \bar{8}\ \bar{4}$

4032

Step 10: $12\,(2bi + 2ch + 2dg + 2ef)$

(j) $= 12[0 + (2 \times \bar{4} \times \bar{4}\,\bar{4})$

$+ (2 \times \bar{3} \times \overline{22}) + (2 \times \overline{11} \times \overline{11}\,]$

$= 12\,(352 + 132 + 242)$

$= 12 \times 726 = 8712$

$\therefore$ part 1 $= 8712$

$1(3b^2h + 6bcg + 6bdf + 3be^2 + 3c^2$

$+ 6cde + d^3)$

$= [0 + 0 + 0 + 0 + (3 \times 16 \times \bar{1}\,\bar{1})$

$+ (6 \times \bar{4} \times \bar{3} \times \overline{11}) + \overline{27}\,]$

$= \overline{528} + \overline{792} + \overline{27} = \overline{1347}$

$\therefore$ part 2 $= 1347$

$\therefore\ 80 + \overline{8712} + 1347 = \overline{7445}$

$48\ )\ \overline{7445}\ (\ \overline{155}\,(i)\ \therefore j = \overline{155}$

$\overline{7440}$

Step 11: $12\,(2bj + 2ci + 2dh + 2eg + f^2)$

(k) $= 12[0 + (2 \times \bar{4} \times \bar{8}\,4)$

c i

$+ (2 \times \bar{3} \times \bar{4}\,\bar{4}) + (2 \times \overline{11} \times \overline{22} + 121]$

e g f^2

$= 12\,(672 + 264 + 484 + 121) = 12 \times 1541 = 18492$

$\therefore$ part 1 $= 18492$

and

$1(3b^2i + 6bch + 6bdg + 6bef + 3c^2g$

$+ 6cdf + 3ce^2 + 3d^2e)$

$= [0 + 0 + 0 + 0\,(3 \times 16 \times \overline{22})$

$+ (6 \times \bar{4} \times \bar{3} \times \overline{11}) + (3 \times \bar{4} \times 121)$

c d f c e^2

$+ (3 \times 9 \times \overline{11})]$

d^2 e

$= 1056 + 792 + 1452 + 297$ 3597

$\therefore$ part 2 $= 3597$

$\therefore\ 50 + 18492 + 3597 = 14945$

48) $\overline{14945}$ ($\overline{311}$(k) $\therefore$ k = $\overline{311}$

$\overline{14928}$

$\bar{1}\bar{7}$

Step 12 :

(l)

$f''(a) = 2cj + 2di + 2eh + 2fg$

$f'''(a) = 3c^2h + 6cdg\ 3d^2f + 3de^2 + 6cef$

$12[(2 \times \bar{4} \times \overline{155}) + (2 \times \bar{3} \times \overline{84}) + (2 \times \overline{11} \times \overline{44}) + (2 \times \overline{11} \times \overline{22})]$

c j d i e h f g

$12(1240 + 504 + 968 + 48) = 12 \times 3196 = 38352$

$\therefore$ 1st part = $\overline{38352}$

$1[(3 \times \overline{16} \times \overline{44}) + (6 \times \bar{4} \times \bar{3} \times \overline{22}) + (3 \times 9 \times \overline{11}) + (3 \times \bar{3} \times 121) + (6 \times \bar{4} \times \overline{11} \times \overline{11}$

c^2 h c d g d^2 f d e^2 c e f

$= 2112 + 1584 + 297 + 1089 + 2904 = 7986$

$\therefore$ 2nd part = 7986

$170 + \overline{38352} + 7986 = \overline{30536}$

48) $\overline{30536}$ ($\overline{636}$ $\therefore$ l = $\overline{636}$

$\overline{30528}$

$\overline{8}$

Step 13 :

(m)

$12[(2bl + 2ck + 2dj + 2ei + 2fh + g^2)]$

$12[0 + (2 \times \bar{4} \times \overline{311}) + (2 \times \bar{3} \times \overline{155}) + (2 \times \overline{11} \times \overline{84}) + (2 \times \overline{11} \times \overline{44}) + (\overline{22} + \overline{22})]$

$12(2488 + 930 + 1848 + 968 + 484) = 12(6718) = 80616$

$\therefore$ 1st part = $\overline{80616}$ ⟶ (1)

$1(3b^2k + 6bcj + 6bdi + 6beh + 6bfg + 3c^2i + 6cdh + 6ceg + 3cf^2 + 3d^2g + 6def + e^3)$

$= 1[(0 + 0 + 0 + 0 + 0 + 3 \times \overline{16} \times \overline{84}) + (6 \times \bar{4} \times \bar{3} \times \overline{44}) + (6 \times \bar{4} \times$

$\bar{3} \times \overline{22}) + (3 \times \bar{4} \times 121) + (3 \times 9 \times \overline{22})(6 \times \bar{3} \times 121) + (121 \times \overline{11}]$

$= [(\overline{4032} + \overline{3168} + \overline{5808} + \overline{1452} + \overline{594} + \overline{2178} + \overline{1331})] = \overline{18563}$

$\therefore$ Part 2 = 18563 ⟶ (2)

R + (1) + (2) = $\overline{80} - 80616 + 18563 = \overline{62133}$

48) $\overline{62133}$ ($\overline{1294}$ $\therefore$ m = $\overline{1294}$

62112

$\overline{21}$ Remainder = 21

a	b	c	d	e	f	g	h	i	j	k	l	m
4.	0	$\bar{4}$	$\bar{3}$	$\overline{11}$	$\overline{11}$	$\overline{22}$	$\overline{44}$	$\overline{84}$	$\overline{155}$	$\overline{311}$	$\overline{636}$	$\overline{1294}$

$\Rightarrow 4.0\overline{44237433754} \Rightarrow 3.955762566246$

6. Cube Root of $\overline{61.9}$ Let $x^3 = \overline{61.9}$

Swamiji's Method

$a = \bar{4}$ be the nearest CR of $\overline{61.9}$

$3a^2 = 48$, $3a = -12$

		1	2	3	4	5	6	7	8	9	10	11	12
			0	0	0	0	0	0	0	0	0	0	0
CD=48	$\overline{61}$	$\bar{9}$=21	21	18	36	24	0	44	32	8	5	17	8
	$\overline{64}$	3	0	0	192	0+0 +288= 288	$\overline{64}$+0 +1056 +108 =1100	0+0+$\overline{1144}$ +0+0+ 1056+792 =1704	0+$\overline{528}$+ $\overline{108}$+0+ 0+2112 +792 +1452+ =3720	$\overline{27}$+0+0+ $\overline{528}$+$\overline{792}$ +4224+ 1584+ 2904 =7365	0+$\overline{1056}$+ $\overline{1452}$+ $\overline{297}$ +$\overline{792}$ +8064+ 3168+ 5808+ 1452+ =14895	0+0+ $\overline{2112}$+ $\overline{297}$+ $\overline{1089}$+ $\overline{2904}$+ $\overline{1584}$ +14880 +6048 +11616 +5808 =30366	$\overline{1331}$+0+ $\overline{4032}$+ $\overline{1452}$+$\overline{594}$ +$\overline{3168}$+ $\overline{5808}$+ $\overline{2178}$+ 29856 +11160 +22176+ 11616+ 5808= 62053
$\bar{4}$		0	4	3	11	11	22	44	Q	155	311	636	1294
a		b	c	d	e	f	g	h	i	j	k	l	m

Note: The subtraction terms are given in the next page under the corresponding columns

1	2	3	4	5	6	7	8	9	10	11	12
	$3ab^2$	b^3+ $3a.2bc$	$3b^2c$ $+3a.2bd$ $+3a.c^2$	$3b^2d$ $+3bc^2$ $+3a.2be$ $+3a.2cd$	c^3+3b^2e $+6bcd+$ $+3a.2bf$ $+3a.2ce$ $+3a.d^2$	$3b^2f+3bd^2$ $+3c^2d+$ $+6bce+$ $+3a.2bg$ $+3a.2cf$ $+3a.2de$	$3b^2g+3cd^2$ $+3c^2e$ $+6bcf$ $+6bde+$ $+3a.2bh$ $+3a.2cg$ $+3a.2df+$ $3a.e^2$	d^3+3b^2h $+3be^2$ $+3c^2f$ $+6bcg$ $+6bdf$ $+6cde$ $+3a.2bi$ $+3a.2ch$ $+3a.2dg$ $+3a.2ef$	$3b^2i+3c^2g$ $+3ce^2+3d^2e$ $+ +6bch$ $+6bdg$ $+6cdf+6bef$ $+3a.2bj$ $+3a.2ci$ $+3a.2dh$ $+3a.2cg$ $+3a.f^2$	$3b^2j+3bf^2$ $+3c^2h+3d^2f$ $+3de^2+6bci$ $+6bdh+6cdg$ $+6beg+$ $+6cef+$ $3a.2bk+$ $+3a.2cj$ $+3a.2di$ $+3a.2eh$ $+3a.2fg$	E^3+3b^2h $+3c^2i+3cf^2$ $+3d^2g+6bcj$ $+6bdi+6cdh$ $+6beh$ $+6ceg+6def$ $+6bfg$ $+3a.2bl$ $+3a.2ck$ $+3a.2dj$ $+3a.2ei$ $+3a.2fh$ $+3a.g^2$

Upto m (12 decimals) $x = \bar{4}.0\,4\,\dot{3}\,11\,11\,22\,44\,84\,155\,311\,636\,1294 = \bar{4}.044237433754 = \bar{3}.\overline{955762566246}$

Taylor's Method

$x^3 = \overline{61.9}$ $\quad a = \bar{4}$ $\quad x = -(a.\ bcdefghijklm)$

		b	c	d	e	f	g	h	i	j	k	l	m	
$f(4)=48$	$\overline{61}$	$_3\bar{9}$	21^0	18^0	36^0	$24.^0$	0^0	44^0	32^0	8^0	5^0	17^0	8^0	21^0
$\frac{f''}{2}(4)=\overline{12}$			0	0	192	288	1164	1848	4356	8712	18492	38352	80616	
$\frac{f'''}{6}(4)=1$				0	0	0	$\overline{64}$	$\overline{144}$	$\overline{636}$	$\overline{1347}$	$\overline{3597}$	$\overline{7986}$	$\overline{18563}$	
	$\bar{4}$	0	4	3	11	11	22	44	84	155	311	636	1294	
	a	b	c	d	e	f	g	h	i	j	k			

Step 1 f(x) – f(a) = 61+64 = 3

Step 2 39 = 21

48) 21 (0 ∴ b = 0
0
21

Step 3 : (c) $\overline{12} \times b^2 = 0$

48) 210 (4 ∴ c = 4
192
18

Step 4 (d) 12 (2bc) = 0
∴ Part 1 = 0 → (1)
$1 \times b^3 = 0$ → (2)

48) 180 (3 d = 3
144
36

Step 5 : (e) $\overline{12}(2bd + c^2) = \overline{12} \times 16 = \overline{192}$
∴ Part 1 = 192 → (1)
$1(3b^2c) = 0$ → (2)
360 + 192 = 552
b = 0, c = 4, d = 3,
e = 11, f = 11, g = 22

48) 552 (11 e = 11
528
24

Step 6 : (f) $\overline{12}(2be + 2cd) = \overline{12}(2 \times 4 \times 3) = \overline{288}$
∴ Part 1 = 288 → (1)
$1(3b^2d + 3bc^2) = 0$ → (2)
288 + 240 = 528

48) 528 (11 ∴ f = 11
528

Step 7 : (g) $\overline{12}(2bf + 2ce + d^2) =$
$\overline{12}(0 + (2 \times 4 \times 11) + 9)$
c e d^2
1164 ∴ Part 1 = 1164
$1(3b^2e + 6bcd + c^3)$
$= (0 + 0 + \overline{64})$
∴ Part 2 = $\overline{64}$ → (2)
1164
$\overline{64}$
1100

48) 1100 (22 ∴ g = 22
1056
44

b = 4, c = 4, d = 3, e = 11, f = 11,
g = 22, h = 44, i = 84

Step 8 (h) $\overline{12}(2bg + 2cf + 2de)$
$= \overline{12}[0(2 \times 4 \times 11) + (2 \times 3 \times 11)] = \overline{1848}$
∴ Part 1 = 1848 → (1)
$1(3b^2f + 6bce + 3bcd^2 + 3c^2d)$
$= [0+0+0(3 \times 16 \times 3)] = 144$
∴ Part 2 = $\overline{144}$ → (2)
R + 1 + 2 = 440 + 1848 + $\overline{144}$
= 2288 + $\overline{144}$ = 2144

48) 2144 (44 ∴ h = 44
2112
32

Step 9 (i) $\overline{12}(2bh + 2cg + 2df + e^2)$
$= \overline{12}[0 + (2 \times 4 \times 22) + (2 \times 3 \times 11) + 121]$
$= \overline{12}(176 + 66 + 121)$
$= \overline{12}(363) = \overline{4356}$
∴ Part 1 = 4356 → (1)
$1(3b^2g + 6bcf + 6bde + 3c^2e + 3cd^2)$
$= [0+0+0(3 \times 16 \times 11) + (3 \times 4 \times 9)]$
c^2 e c d^2
= 528 + 108 = 636
∴ Part 2 = $\overline{636}$ → (2)
∴ R + (1) + (2) = 320 + 4356 + $\overline{636}$ = 4040

48) 4040 (84 ∴ i = 84
4032
8

Step 10 : $\overline{12}$ (2bi + 2ch + 2dg +2ef)

(j) $= \overline{12}$ [0 + (2 x 4 x 44) +

c h

(2 x 3 x 22) +(2 x 11x 11)]

d e

$= \overline{12}$ (352 + 132 + 242)

$= \overline{12} \times 726 = \overline{8712}$

$\therefore$ Part 1 = 8712 → ①

1 $(3b^2h + 6bcg + 6bdf + 3be^2 + 3c^2f + 6cde + d^3)$

= [0 + 0 + 0 + 0(3 x 16 x 11)

+ (6 x 4 x 3 x 11) + 27]

c

= 528 + 792 + 27 = 1347

$\therefore$ Part 2 = $\overline{1347}$ → ②

$\therefore$ R +① +② = 80 + 8712 +

1347 = 7445

48) 7445(155 $\boxed{\therefore j = 155}$

7440

5

Step 11 $\overline{12}$ $(2bj + 2ci + 2dh + 2eg + f^2)$

(k) = 12[0 + (2 x 4 x 84) + (2 x 3 x 44) + (2 x 11x 22) + 121]

c i d h e g f^2

$= \overline{12}$ (672 + 264 + 484 + 121) = $\overline{12}$ x 1541

$= \overline{18492}$ $\therefore$ Part 1 = 18492 → ①

1 $(3b^2i + 6bch + 6bdg + 6bef + 3c^2g + 6cdf + 3ce^2 + 3d^2e)$

= [0 + 0 + 0 + 0 + (3 x 16 x 22) + (6 x 4 x 3 x 11) + (3 x 4x 121) + (3 x 9 x 11)]

c^2 g c d f c e^2 d^2 e

= 1056 + 792 + 1452 + 297 = 3597

$\therefore$ Part 2 = $\overline{3597}$ → ②

$\therefore$ R +① +② = 50 + 18492 – 3597 = 14945

48) 14945 (311

144

54

48

65

48

17

Step 12 $\overline{12}$ (2bk + 2cj + 2di + 2eh + 2fg)

(l) = $\overline{12}$[(2 x 4 x 155) + (2 x 3 x 84) + (2 x 11x 44) + (2 x 11 x 22)]

= 12 (1240 + 504 + 968 +484) = 12(3196) = 38352

∴ Part 1 = 38352 ⟶ ①

1 ($3b^2j$ + 6bci + 6bdh + 6beg + $3bf^2$ + $3c^2h$ + 6cdg + 6cef + $3d^2f$ + $3de^2$)

= [0 + 0 + 0 + 0 + 0 + (3 x 16 x 44) + (6 x 4 x 3 x 22) + (6 x 4x 11x 11) + (3 x 9 x 11) + (3 x 3 x 121)]

= 1(2112 + 1584 + 2904 + 297 + 1089) = 7986

∴ Part 2 = $\overline{7986}$ ⟶ ②

∴ R + ① + ② = 170 + 38352 – 7986

= 30536

48) 30536 (636
30528
8

∴ l = 636

Remainder = 8

Step 13 : $\overline{12}$ (2bl + 2ck + 2dj + 2ei + 2fh + g^2)

(m) = $\overline{12}$[0 + (2 x 4 x 311) + (2 x 3 x 155) + (2 x 11x 84) + (2 x 11 x 44) + 484]

= $\overline{12}$ (2488 + 930 + 1848 + 484)

= $\overline{12}$(6718)

= $\overline{80616}$

∴ Part 1 = 80616 ⟶ ①

1 ($3b^2k$ + 6bcj + 6bdi + 6beh + 6bfg + $3c^2i$ + 6cdh + 6ceg + $3cf^2$ + $3d^2g$ + 6def + e^3)

= 1[0 + 0 + 0 + 0 + 0 + (3 x 16 x 84) + (6 x 4 x 3 x 44) + (6 x 4x 11x 22) + (3 x 4 x 121) + (3 x 9 x 22) + (6 x 3 x 121) + (121 x 11)]

= 1(4032 + 3168 + 5808 + 1452 +594 + 2178 + 1331)

= 18563

∴ Part 2 = 18563 ⟶ ②

∴ R + ① + ② = 80 + 80616 – 18563

= 62133

48) 62133 (1294
62112
21

∴ m = 1294

Remainder = 21

a	b	c	d	e	f	g	h	i	j	k	l	m
$\bar{4}$.	0	4	3	11	11	22	44	84	155	311	636	1294

⇒ $\bar{4}$. 044237433754

⇒ $\bar{3}$. $\overline{955762566246}$

Cube Roots of Polynomials

The method of solving the roots can be extended to Polynomials as well. Here interestingly the cube root of the term can be written in a form without remainder and hence need not be carried to the next term. Thus the working of the problem is simpler.

Swamiji's Straight Division Method

(1) Find the Cube root of $8x^6 + 12x^5 + 42x^4 + 37x^3 + 63x^2 + 27x + 27$

$$12x^4 \;\Big|\; 8x^6 + 12x^5 + 42x^4 + 37x^3 + 63x^2 + 27x + 27$$

$$\underset{Q_1}{2x^2} + \underset{Q_2}{x} + \underset{Q_3}{3} + \frac{0}{x} + \frac{0}{x^2} + \frac{0}{x^3} + \frac{0}{x^4}$$

Step1: Cube root of $8x^6 = 2x^2$ (Q_1)

Step2: Common Divisor $= 3 \times Q_1^{\,2} = 3 \times 4x^4 = 12x^4$

Step3: $\dfrac{12x^5}{12x^4} = x$ (Q_2)

Step4: $42x^4 - 3Q_1Q_2$

$= 42x^4 - 3(2x^2)(x^2) = 36x^4$ $\dfrac{36x^4}{12x^4} = 3$ (Q_3)

Step5: $37x^3 - (6Q_1Q_2Q_3 + Q_2^3)$

$= 37x^3 - [(6)(2x^2)(x)(3) + x^3] = \dfrac{0}{12x^4} = 0\ (Q_4)$

Step6: $63x^2 - (6Q_1Q_2Q_4 + 3Q_1Q_3^2 + 3Q_2^2Q_3)$

$= 63x^2 - [6(2x^2)(x)(0) + 3(2x^2)(9) + 3(x^2)(3)] = \dfrac{0}{12x^4} = 0\ (Q_5)$

Step7: $27x - (6Q_1Q_2Q_5 + 6Q_1Q_3Q_4 + 3Q_2Q_3^2)$

$= 27x - [0 + 0 + (3)(x)(9)] = \dfrac{0}{12x^4} = 0\ (Q_6)$

$27 - (Q_3^3) = \dfrac{0}{12x^4} = 0\ (Q_7)$

$\therefore$ $2x^2 + x + 3$ is one cube root and the other two roots are obtained by multiplying with cube roots of unity, $\dfrac{-1 \pm \sqrt{3}i}{2}$

(2) Find the Cube root of $x^6 + 3x^5 + 9x^4 + 13x^3 + 18x^2 + 12x + 8$

$$3x^4 \mid x^6 + 3x^5 + 9x^4 + 13x^3 + 18x^2 + 12x + 8$$

$$x^2 + x + 2 + \frac{0}{x} + \frac{0}{x^2} + \frac{0}{x^3} + \frac{0}{x^4}$$

Step1: Cube root of $x^6 = x^2$ (Q_1)

Step2: Common Divisor $= 3Q_1^2 = 3x^4$

Step3: $\frac{3x^5}{3x^4} = x$ (Q_2)

Step4: $9x^4 - 3Q_1Q_2^2$

$= 9x^4 - 3(x^2)(x^2) = 6x^4 = \frac{6x^4}{3x^4} = 2$ (Q_3)

Step5: $13x^3 - (6Q_1Q_2Q_3 + Q_2^3)$

$= 13x^3 - [6(x^2)(x)2 + x^3]$

$= 13x^3 - (12x^3 + x^3) = \frac{0}{3x^4} = 0$ (Q_4)

Step6: $18x^2 - (6Q_1Q_2Q_4 + 3Q_1Q_3^2 + 3Q_2^2Q_3)$

$= 18x^2 - [0 + 3(x^2)(4) + 3(x^2)(2)]$

$= 18x^2 - (12x^2 + 6x^2) = \frac{0}{3x^4} = (Q_5)$

Step7: $12x - (6Q_1Q_2Q_5 + 6Q_1Q_3Q_4 + 3Q_2Q_3^2)$

$= 12x - (0 + 0 + 3(x)4)$

$= 12x - 12x = \frac{0}{3x^4} = 0$ (Q_6)

Step8: $8 - (Q_3^3) = \frac{0}{3x^4}$ 0 (Q_7)

$\therefore x^2 + x + 2$ is the Cube root and the other roots are obtained by multiplying cube roots of unity i.e., $\frac{-1 \pm \sqrt{3}i}{2}$

(3) Obtain Cube root of $x^3 + 6x^2 + 15x + 27$

In decreasing power series of x.

$$3x^2 \quad x^3 + 6x^2 + 15x + 27 + 0$$

$$x + 2 + \frac{-}{} + \frac{7}{3x^2} \quad \frac{43}{3x^3}$$

Step1: Cube root of $x^3 = x$ (Q_1)

Step2: Common Divisor $= 3Q_1^2 = 3x^2$

Step3: $\frac{6x^2}{3x^2} = 2 \ (Q_2)$

Step4: $15x - 3Q_1Q_2^2$

$= 15x - 3(x)4 = 15x - 12x = 3x = \frac{3x}{3x^2} = \frac{1}{x} \ (Q_3)$

Step5: $27 - (6Q_1 Q_2 Q_3 + Q_2^3)$

$= 27 - (6(x)(2)\left(\frac{1}{x}\right) + 8) = 27 - (12 + 8)$

$= 27 - 20 = 7$

$\frac{7}{3x^2} \ (Q_4)$

Step6: $0 - (6Q_1Q_2Q_4 + 3Q_1Q_3^2 + 3Q_2^2Q_3)$

$= 0 - \left(6(x)2\left(\frac{7}{3x^2}\right) + 3(x)\frac{1}{x^2} + 3(4)\times\frac{1}{x}\right) = \frac{-43}{x}$

$\frac{-43}{x} \times \frac{1}{3x^2} = \frac{-43}{3x^3} \ (Q_5)$

Increasing power series of x

$$27 \,\lfloor\, 27 + 15x + 6x^2 + x^3$$

$$3 + \frac{5x}{9} + \frac{29x^2}{243} - \frac{752x^3}{6561} +$$

Step1: Cube root of $27 = 3 \ (Q_1)$

Step2: Common Divisor $= 3 \times 3^2 = 27$

Step3: $\frac{15x}{27} = \frac{5x}{9} \ (Q_2)$

Step4: $6x^2 - 3Q_1Q_2^2$

$= 6x^2 - 3(3)\left(\frac{25x^2}{81}\right) = 6x^2 - \frac{25x^2}{9} = \frac{29x^2}{9}$

$\frac{29x^2}{9 \times 27} = \frac{29x^2}{243} \ (Q_3)$

Step5: $x^3 - (6Q_1 Q_2 Q_3 + Q_2^3)$

$$= x^3 - \left[6(3)\left(\frac{5}{9}\right)\left(\frac{29}{243}\right) + \frac{125}{729}\right] = x^3 - \left(\frac{995x^3}{729}\right) = \frac{-266x^3}{729}$$

$$\frac{-266x^3}{729 \times 27} = \frac{-266x^5}{19683} = (Q_4)$$

4) Cube Root of $8 + 36x + 66x^2 + 87x^3 + 93x^4 - 21x^5$ (ascending order)

a) Swamijis Method

CD=$3a^2$=12	8	+36x	$+66x^2$	$+87x^3$	$+93x^4$	$-21x^5$	
			$-54x^2$	$-(27x^3+36x^3)$ $=-63x^3$	$-(6x+27x^4+72x^4)$ $=-105x^4$	$-(54x^5+9x^5$ $-36x^5+24x^5)$ $=-51x^5$	
			$3ab^2$	b^3+6abc	$3ac^2+3b^2c+6abd$	$3b^2d+3bc^2+$ $6abe+6acd$	
	2	+3x	$+x^2$	$+2x^3$	$-x^4$	$-6x^5$	+....
	a	b	c	d	e	f	

Cube Root of $(8 + 36x + 66x^2 + 87x^3 + 93x^4 - 21x^5)$ is $(2 + 3x + x^2 + 2x^3 - x^4 - 6x^5)$

Taylor's series with duplex and Triplex terms

Ascending Power Series of x

$3a^2 = 12$	$8 + 36x + 66x^2 + 87x^3 + 93x^4 - 21x^5$					
answer:	$2 + 3x + x^2 + 2x^3 - x^4 - 6x^5 + \ldots$					
3a = 6		9	6	13	−2	(Duplex terms)
1			27	27	63	(Triplex)

Step1: Cube root of 8 = 2 (a)

Step2: Common Divisor = $3a^2 = 3 \times 2^2 = 12$

Step3: $\frac{36x}{12} = 3x$ (b)

Step4: Duplex term $= b^2 = (3x)^2 = 9x^2$

$3ab^2 = 54x^2 = 12x^2$

ND $= 66x^2 - 54x^2$

$\frac{12x^2}{12} \cdot x^2$ (c)

Step5: Duplex term = 2bc

$= 2(3x)(x^2) = 6x^3$

$3a \times 2bc = 36x^3$

Triplex term $= b^3 = (3x)^3 = 27x^3$

$3a(2bc) + b^3 = 36x^3 + 27x^3 = 63x^3$

$$ND = 87x^3 - 63x^3 = 24x^3$$

$$\frac{24x^3}{12} = 2x^3 \text{ (d)}$$

Step6: Duplex term = $2bd + c^2$

$= 2(3x)(2x^2) + (x^2)^2$

$= 12x^4 + x^4 = 13x^4$

Triplex term = $3b^2c = 3(3x)^2 x^2 = 27x^4$

$3a(2bd + c^2) = 78x^4$

$3a(2bd + c^2) + 3b^2c = 105x^4$

$ND = 93x^4 - 105x^4 = -12x^4$

$$\frac{-12x^4}{12} = -x^4 \text{ (e)}$$

Step7: Duplex term = $2be + 2cd$

$= 2(3x)(-x^4) + 2(x^2)(2x^3)$

$= -6x^5 + 4x^5 = -2x^5$

$3a(2be + 2cd) = -12x^5$

Triplex term = $3b^2d + 3bc^2$

$= 3(3x)^2(2x^3) + 3(3x)(x^2)^2$

$= 54x^5 + 9x^5 = 63x^5$

$3a(2be + 2cd) + 3b^2d + 3bc^2$

$= -12x^5 + 63x^5 = 51x^5$

$ND = -21x^5 - 51x^5 = -72x^5$

$$\frac{-72x^5}{12} = 6x^5 \text{ (f)}$$

3rd root is $2 + 3x + x^2 + 2x^3 - x^4 - 6x^5 +$

5) $x^3 + 12x^2 + 48x + 64$

Swamiji's

$3x^2$	x^3	+	$12x^2$	+	$48x$	+	64
				$-$	$48x$	$-$	64
	x	+	4		$\frac{0}{3x}$		$\frac{0}{3x^2}$

$(x + 4)$ is the CR

Taylor's

$3x^2$	x^3	+	$12x^2$	+	$48x$	+	64
$3x$					$\overline{48x}$		0
1							$\overline{64}$
	x	+	4	+	$\frac{0}{3x}$	+	$\frac{0}{3x^2}$
	a		b		c		

$(x + 4)$ is the CR

6) $x^6 + 6x^5 + 3x^4 - 28x^3 - 9x^2 + 54x - 27$

Swamiji's

$3x^4$	x^6	$+\ 6x^5$	$+\ 3x^4$	$-\ 28x^3$	$-\ 9x^2$	$+\ 54x$	$-\ 27$
			$-\ 12x^4$	$-\ 28$	$-\ 9x^2$	$+\ 54x$	$-\ 27$
	x^2	$+\ 2x$	$-\ 3$	$\frac{0}{x}$	$\frac{0}{3x^2}$	$\frac{0}{3x^3}$	$\frac{0}{3x^4}$
	a	b	c	d	e	f	g

$(x^2 + 2x - 3)$ is the CR

Taylors

$3x^4$	x^6	$+\ 6x^5$	$+\ 3x^4$	$-\ 28x^3$	$-\ 9x^2$	$+\ 54x$	$-\ 27$
$3x^2$			$-\ \overline{12}x^4$	$-\ 36x^3$	$-\ \overline{27}x^2$	$+\ 0$	$-\ 0$
1				$\bar{8}x^3$	$36x^2$	$\overline{54}x$	$+\ 27$
	x^2	$+\ 2x$	$-\ 3$	$\frac{0}{3x}$	$\frac{0}{3x^2}$	$\frac{0}{3x^3}$	$\frac{0}{3x^4}$
	a	b	c	d	e	f	g

$(x^2 + 2x - 3)$ is the CR

7) $\sqrt[3]{x^3 + 12x^2 + 48x + 64}$

Swamiji Method

$CD = 3a^2 = 3x^2$	X^3	$+\ 12x^2$	$+\ 48x$	$+\ 64$
			$\overline{48}\ x$	$\overline{64} + 0$
				$= \overline{64}$
			$-\ 3ab^2$	$b^3 + 6abc$
	x	$+\ 4$	$+\ \frac{0x}{3x^2} = 0$	$+\ \frac{0}{3x^2} = 0$
	Q_1	Q_2	Q_3	Q_4
	a	b	c	d

Taylors Series Method

	x^3	$+\ 12x^2$	$+\ 48x$	$+\ 64$	
$f'(a) = CD = 3a^2 = 3x^2$			$-\ 48x$	0	
$\frac{1}{2} f''(a) = 3a = 3x$			$-\ b^2$	$-\ 2bc$	Duplex Terms
$\frac{1}{6} f'''(a) = 1$				$\overline{64}$	Triplex Terms
				b^3	
	x	$+\ 4$	$+\ 0$	$+\ 0$	
	Q_1	Q_2			
	a	b	c	d	

Note: Duplex terms are multiplied by $\frac{1}{2} f''(x)$ at $x = a$ and shown directly in the table.

Triplex terms are multiplied by $\frac{1}{6} f'''(x)$ at $x = a$ and shown directly in the table.

8) $\sqrt[3]{x^9+6x^8+3x^7-25x^6+3x^5+50x^4-66x^3+33x^2-9x+1}$

Swamiji's Method

$CD=3a^2$ $=3x^6$	x^9 +	$6x^8$ +	$3x^7$ -	$25x^6$ +	$3x^5$ +	$48x^4$ -	$60x^3$ +	$33x^2$ -	$9x$ +	1
			$-12x^7$	$-(8x^6-36)=28x^6$	$-(-36x^5+27x^5+12x^5)=-3x^5$	+ $-(12x^4+54x^4+0-18x^4)=-48x^4$	$-(-27x^3+0-36x^3+0+0+3x^3)=-60x^4$	$-(0+6x^2+27x^2)=-33x^2$	$-(-9x+0)=-9x$	$-(1+0)=-$
			$-(3ab^2)$	$-(b^3+6abc)$	$-(3b^2c+3ac^2+3a.2bd)$	$(3b^2d+3bc^2+3a.2be+3a.2cd)$	$-(c^3+3b^2e+6bcd+3a.2bf+3a.2ce+3a.2d^2)$	$-(3b^2f+3bd^2+3c^2d+6bce+3a.2bg+3a.2cf+3a.2de)$	$-(3b^2g+3c^2e+3cd^2+6bcf+6bde+3a.2bh+3a.2cg+3a.2df+3a.e)$	$-(d^3+3b^2h+3be^2+3c^21+6bcg+6bdf+6cd+3a.2bi+3a.2ch+3a.2dg+3a.2ef)$
	x^3 +	$2x^2$ -	$3x$ +	1 +	$\frac{0}{x}$ +	$\frac{0}{x^2}$	$\frac{0}{x^3}$	$\frac{0}{x^4}$	$\frac{0}{x^5}$	$\frac{0}{x^6}$
	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}
	a	b	c	d	e	F	g	h	i	j

Taylors Series Method

$(x)=3a^2=CD=3x^6$	x^9 +	$6x^8$ +	$3x^7$ −	$25x^6$ +	$3x^5$ +	$48x^4$ −	$60x^3$ +	$33x^2$ −	$9x$ +	1
$\frac{1}{2}f''(x)=3a=3x^3$			$-12x^7$	$36x^6$	$\overline{39}x^5$	$18x^4$	$-3x^3$	0	0	0
			(b^2)	$(2bc)$	$(2bd+c^2)$	$(2be+2cd)$	$(2bf+2ce+d^2)$	$(2bg+2cf+2de)$	$(2bh+2cg+2df+e^2)$	$(2bi+2ch+2dg+2ef)$
$\frac{1}{6}f'''(x)=1$				$-8x^6$	$36x^5$	$-66x^4$	$63x^3$	$-33x^2$	$+9x$	-1
				(b^3)	$(3b^2c)$	$(3b^2d+3bc^2)$	$(c^3+3b^2ed+6bcd)$	$(3b^2f+3bd^2+3c^2d+6bce)$	$(3b^2g+3c^2e+3cd+6bcf+6bde)$	$(d^3+3b^2h+3be^2+3c^2f+6bcg+6bdf+6cde)$
	x^3 +	$2x^2$ −	$3x$ +	1 +	0 +	0 +	0 +	0 +	0 +	0
	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}
	a	b	c	d	e	f	g	h	i	J

Note: Duplex terms are multiplied by $\frac{1}{2}f''(x)$ at x = a and shown directly in the table

Triplex terms are multiplied by $\frac{1}{6}f'''(x)$ at x = a and shown directly in the table

Swamiji's Method

48	64	48x	$12x^2$	x^3
			$-12x^2$	$-x^3$
	4 +	x	$\frac{0x^2}{48}$	$\frac{0x^3}{48}$

(4 + x) is the CR

It the given polynomial has the perfect cube root, then it will be the same either we proceed in the ascending order or descending order on the other hand it is not a perfect cube the CR obtained from descending order is different from the CR obtained in ascending order.

For example:

(Ascending Order)

48	64	46x	$12x^2$	x^3
			$-\frac{529}{48}x^2$	$\frac{323495}{13824}x^3$
			$3ab^2$	$b^3 + 6abc$
	4	$\frac{23}{24}x$	$\frac{47}{48}x^2$	$-\frac{309671}{663552}x^3$

(Descending Order)

48	x^3	$12x^2$	46x	64
			48x	67
	x	4	$\frac{-2}{3x}$	$\frac{-16}{3x^2}$

Higher Roots of Numbers

All the methods used for finding out the Cube Roots can be extendable for higher roots as well

1) Find the 4^{th} root of N = 331776

(33) (1776)

The number is grouped into 2 $\therefore n = 2$

Therefore the 4^{th} root contains 2 digits as KL

From the first group F one has to determine the nearest 4^{th} root of the group value.

F = K = 2

As the given number ends in 6, L = 2, 4, 6, 8 (Refer Table U)

		33 1776
Let L = 2	$-L^4$	– 16
$4L^3K$ = 32K ends in 6		176

K = 3 or 8 $\therefore$ it is invalid

$\therefore$ The 4^{th} root is KL nearest 4^{th} root of the 1^{st} group 33 is K = 2

Let us try L = 4

		33 1776
	$-L^4$	– 256
$4L^3K$ = 256K ends in 2		33 152

K = 2 or 7

Since K = 2, is valid, 4^{th} root of N is 24

2) Find the fourth Root of N = 28 4739 6321

The number is grouped into 3 $\therefore$ n = 3

Therefore the 4^{th} root contains 3 digits as JKL

From the first group F one has to determine the nearest 4^{th} root of the group value.

F = J = 2

As the given number ends in 1

L = 1, 3, 7, 9 (Refer Table U)

Let L = 3		28 4739 6321
	$-L^4$	– 81
$4L^3K$ = 108K ends in 4		39624
K = 3, 8 If K = 3		– 324
$4L^3J + 6L^2K^2$		3930
	$-6L^2K^2 = -486$	– 486
108J ends in 4		444

J = 3, 8 is invalid

Trial with K = 8		39624
$4L^3K = 108K$	$\therefore\ -4L^3K$	– 864
$4L^3J + 6L^2K^2$		3876
108J + 3456		– 3456
108J ends in zero		70420
∴ J = 0, 5 not valid		

Let L = 7		28 4739 6321
	$-L^4$	– 2401
$4L^3K$ ends in 2		39392
1372 K ends in 2 ∴ K = 1, 6		
Let K = 1, $4L^3K = 1372$		39392
	$-4L^3K$	– 1372
$4L^3J + 6L^2K^2$		3802
	1372J+ 294	– 294
1372J ends in 8		3508

J = 4, 9 is not valid		
Trial with K = 6		739392
	$4L^3K = 8232$	– 8232
$4L^3J + 6L^2K^2$		73116
	= 1372J + 10584	– 10584
1372J ends in 2		62532

J = 1, 6 (invalid)		
Let L = 9		28 4739 6321
	$-L^4$	– 6561
$4L^3K$ = 2916K ends in 6		738976
K = 1 or 6		
Trial with K=1	–2916K	– 2916
$4L^3J + 6L^2K^2$		73606
	= 2916J + 486	– 486
2916 J ends in 0		73120
J = 5 not valid		
K = 6		738976
	–2916K	– 17496
$4L^3J + 6L^2K^2$		72148
	= 2916J + 17496	– 17496
2916 J ends in 2		54702
⇒ J = 2 or 7		

J = 2 is valid

But $(219)^4 \neq N$

Hence one has to try with

		28 4739 6321
	$-L^4$	-1
$4L^3K = 4K$ ends in 2		9632
∴ K = 3 or 8 Let K = 3.	$-4L^3K$	-12
$4L^3J + 6L^2K^2$ ends in 2		962
	$-6L^2K^2$	-54
4J ends in 8		908
This results in J = 2 or 7		
J = 2 is valid	4^{th} root is 231	
∴ 4^{th} root of N = 231		

3) Find the fourth root of N = 2927 971750461996

Given number is grouped into 4 ∴ n = 4

(2927) (9717) (5046) (1696)

Therefore the fourth root contains 4 digits as HJKL

From the first group F one has to determine fourth root of the group value F = H = 7

As the number ends in 6. L = 2 or 4 or 6 or 8 (Refer Table U)

If L = 2. $L^4 = 16$		50461696
	$-L^4$	-16
$4L^3K = 32K$ ends in 8	K = 4 or 9	5046168
If K = 4 then $4L^3K = 128$	$-4L^3K$	-128
		504604
$4L^3J + 6L^2K^2 = 32J + 384$	$-6L^2K^2$	-384
32J ends in 0		29 504220
∴ J = 0 or 5		
If J = 0 32J = 0	$-32J$	-0
$4L^3H + 4LK^3 + 12L^2KJ = 32H + 512 + 0$		29 50422
$= 32H + 512$		-512
32H ends in '0'		29 49910

H = 0 or 5 Both are not valid as H is already shown as 7 L = 2, K = 4, J = 5

If J = 5		504220
32J = 160	$-32J$	-160
$4L^3H + 4LK^3 + 12L^2KJ = 32H + 512 + 960$		50406
$= 32H + 1472$		
		-1472
32H end in 4		48934

H = 2 or 7

∴ 4^{th} root of N is 7542

This is not giving the exact 4^{th} power of N.

H = 2 is invalid because the nearest fourth root of first group is 7

So let us again start with Stage II with K = 9, L = 2

		5046168
$4L^3K = 288$	$-4L^3K$	-288
$4L^3J + 6L^2K^2$		504588
$= 32J + 1944$	$6L^2K^2$	-1944
32J ends in 4		502644
J = 2 or 7	32J	-64
Let J = 2 then 32J = 64		
$4L^3H+4LK^3+12L^2KJ=32H+5832+864$		50258
	$= 32H + 6696$	-6696
32H ends in 2 ∴ H = 1 or 2		3562

But H = 7 by the first group. This is not valid

Let us again start with L = 4

L = 4		50461696
$L^4 = 256$	$-L^4$	-256
$4L^3K = 256K$ ends in 4		5046144
K = 4 or 9		
If K = 4	256K = 1024, – 256K	-1024
		504512
$4L^3J + 6L^2K^2 = 256J + 1536$		-1536
256J ends in 6 ∴ J = 1 or 6		502976
If J = 1		
	$-256J$	-256
$4L^3H+4LK^3+12L^2KJ=256H+1024+768$		50272
$= 256H + 1792$		-1792
256 H ends in 0		48480

∴ H = 0 or 5 is invalid

Now let us start with

J = 6		750 2976
	256J = 1536	-1536
$4L^3H+4LK^3+12L^2KJ=256H+1024+4608$		750144
	$= 256H + 5632$	-5632
256H ends in 2		744512
H = 2 or 7		
If H = 7 then 256 H = 1792	$-256H$	-1792
$4L^3I+6L^2J^2+12L^2KH+12K^2LJ+K^4$		74272
$=256I+3456+5376+4608+256$		
$256I + 13696$		-13696
256I ends in 6		60576

$\therefore$ I = 1 or 6 is invalid since I does not exist

Let us try with

L = 4, K = 9

		5046144
$4L^3K = 256K = 2304$	$-4L^3K$	-2304
$4L^3J + 6L^2K^2$		504384
$= 256J + 7776$	$-6L^2K^2$	-7776
256J ends in 8 $\Rightarrow$ J = 3 or 8		496608
If J = 3		
	$-256J$	-768
$4L^3H+4LK^3+12L^2KJ=256H+11664+5184$		49584
$= 256H + 16848$		-16848
256H ends in 6		32736
H = 1 or 6 is invalid		
Now let us try with L = 4, K = 9	and J = 8	
		496608
	$-256J$	-2048
$4L^3H+4LK^3+12L^2KJ=256H+11664+13824$		49456
$= 256h + 25488$		25488
256H ends in '8'		2368
H = 3 or 8 is invalid		

So one has to try with L = 6 or 8

Let us again start with L = 6

		50461696
	$-L^4$	-1296
$4L^3K$ = 864 K ends in 0		50460410
K = 0 or 5		
If K = 5	864K = 4320	-4320
		7504172
$4L^3J + 6L^2K^2 = 864J + 5400$		-5400
864 J ends in 2		7498772
J = 3 or 8		
If J = 3	864J = 2592	-2592
$4L^3H + 4LK^3 + 12L^2KJ$		749618
=864H + 3000 + 6480		-9480
864H ends in 8	$\therefore$ H = 2 or 7	740138
If H = 7	864H = 6048	
		-6048
$4L^3I + 6L^2J^2 + 12L^2KH + 12K^2LJ + K^4$		73409
= 864I + 1944 + 15120 + 5400 + 625		-23089
864I ends in 0 is absurd		50320

∴ 864 I ends ⇒ I = 0 or 5

I = 0 automatically satisfies condition that I = 5 cannot exist as it cannot have 5 digits as the fourth root

∴ 7356 is the 4^{th} root of N.

Note : The number contains 4 groups one should expect a 4 digited 4^{th} root. On this score existence of I is not valid. The value taking 4 digits B to be considered. When one expand the 4 digits it comes out to be exactly the given number

The table so prepared for expansion is also useful for the evaluation of the roots as well.

Working out the 4^{th} root using the corresponding table D. (Swamiji's method) is shown below

Swamiji's one digit method.

	2927	9	7	1	7	5	0
$CD=4a^3=1372$	2401	526	1153	659	1131	498	665
				2031	2503	1870+3242	
			$\overline{2646}$	$\overline{9576}$	$\overline{21795}$	$\overline{29016}$	
			$6a^2b^2$	$4ab^3+12abc$	$b^4+6a^2c^2$ $+12a^2bd+12b^2ac$	$4b^3c+12a^2be$ $+12a^2cd$ $+12b^2ad$ $+12c^2ab$	
	(7)	(3)	6	7	2	2	
	a	b	c	d	e	f	
			(5)	(6)	1 (0)		

On the basis that it is a 4^{th} root, the number is grouped such that each containing 4 digits, the 4^{th} root consists of only 4 digits it is a perfect one. One can still proceed to show that from 'e' onwards all the coefficients vanish, which is a bit laborious.

The four groups are

2927 9717 5046 1696

1^{st} Step: The nearest 4^{th} root of 2927 is 7 which is a value of 'a' 7^4 =2401 giving a difference of 526 from (2927) as such 526 with '9' forms the 1^{st} dividend as 5269 which is the first ID and is also ND

2^{nd} Step: Common Divisor $CD = 4a^3 = 1372$

3^{rd} Step: The 1^{st} ND when divided by 1372 gives the value 3 for 'b'. with he remainder 1153 forming the next ID as 11537

4^{th} Step: From 11537, $6a^2b^2$ is subtracted to get the ND this is divided by 1372 which gives the value 'c' as 6 with the remainder 659 and forming the next ID as 6591

5th Step: From 6591 $(4ab^3 + 12a^2bc)$ is subtracted which gives a negative ND (–4749) and hence c is reduced by 1 to give 5

6th Step: Consequent upon the remainder is (659 + 1372) = 2031 the modified ID is 20311 from this one has to subtract the value $(4ab^3 + 12a^2bc)$ to give the ND which when divided by CD gives the value 7 as 'd'. with the remainder 1131 which gives the ID as 11317 proceeding in the same manner one finds a number of reductions in each value. At this stage Swamiji has suggested that a better working details are possible when we considers the first two groups as a single unit to start with which is called as two digit method. From the above results it is easy to get the nearest 4th root of the two groups put together. This can be easily read as 73 which is designated as the starting point 'a' with this one can proceeds to work, out the complete details and it can be easily seen that 7356 are the four digits representing the 4th root. At this stage one can verify $(7356)^4$ is exactly as the given number in the same manner. If not one will have to proceed to get the decimals of our choice.

(29279717) (5046) (1696)

29279717 5 0 4 6 1 6

For the first group 73 is the nearest 4th root which can be taken as a. proceeding in the above manner the following is the table.

	29279717	5	0	4	6	1	6
$CD = 4a^3 = 1556068$	28398241	881476	1034425	208492	129984	16757	
			799350	1954940	1283089		
			$(6a^2b^2)$	$4b^3$	$b^4+6a^2c^2$		
				$+12a^2bc$	$+12b^2ac$		
	73	5	6	0	0	0	
	a	b	c	d	e	f	

This very clearly shows that two digit method gives the 4th root with less working details for a Still larger number one can work out with first three groups as one unit and get the value of 'a' and proceed to evaluate the required root. We can call it as three digit method and so on

British authors have solved the cube roots of numbers using the Taylors Series together with Swamijis concept of duplex and triplex terms and expressing the root in general as a. b c d e f ….. where b, c, d, e …. are the first, second, third… decimals. They have used the three differentials in case of cube. They have extended also the same method to fifth order and 7th order equations as well. In each case all the differentials up to the order of the equations are used coupling with the Taylors expansion.

A comparison between the method followed by Swamiji and the method that is given by British authors is given.

The present work on general expansion table and their use in evaluating the powers and also the roots is considered to be novel. It is further applicable to non perfect roots leading to any required decimal of ones choice.

Example Find the 4^{th} root of 21262591725153678928421696

1) Divide the entire result into sub groups consisting of 4 each from RHS. Here the first group has only two digits ie 21 the following steps are worked out
   Complete 21 262591725153678928421696
   Single Digit Method:
   The nearest 4^{th} root of first group is 2.
   Let this be a then subtract a^4 from the first group ie 21. We get remainder (27 – 16 = 5). Next dividend is 52.
2) Common is divisor is $4a^3 = 32$
3) Divide 52 by 32 we get quotient b as 1 leaving the second dividing as 206
4) Subtracts $6a^2b^2$ from 206 leaving the next dividend as (206 – 6(4) (1)) = 182. Dividing 182 by CD, 32. We get 5 as the value for C leaving the next dividend 222.
5) Subtracts $(4b^3a + 12a^2bc)$ from 222 gets a negative value not, desirable which complicates further calculation.
   Swamiji Suggested, that in all such cases when the CD is of a Small value then one can considere first two groups as a unit. With this one can proceed to evaluate the root. Hence our status point is 212625 which constitutes the first two groups. Starting from this one could upto 21as a
   We come upto 2147 and further working is complicated (reduction is found necessary).

CD = 37044	212625	9	1	7
	194481	18144	33273	31087
	a = 21	4	7	4
		b	c	d

While evaluating e, we come across a negative dividend and hence either to the d value is to be reduced (or) better extend the concept of Swamiji's two – digit method to triple digit method.

We have adopted starting of working with three groups as a unit and 214 is the value for 'a'

CD = 2126259172	2	5	1	5	3
2097273616	28985556	15445930	23391153	26070457	3409118
a = 214	7	3	5	6	0
	b	c	d	e	f

At this stage one can stop because it is satisfying Seven digits (seven groups) as the 4^{th} root one can try the verification with this seven digits by working out the expansion of its 4^{th} power.

If its 4^{th} power comes out to be exactly, the number with which we have started, then it is a perfect root other wise one has to work out in continuation to obtain the required decimal points.

4) Find Fifth Root of N = 6 9 3 4 3 9 5 7
(693) (43957)
The number is grouped into 2 $\therefore$ n = 2
Therefore the 5^{th} root contains 2 digits as KL
From the first group F one has to determine the nearest 5^{th} root of the group value.
F = K = 3
As the given number ends in 7 L = 7 (Refer Table U)

$$\begin{array}{rr} & 69343957 \\ -L^5 & -16807 \\ \hline & 6932715 \end{array}$$

$5L^4K = 12005K$ ends in 5
$\therefore$ K = 1,3,5,7 or 9. But K is already worked out as 3
$\therefore$ The 5^{th} root of N is (37)

5) Finding out the fifth root of 1036579476493 using Swamiji's both Methods

j k l
103 65794 76493
J = 2 ($2^5 = 32$)
Since 2 is the nearest 5^{th} root of 10^3
The last group is ending in 3 (Refer Table U)

$\therefore$ L = 3	9476493
$L^5 = 243$	– 243
$5L^4k = 405$ ends in 5	947625
$\therefore$ K = 1 or 3 or 5 or 7 or 9	
If K = 1 then $5L^4k = 405$	– 405
	947220
$10L^3K^2 + 5L^4J = 405 + 270$	– 270
405J ends in 2	94452

This is absurd
$\therefore$ K = 1 is not valid
Let us try with K = 3

	947625
$5L^4K = 1215$	– 1215
	94641
$10L^4K^2 + 5L^4J = 2430 + 405J$	– 2430
405J ends in 1	92211

This is absurd
$\therefore$ K = 3 is not valid

Let us try with $K = 5$

$5L^4K = 2025$

$10L^3K^2 + 5L^4J = 6750 + 405J$

405J ends in 0

947625
– 2025
94560
– 6750
87810

$\therefore J = 0$ (or) 2 (or) 4 (or) 6 (or) 8

J = 0 is not possible as I is concerned

With here digited number and J being in the Hundreds place I cannot be zero on the other hand J = 2 is more probable

To confirm J = 2 let us continue with the problem with H.

$10L^2K^3 + 20L^3KJ + 5L^4H = 11250 + 5400 + 405H$

87810
– 16650
71160

405H ends in 0

$\therefore H = 0$

$\therefore JKL = 253$

	103	6	5	7	9	4	7 6 4 9 3
80	32	71	76	35	57		
	71						
	2	8	54	603			
	a	b	c	d			

Two digit method

CD		1036794	7	6	4	9
1953125		9765625	600169	142322	16976	1014
$5a^4$	600169					
a	25	3	0	0		
	a	b	c	d		

As the single digit method is leading to a complicated working details, swamiji suggested in all such cases that one can attempt two digited (groups) method where in the first two groups are to be considered as one unit for starting the work

6) Find the sixth root of 148035889

(148) (035889)

The number is grouped into 2 $\therefore n = 2$

Therefore the 6^{th} root contains 2 digits KL from the first group F one has to determine the nearest 6^{th} root of the group value, F = K = 2

As the given number ends in 9 L = 3 or 7 (Refer Table U)

K	L
148	035889
	– 729
	516

$\Rightarrow$ L = 3 or 7.

Let L = 3 $\quad - L^6$

$6L^5K = 1458K$ ends in 6

K = 2 or 7

∴ Sixth root of 148035889 is 23

If L = 7

		148 035889
	$-L^6$	– 117649
$6L^5K$ = 100842K ends in 4		791824
K = 2 or 7		– 201684
If the 6^{th} root is 27		59014

$(27)^6 \neq N$ ∴ 23 is the 6^{th} root of N

7) Find the 6^{th} root of N = 3297303959104

(3) (297303) (959104)

The number is grouped into 3 ∴ n = 3

Therefore the 6^{th} root contains 3 digits as JKL.

From the first group F. one has to determine the nearest 6^{th} root of the group value.

F = J = 1

As the given number ends in 4

L = 2 or 8 (Refer Table U)

		3 297303 959104
Trial with L = 2	$-L^6$	– 64
$6L^5K$ = 192K ends in 4. K = 2, 7		5904
Trial with K = 2	$-6L^5K$	– 384
$6L^5J + 15L^4K^2$		9552
192J + 960 ends in 2		– 960
192J ends in 2		8592
J = 1, 6	– 192J =	– 192
Let J = 1, 192J		
		840

We can further extend to show that H = 0

$6L^5H + 30L^4KJ + 20L^3K^3 = 6L^5H + 960 + 1280 = 6L^5H + 2240$

192H ends in zero H = 0, 5 But there cannot be 4^{th} digit Hence H = 0

∴ 6^{th} root of given number is 122

8) Find the 7^{th} root of 410338673

(41) (033 8673)

The number is grouped into 2 ∴ n = 2

Therefore the 7^{th} root contains 2 digits as KL

From the first group F one has to determine the nearest 7^{th} root of the group value.

F = K = 1

As the given number ends in 3 L = 7 (Refer Table U)

		K L
		41 0338673
Number ends in 3, ∴ L = 7.	$-L^7$	– 823543
$7L^6K$ = 823543K ends in 3		40951513
⇒ K = 1		
	$-7L^6K$	– 823543
∴ Seventh root of 410338673 is 17		1012797
To show that J = 0	$7L^6J + 21L^5K^2$	– 352947
$7L^6J$ ends in 0 ∴ J = 0		50

9) Find the 7^{th} root of 2054210978157184

(20) (5421097) (8157184)

The number is grouped into 3 ∴ n = 3

Therefore the 7^{th} root contains 3 digits as JKL

From the first group F one has to determine the nearest 7^{th} root of the group value.

F = J = 1

As the given number ends in 4, L = 4 (Refer Table U)

		20 5421097 8157184
	$-L^7$	– 16384
$7L^6K$ = 28672K ends in 0		814080
⇒ K = 0 or 5	Trial with K = 5	
$7L^6J + 21L^5K^2$	$-7L^6K$	– 143360
		109767072
28672J + 537600 ends in 2		– 537600
28672J ends in 2 ∴ J = 1, 6		109229472
∴ 154 is the 7^{th} root of given number		
If K = 0		814080
	$7L^6K$	00
28762J ends in 8 ⇒ J = 4 (or) 9 is invalid		81408

∴ 154 is the 7^{th} root of the given number.

To show that H = 0

When L = 4, K = 5, J = 1

		109229472
		–28672
	$7L^6H + 42l^5K^5 + 35L^4K^3$	10920080
28672H+215040+1120000 ends in 0⇒H=0		1335040
28672H + 1335040		
28672H ends in 0		89585040
⇒ J = 0 or 5		

There is no 4^{th} digit in the seventh root

Higher Roots of Polynomials

1) Fifth root of $243 + 810x + 1080x^2 + 720x^3 + 240x^4 + 32x^5$ (ascending order)

a) Swamiji Method

$CD = 5a^4 = 405$	243	$+810x$	$+1080x^2$	$+720x^3$	$+240x^4$	$+32x^5$
			$-1080x^2$	$-(720 + 0) = -720x^3$	$-(240x^4 + 0) = -240x^4$	$-(32x^5 + 0) = -325x^5$
			$10a^3b^2$	$10a^2b^3 + 10a^3 2bc$	$5ab^4 + 10a^3c^2 + 10a^3.2bd + 10a^2.3b^2c$	$b^5 + 10a^3 2bc + 10a^3 2cd + 5a.4b^3c + 10a^2.3b^2d + 10a^2.3bc^2$
	3	$+2x$	$+0$	$+0$	$+0$	$+0$
	a	b	c	d	e	f

Fifth root of $(243 + 810x + 1080x^2 + 720x^3 + 240x^4 + 32x^5) = (3 + 2x)$

2) Find fifth root of $1 + 15x + 90x^2 + 295x^3 + 700x^4 + 1543x^5$ (ascending order)

a) Swamiji Method

$CD = 5a^4 = 5$		$15x$	$90x^2$	$295x^3$	$700x^4$	$1543x^5$
			$-90x^2$	$-(270x^3 + 0) = -270x^3$	$-(405x^4 + 0 + 300x^4 + 0) = -705x^4$	$-(243x^5 - 60x^5 + 0 + 1350x^5) = -1533x^5$
			$10a^3b^2$	$10a^2b^3 + 10a^3 2bc$	$5ab^4 + 10a^3c^2 + 10a^3.2bd + 10a^2.3b^2c$	$b^5 + 10a^3 2bc + 10a^3 2cd + 5a.4b^3c + 10a^2.3b^2d + 10a^2.3bc^2$
	1	$+3x$	$+0$	$+5x^3$	$-x^4$	$+2x^5 + \ldots$
	a	b	c	d	e	f

Fifth root of $(1 + 15x + 90x^2 + 1295x^3 + 700x^4 + 1543x^5)$ is $(1 + 3x + 0x^2 + 5x^3 - x^4 + 2x^5 + ..$

3) $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

Swamiji's

$5x^4$	x^5	$+ 5x^4$	$+ 10x^3$	$+ 10x^2$	$+ 5x$	$+ 1$
			$- 10x^3$	$- 10x^2$	$- 5x$	$- 1$
	x	$+ 1$	$+ \frac{0}{5x}$	$+ \frac{0}{5x^2}$	$+ \frac{0}{5x^3}$	$+ \frac{0}{5x^4}$
	a	b	c	d	e	f

$(x + 1)$ is the fifth root

Taylors

$5x^4$	x^5	$+ 5x^4$	$+ 10x^3$	$+ 10x^2$	$+ 5x$	$+ 1$
$10x^3$			$- 10x^3$	0	0	—
$10x^2$			—	$- 10x^2$	0	—
$5x$			—	—	$- 5x$	—
1			—	—	—	$- 1$
	x	$+ 1$	$+ \frac{0}{5x}$	$+ \frac{0}{5x^2}$	$+ \frac{0}{5x^3}$	$+ \frac{0}{5x^4}$
	a	b	c	d	e	f

$(x + 1)$ is the fifth root

4) $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

Swamiji's

$5x^4$	x^5	$-$	$10x^4$	$+$	$40x^3$	$-$	$80x^2$	$+$	$80x$	$-$	32		
				$+$	$40x^3$	$+$	$80x^2$	$-$	$80x$	$+$	32		
	x	$-$	2	$+$	$\frac{0}{5x}$	$+$	$\frac{0}{5x^2}$	$+$	$\frac{0}{5x^3}$	$+$	$\frac{0}{5x^4}$		
	a		b		c		d		e		f		

(x – 2) is the 5^{th} root

Taylors

$5x^4$	x^5	$-$	$10x^4$	$+$	$40x^3$	$-$	$80x^2$	$+$	$80x$	$-$	32
$10x^3$				$+$	$40x^3$		$-$		$-$		$-$
$10x^2$					$-$		$80x^2$		$-$		$-$
$5x$							...	$-$	$80x$		$-$
1					$-$		$-$		$-$		32
	x	$-$	2	$+$	$\frac{0}{5x}$	$+$	$\frac{0}{5x^2}$	$+$	$\frac{0}{5x^3}$	$+$	$\frac{0}{5x^4}$
	a		b		c		d		e		f

(x – 2) is the 5^{th} root

5) Find the fifth root of $x^{10} + 15x^9 + 80x^8 + 150x^7 - 95x^6 - 477x^5 - 190x^4 + 640x^3$ $600x^2 + 240x - 32$

Swamiji

^{10}	$+$	$15x^9$	$+$	$80x^8$	$+$	$150x^7$		$95x^6$		$477x^5$		$190x^4$	$+$	$640x^3$	$-$	$600x^2$	$+$	$240x$	32
			$-$	$90x^8$															
^{2}	$+$	$3x$	$-$	2	$+$	$\frac{0}{x}$	$+$	-----											
a		b		c		d		e		f		g		h		i		j	

Taylors

x^{10}	$+$	$15x^9$	$+$	$80x^8$	$+$	$150x^7$	$-$	$95x^6$	$-$	$477x^5$	$-$	$190x^4$	$+$	$640x^3$	$-$	$600x^2$	$+$	$240x$	$-$	32	
			$-$	$90x^8$	$+$	$120x^7$															
					$-$	$0x^7$															
x^2	$+$	$3x$	$-$	2	$+$	$\frac{0}{x}$	$+$	--													
a		b		c		d															

6 nd Seven oo 28 + 448x + 672x 560x^3 + 280x^4 84x 4x^6 + x

a Swami Method

CD=$7x^6$	$+128$	$+448x$	$+672x^2$	$+560x^3$	$+280x^4$	$+84x^5$	$-14x^6$	$+x^7$
=448								
			$-672x^2$	$-560x^3$	$-280x^4$	$-84x^5$	$-14x^6$	$-x^7$
			$21a^5b^2$	$35a^4b^3$	$21a^5c^2$	$21a^2b^5+42a^5be$	$21a^5d^2+35a^4c^3$	b^7+42a^5bg
				$+42a^5bc$	$+35a^3b^4$	$+42a^5cd$	$+42a^5bf+42a^5ce$	$+42a^5cf+42a^5de$
					$+42a^5bd$	$+105a^4b^2d$	$+105a^4b^2e$	$+42ab^5c+105a^4b^2f$
					$+105a^4b^2c$	$+105a^4c^2b$	$+105a^2b^4c$	$+105a^4c^2d+105a^4d^2b$
						$+140a^3b^3c$	$+140a^3b^3d$	$+105a^2b^4d$
							$+210a^3b^2c^2$	$+140a^3b^3e$
							$+210a^4bcd+7ab^6$	$+140a^3c^3b$
								$+210a^2b^3c^2$
								$+210a^2b^3c^2$
								$+210a^4bce$
								$+420a^3b^2cd$
	2	+x	+0	+0	+0	+0	+0	+0
	a	b	c	d	e	f	g	h

Seventh oot of $(128 + 448x + 672x^2 + 560x^3 + 280x^4 + 84x^5 + 14x^6 + x^7)$ is $(2 + x)$

7) Find the Seventh root of $2187 + 25515x + 132678x^2 + 4054x^3 + 818559x^4 + 1228122x^5 + 1713117x^6 + 2907322x^7$

a) **Swamiji Method**

$CD=7a^6=5103$	2187	25515x	$132678x^2$	$4054x^3$	$818559x^4$	$1228122x^5$	$1713117x^6$	$2907322x^7$
			$-27575x^2$	$-(354375x^3$ $+51030x^3)$ $= -405405x^3$	$-(5103x^4$ $+590625x^4$ $+0+212625)$ $= -808353x^4$	$(590625x^5+102060x^5$ $+42525x^5+472500x^5)$ $= -1207710$	$x^6(328125+0+2835$ $+204120+20412$ $+425250+590625$ $+0+141750+0)$ $= -1713117x^6$	$x^7(78125+0+40824$ $+0+393750+850500$ $+0+0++0+945000$ $+18900+236250$ $+170100)$ $= -2733449x^7$
			$21a^5b^2$	$35a^4b^3$ $+42a^5bc$	$21a^5c^2$ $+35a^3b^4$ $+42a^5bd$ $+105a^4b^2c$	$21a^2b^5+42a^5be$ $+42a^5cd+105a^4b^2d$ $+105a^4c^2b+140a^3b^3c$	$7ab^6+21a^5d^2+35a^4c^3$ $+42a^5bf+42a^5ce$ $+105a^4b^2e+105a^2b^4c$ $+140a^3b^3d+210a^3b^2c^2$ $+210a^4bcd$	b^7+42a^5bg $+42a^5cf+42a^5de$ $+42ab^5c+105a^4b^2f$ $+105a^4c^2d+105a^4d^2b$ $+105a^3b^4d+140a^3b^3e$ $+140a^3c^3b+210a^4bce$ $+210a^2b^3c^2+210a^2b^3c^2$ $+420a^3b^2cd$
	3	5x	x^2	$0.x^3$	$2x^4$	$4x^5$	$0x^6$	$173875x^7$
	a	b	c	d	e	f	g	h

6 b) Find the Seventh root of $128 + 448x + 672x^2 + 560x^3 + 280x^4 + 84x^5 + 14x^6 + x7$

Taylor's Method

Common Divisor = $7a^6$

448									
answer:	2 +	x +	$0.x^2$ +	$0.x^3$ +	0.x4 +	0.x5 +	0.x6 +	0.x7	
a	a	b	c	d	e	f	g	h	
$21a^5$ 672			$1x^2$	0	0	0	0	0	
			b^2	2bc	c^2+2bd	2be+2cd	$2bf+d^2$ +2ce	2bg+2de +2cf	Duplex
$35a^4$ = 560			$1x^3$	0	0	0	0		
				b^3	$3b^2c$	$3b^2d+$ $3bc^2$	$3b^2e+c^3$ +6bcd	$3b^2f+3c^2d$ $+3bd^2+6bce$	Triplex
$35a^3$ = 280				$1x^4$	0	0	0		
					b^4	$4b^3c$	$4b^3d+6b^2c^2$	$12b^2cd$ $+4bc^3+4b^3e$	Quadruplex
$21a^2$ = 84						$1x^5$	0	0	
						b^5	$5b^4c$	$10b^3c^2$ $+5b^4d$	Quintuplex
7a = 14							$1x^6$	0	
							b^6	$6b^5c$	Sextets
$11x^7$									
								b^7	Heptets

∴ Answer = $x + 2$

All Seven roots are given by multiplying by the roots of unity, i.e., 1, seventh.

<u>Step1:</u> Seventh root of 128 = 2 (a)

<u>Step2:</u> Common Divisor = $7a^6 = 7 \times 2^6 = 448$

<u>Step3:</u> $\frac{448x}{448} = x$ (b)

<u>Step4:</u> $21a^5b^2 = 672x^2$

$$ND = 672x^2 - 672x^2 = \frac{0}{448} = 0 \text{ (c)}$$

<u>Step5:</u> $21a^5(2bc) + 35a^4b^3 = 560x^3$

$$ND = 560x^3 - 560x^3 = \frac{0}{448} = 0 \text{ (d)}$$

<u>Step6:</u> $21a^5(c^2 + 2bd) + 35a^4(3b^2c) + 35a^3b^4 = 280x^4$

$$ND = 280x^4 - 280x^4 = \frac{0}{448} = 0\ (c)$$

<u>Step7:</u> $21a^5(2be + 2cd) + 35a^4(3b^2d + 3bc^2) + 35a^3(4b^3c) + 21a^2b^5 = 84x^5$

$$ND = 84x^5 - 84x^5 = \frac{0}{448} = 0\ (f)$$

<u>Step8:</u> $21a^5(2bf + d^2 + 2ce) + 35a^4(3b^2e + 6bcd + c^3) + 35a^3(4b^3d + 6b^2c^2) + 21a^2\,5b^4c + 7ab^6 = 14x^6$

$$ND = 14x^6 - 14x^6 = \frac{0}{448} = 0\ (g)$$

<u>Step9:</u> $21a^5(2bg + 2de + 2cf) + 35a^4(3b^2f + 3c^2d + 3bd^2 + 6bce) + 35a^3(12b^2cd + 4bc^3 + 4b^3e) + 21a^2(10b^3c^2 + 5b^4d) + 7a(6b^5c) + b^7 = x^7$

$\therefore ND = x^7 - x^7 = 0$

$$\frac{0}{448} = 0\ (h)$$

7^{th} root is $(2 + x)$

7 b) Obtain Seventh root of
$2187 + 25515x + 132678x^2 + 405405x^3 + 818559x^4 + 1228122x^5 + 1713117x^6 + 2907322x^7$

Taylor's Method

in increasing power series of x.

CD $7a^6 = 5103$	2187	$+ 25515x$	$+ 132678x^2$	$+ 405405x^3$	$+ 818559x^4$	$+ 1228122x^5$	$+ 1713117x^6$	$+2907322x^7$
answer:	3	$+ 5x$	$+ x^2$	$+ 0.x^3$	$+ 2.x^4$	$+ 4.x^5$	$+ 0.x^6$	$+ \frac{173873}{5103}x^7$
$21a^5$ 5103			$25x^2$	$10x^3$	$1x^4$	$20x^5$	$44x^6$	$8x^7$
$35a^4$ 2835				$125x^3$	$75x^4$	$15x^5$	$151x^6$	$360x^7$
$35a^3$ 945					$625x^4$	$500x^5$	$150x^6$	$1020x^7$
$21a^2$ 189						$3125x^5$	$3125x^6$	$1250x^7$
7a 21							$5625x^6$	$18750x^7$
1								$78125x^7$

$\therefore$ Answer is $3 + 5x + x^2 + 2x^4 + 4x^5 + x^7 +$

<u>Step1:</u> Seventh root of 2187 = 3 (a)

<u>Step2:</u> Common Divisor = $7a^6 = 7\ 3^6 = 5103$

<u>Step3:</u> $\frac{25515x}{5103} = 5x$ (b)

<u>Step4:</u> $21a^5b^2 = 127575x^2$

$$ND = 132678x^2 - 127575x^2 = 5103x^2 = \frac{5103x^2}{5103} = x^2\ (c) \quad c = 1$$

<u>Step5:</u> $2bc = 2(5x)(x^2) = 10x^3$

$21a^5\ (2bc) + 35a^4b^3 = 405405x^3$

$$ND = 405405x^3 - 405405x^3 = \frac{0}{5103} = 0\ (d)$$

<u>Step6:</u> $21a^5\ (c^2 + 2bd) + 35a^4\ (3b^2c) + 35a^3\ (b^4) = 808353x^4$

$$ND = 818559x^4 - 808353x^4 = 10206x^4\ \frac{10206x^4}{5103} = 2x^4\ (e)$$

<u>Step7:</u> $21a^5\ (2be + 2cd) + 35a^4\ (3b^2d + 3bc^2) + 35a^3\ (4b^3\ c) + 21a^2b^5 = 1207710x^5$

$$ND = 1228122x^5 - 1207710x^5 = 20412x^5\ \frac{20412x^5}{5103} = 4x^5\ (f)$$

<u>Step8:</u> $21a^5\ (2bf + d^2 + 2ce) + 35a^4\ (3b^2e + c^3 + 6bcd) + 35a^3\ (4b^3d + 6b^2c^2) + 21a^2\ (5b^4c) + 7a\ (b^6) = 1713117x^6$

$ND = 1713117x^6 - 1713117x^6$

$$\frac{0}{5103} = 0\ (g)$$

<u>Step9:</u> $21a^5\ (2bg + 2de + 2cf) + 35a^4\ (3b^2f + 3c^2d + 3bd^2 + 6bce) + 35a^3\ (12b^2cd + 4bc^3 + 4b^3e) + 21a^2\ (10b^3c^2 + 5b^4d) + 7a\ (6b^5c) + b^7 = 2733449x^7$

$ND = 2907322 - 2733449x^7 = 173873x^7$

$$\frac{173873}{5103}x^7 = (h)$$

Seventh root is $3 \cdot 5x + x^2 + 2x^4 + 4x^5 + \frac{173873}{5103}x^7 + \ldots..$

PART – II EQUATIONS (Contd.)

Section – C Swamiji's Method for Cubic and Higher Order Equations

Notes on a method adopted to solve the equations by using Swamiji's Method (Cubic).

1) The given equation in 'x' is written as f(x) is equated to the given number.

2) By using Vilokanam method, the solution of it as the nearest integer 'a' satisfying the equation, is to be obtained.

3) In order to get the nearest solution with the decimals of required extent, the following procedure is adopted.

4) For a cubic equation the terms of the expansion $(a + b + c)^3$ is made use of wherein the placements (Refer Table B) are shown. The Same table is applicable to the decimals as well, treating that b is in 10^{-1}, c is in 10^{-2} and so on in the solution, where 'a' is the integer obtained from the above Vilokanam method. As such the solution will be a. b c d......

5) A few evaluations have to be first carried out in order to proceed to get the solution

   The method is similar to Swamiji's Straight Division method (Lecture Notes – II)

   i. Evaluation of Common Divisor (CD): In case of a number in the form of x^3 =N the common divisor will be only $3x^2$ at x ='a' but in the Cubic Equation the $3a^2$ represents the first differential of the f(x) at 'x = a' as such it is called $3a^2$ representation.

   ii. In order to get the Intermediate Dividend ID; the difference between RHS and the LHS with (x = a) is to be multiplied by 10 and added to the existing Dividend. This is the first ND.

   iii. The first ND is divided by the common divisor which gives the quotient 'b' with the remainder 'R'. The Remainder is multiplied by 10 to get the next intermediate dividend. From this dividend one has to subtract '$3ab^2$' to get the ND.

   But in case of an Equation, '3a' is **represented by the Second** derivative. **In order to identity this expression with '3a', we have to** divide the second differential at **'x = a' by '2'. The value of this '3a'** is then multiplied **by b^2 to get the '$3ab^2$'.**

iv. (Intermediate dividend – $3ab^2$) is the new dividend and this is to be divided by the CD to get the value c, and the remainder R is to be multiplied by 10 and added to the next dividend to get the ID.

v. In order to get the value of 'd', the third decimal point, from the table it is clear that one has to subtract ($b^3 + 6abc$) which can be written as $b^3 + 3a(2bc)$. The value of 3a is already obtained (as representation) in the above step which is to be made use of, where as b^3 can be obtained from 'b' value. The ND thus obtained is divided by CD to get the value of d and corresponding R.

Care is to be taken in case of terms of $3a^2$ or 3a in a cubic equation which are to be read directly from corresponding differentials followed by the Substitution of x = a in the entire differential expressions

vi. In order to get the next ID which leads to the next decimal point, e, from the table, the following expression is the subtraction. $S = (3a)c^2 + 3a(2bd) + 3b^2c$.

Same principle is adopted for the substitution of 3a. The S value is subtracted from the ID and the result (ND) is divided by CD to get the value e and so on. This procedure is continued to get the corresponding decimals and remainders which leads to the ID and when the respective combinations from them are subtracted and the result ND's are divided by the CD to give the corresponding decimal points of one's choice.

Eg $x^3 + 29x - 97 = 0$

$f(x) = x^3 + 29x = 97$

Solution x = a.bcde....

From Vilokanam x = 3

if x = 3 → LHS = 114 RHS – LHS = 97 – 114 = – 17, x = 3 the nearest integer (a) for the solution.

The Common Divisor (CD) = $f'(x)_{x=a=3} = 3a^2 + 29 = 56$

	LHS	RHS	RHS – LHS
x = 1	30	97	67
x = 2	66		31
x = 3	114		$\overline{17}$

$f''(x) = 6x$ at x = 3 is 18 but 3a representation is needed

Hence 3a representation at x = 3 is $\frac{18}{2} = 9$

	0	0	0	0	0	
50	$\overline{17}$	$\bar{2}$	$\overline{45}$	$\overline{29}$	$\overline{32}$	Subtraction Terms
		$\overline{81}$	$27+\overline{54}$	$\bar{9}+\overline{432}+27$	$\overline{648}+\overline{144}+216+9$	
		$3a(b^2)$	$b^3+3a(2bc)$	$3a(c^2)+3a(2bd)+3b^2c$	$3a(2be)+3a(2cd)+3b^2d+3bc^2$	
3	$\bar{3}$	$\bar{1}$	$\bar{8}$	$\overline{12}$	$\overline{15}$	
a	b	c	d	e	f	

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND $\div$ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a. For example $3ab^2 = 9 \times 9 = 81$

Upto f (5 decimals) $x = 3.\ \bar{3}\ \bar{1}\ \bar{8}\ \overline{12}\ \overline{15} = 2.68065$

Eg : $x^3 - 3x^2 + 17x - 52 = 0$

$f(x) = x^3 - 3x^2 + 17x = 52$

Solution x = a.bcde ....

From Vilokanam a = 3

Let $x = 3 \Rightarrow$ RHS – LHS = 1

$3a^2$Representation = CD = $f'(x)$at $x = 3 \Rightarrow (3x^2 - 6x + 17)$ at $x = 3 \Rightarrow 26$

3a Representation at $x = 3 \Rightarrow \frac{1}{2} f''(x) = \left(\frac{1}{2}(6x-6)\right)$ at $x = 3 \Rightarrow 6$

CD=26	0	0	0	0	0	
	1	10	22	12	14	
		0	0	$\overline{54}$	$\overline{288}$	Subtraction Terms
		$3ab^2$	$b^3+3a.2bc$	$3b^2c+3a.2bd+3a.c^2$	$3b^2d+3bc^2+3a.2be+3a.2ce$	
3	0	3	8	2	$\bar{5}$	
a	b	c	d	e	f	

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND $\div$ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.. For example $3ab^2 = 6 \times 0 = 0$

Upto f (5 decimals) $x = 3.0382\bar{5} = 3.03815$

A few more examples

1) $E = 2.3x^3 + 8.7x^2 - 0.01x = 5.2$,

$f(x) = 230x^3 + 870x^2 - x = 520$ Solution x = a.bcde ....

	L.H.S.	R.H.S.	Diff: R.H.S. – L.H.S.
x = 0.3	84.21	520	+ 435.79
x = 0.5	245.75		+ 274.25
x = 1	1099		– 579
x = 2	5318		– 4798
x = 3	14037		– 13517
x = – 0.5	189.25		+ 330.75
x = – 0.8	439.84		+ 80.16
x = – 1	641		– 121
x = – 2	1642		– 1122
x = – 3	1623		– 1103
x = – 4	– 796		+ 1316
x = – 5	– 6995		+ 7515

Three regions of solution are indicated. Starting with one of them i.e. with x = – 1 all solutions are calculated. An attempt is also made to calculate independently starting with x = – 4. The results are compared with a further refinement of x to – 3.6.

From Vilokanam a = – 1

Let x = – 1. then RHS = (– 230 + 870 + 1 = 520) RHS – LHS = $\overline{121}$

CD = $3a^2$ Representation is $(230\,(3x^2) + 870\,(2x) - 1)$ At x = – 1 it is – 1051 $\therefore$ effective $3a^2 = \overline{1051}$ = CD

3a Representation is = $\frac{1}{2}$ [230 (6x) + 870 (2)] At x = – 1 it is $\overline{180}$ effective 3a = $\overline{180}$

	0	0	0	0	0	0	0	0
CD 05	$\overline{2}$	$\overline{59}$	$\overline{719}$	$\overline{423}$	$\overline{263}$	$\overline{578}$	$\overline{426}$	$\overline{450}$
		$\overline{80}$	$\overline{230}+\overline{3(}$ $=\overline{590}$	$\overline{80}+\overline{2520}$ $+\overline{690}$ $=\overline{3390}$	$\overline{4830}+\overline{690}$ $\overline{2520}+\overline{2520}$ $\overline{05}$	$\overline{230}+\overline{4830}$ $+\overline{9660}+\overline{8820}$ $+\overline{4320}+\overline{2520}$ $=\overline{30380}$	$\overline{8280}+\overline{33810}$ $+\overline{4830}+\overline{9660}$ $+\overline{12240}+\overline{4320}$ $+17640=90780$	
		$3ab^2$	$230(b$ $+3a.2bc$	$3ac^2+3a.2bd$ $+230(3b^2c)$	$230(3b^2d)+$ $230(3bc^2)$ $+3a.2be+3a.2cd$	$230(c^3)$ $+230(3b^2e)$ $+230(6bcd)$ $+3a.d^2+$ $3a.2bf$ $+3a.2ce$	$230(3b^2f+3bd^2$ $+3c^2d+6bce)+$ $3a.2bg+3a.2cf$ $+3a.2de$	
-1.	1	1	7	7	12	34	90	
a	b	c	d	e	f	g	h	

Subtraction Terms

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal are to be obtained and the usual manner, ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a. For example $3ab^2 = \overline{180} \times 1 = \overline{180}$

Upto h (7 decimals) x $= -1.\ 1\ 1\ 7\ 7\ 12\ 34\ 90 = -1.\ 1178630 = 0.\ \overline{8821370}$

$E = (x + 0.8821370)\ A$, A should contain x^2, x and constant terms

Applying Adyamadyena Antyamantyena and by Argumentation

$\therefore\ E = (x + 0.8821370)\ (230x^2 + \alpha x - 589.4775982)$

Comparing coefficients of like terms on both sides.

$-589.4775982 + 0.8821370\alpha = -1$

$\alpha = \frac{588.4775982}{0.8821370} = 667.1045407$

$E = (x + 0.8821370)(230x^2 + 667.1045407x - 589.4775982)$

Applying Gunita Samuccaya for verification

$S_c(E) = 579 \simeq 578.9960494$

This is satisfied

The Quadratic expression $(230x^2 + 667.1045407x - 589.4775982)$

Can be further factorized using Differential relation.

$460x + 667.1045407 \quad = \pm\sqrt{445028.4682 + 542319.3903} = \pm 993.6537921$

$x = -3.610344202, 0.709889677$ $\quad E = 0.00051470, 0.00001990$ Respectively

$E = 2.3(x + 0.8821370)(x + 3.610344202)(x - 0.709889677)$

Final verification by Gunitha Samuccaya Sutram

$S_c = 5.79 = (2.3)(1.8821370)(4.3610344202)(0.290110323) = 5.789960278 \sim 5.79$

Purana Apuranabhyam Method can also be applied for the problems of this type having decimal co-efficients as well.

$2.3x^3 + 8.7x^2 - 0.01x - 5.2 = 0$

$$x^3 + \left(\frac{8.7}{2.3}\right)x^2 - \left(\frac{0.01}{2.3}\right)x - \left(\frac{5.2}{2.3}\right) = 0$$

Given equation $x^3 + 3.78261x^2 - 0.0043478x - 2.26087 = 0$ ———— (1)

We have

$(x + 1.26087)^3$ (standard) $= x^3 + 3x^2(1.26087) + 3x(1.26087)^2 + (1.26087)^3$

$(x + 1.26087)^3 = x^3 + 3.78261x^2 + 4.76938x + 2.00452$ ———— (2)

From the given equation (1) we have

$x^3 + 3.78261x^2 = (0.0043478x + 2.26087)$

using this (2) can be written as $(x + 1.26087)^3$ which is

$= 0.0043478x + 2.26087 + 4.76938x + 2.00452$

or $(x + 1.26087)^3 = 4.773728x + 4.26539$

If $x + 1.26087 = y \Rightarrow x = (y - 1.26087)$

Then $y^3 = 4.773728(y - 1.26087) + 4.26539$

or $y^3 = 4.773728y - 6.019050 + 4.26539$

$\therefore \; y^3 - 4.773728y + 1.75366 = 0$ ——— (3)

Considering upto 5 decimals

$E_1 = y^3 - 4.77373y + 1.75366 = 0$

$f(y) = y^3 - 4.77373y = -1.75366$

	LHS	RHS	RHS – LHS
$y = 1$	-3.77373	-1.75366	2.02007
$y = 2$	-1.54746		-0.2062
$y = -1$	3.77373		-5.52739

$3a^2$ representation $y = 2 = (3y^2 - 4.77373)$ at $y = 2 \Rightarrow 7.22627$

$3a$ representation at $y = 2 = \frac{1}{2}(6y)$ at $y = 2 \Rightarrow 6$

CD = 7.22627						
	$\overline{2.062}$	$\overline{20.62}$	$\overline{61.6746}$	$\overline{38.6444}$	$\overline{48.3424}$	$\overline{18.7549}$
		0	$0 + 0 = 0$	$0 + 0 + \overline{24} = \overline{24}$	$0 + 0 + 0 + \overline{192} = \overline{192}$	$8 + 0 + 0 + 0 + \overline{192} + \overline{384} = \overline{568}$
		$3ab^2$	$b^3 + 3a.2bc$	$3b^2c + 3a.2bd + 3a.c^2$	$3b^2d + 3bc^2 + 3a.2be + 3a.2cd$	$c^3 + 3b^2e + 6bcd + 3a.2bf + 3a.2ce + 3a.d^2$
a = 2	0	$\bar{2}$	$\bar{8}$	$\bar{8}$	$\overline{33}$	$\overline{31}$
	b	c	d	e	f	g

Upto g (6 decimals), $y = 2.0\,\bar{2}\;\bar{8}\;\bar{8}\;\overline{33}\;\overline{81} = 2.0\;\bar{2}\;\bar{9}\;\bar{2}\;\bar{1}\;\bar{1} = 1.970789$

$\therefore$ (y – 1.970789) is a factor of E_1

$\Rightarrow$ E_1 = (y – 1.970789) A. A should have y^2, y and constant terms.

Applying Adyamadyena Antyamantyena and Argumentation

$E_1 = (y - 1.970789)(y^2 + \alpha y - 0.889826358)$

Comparing y-coefficient on both sides

$-0.889826358 - 1.970789\alpha = -4.77373$

$\alpha = \dfrac{-3.883903642}{-1.970789} = 1.970735397$

$\therefore\ E_1 = (y - 1.970789)(y^2 + 1.970735397y - 0.889826358)$

The Quadratic expression $(y^2 + 1.970735397y - 0.889826358)$ is further factorised by using differential relation

$2y + 1.970735397 = \pm\sqrt{3.883798004 + 3.559305432} = \pm 2.728205168$

$y = -2.349470282, 0.378734885$

$E_1 = (y - 1.970789)(y + 2.349470282)(y - 0.378734885)$

Applying Gunita Samuccaya Sutram for final verification

$S_c = -2.02007 = (-0.970789)(3.349470282)(0.621265115) = -2.020123603 \simeq S_c$

$y = x + 1.26087 \Rightarrow x = y - 1.26087$

$y = 1.970789 \Rightarrow x = 0.709919$

$y = -2.349470282 \Rightarrow x = -3.610340282$

$y = 0.378734885 \Rightarrow x = -0.882135115$

These are almost same values which are determined by solving the equation, directly.

Straight Division Method	Purana Method
– 0.8821370	– 0.882135115
– 3.61034	– 3.610340282
0.70992	0.709919

2. $E = x^3 - 7x^2 + 20x - 37 = 0,\ f(x) = x^3 - 7x^2 + 20x = 37$

Solution $x = a.bcde...$

x value	f(x)	RHS 37	RHS – LHS
			23
1	14		17
2	20		13
3	24		5
4	32		– 13
5	50		

From Vilokanam a = 4

Let $x = 4 \Rightarrow$ RHS – LHS = 5

$CD = 3a^2$ Representation at $x = 4$, $(3x^2 - 14x + 20)$ at $x = 4 \Rightarrow 12$

$3a$ Representation at $x = 4$, $\frac{1}{2}(6x - 14)$ at $x = 4 \Rightarrow 5$

CD = 12	0	0	0	0	0	0	0
	5	2	0	4	$\overline{9}$	0	$\overline{2}$
		$\overline{80}$	$\overline{64} + 200 = 136$	$240 + \overline{440} + \overline{125} = \overline{325}$	$\overline{528} + \overline{300} + 920 + 550 = 642$	$125 + 1104 + 1320 + \overline{1840} + \overline{1150} + \overline{605} = \overline{1046}$	
		$3ab^2$	$(b^3) + 3a.2bc$	$3b^2c + 3a.2bd + 3ac^2$	$3b^2d + 3bc^2 + 3a.2be + 3a.2cd$	$c^3 + 3b^2e + 6bcd + 3a.2b[f] + 3a.2ce + 3a.d^2$	
4	4	$\overline{5}$	11	$\overline{23}$	46	$\overline{87}$	
a	b	c	d	e	f	g	

Subtraction Terms

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD

2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms

3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a. For example $3ab^2$ 5 × 16 = 80

Upto g (6 decimals) $x = 4.\ 4\ \bar{5}\ 11\ \overline{23}\ 46\ \overline{87} = 4.\ 4\ \bar{4}\ \bar{1}\ 1\ \bar{2}\ \bar{7} = 4.359073$

Upto f (5 decimals) $x = 4.\ 4\ \bar{5}\ 11\ \overline{23}\ 46 = 4.\ 4\ \bar{4}\ \bar{1}\ 1\ 6 = 4.35916$

The values between x_1 and x_2 comparable with respect to x_1 and x_2

$x_1 = 4.35916$ E = 0.001222

$x_2 = 4.358899$ E = – 0.002

Considering upto 5 decimals,

(x – 4.35916) is a factor of E, Applying Adyamadyena Antyamantyena and Argumentation,

$\therefore$ $E = (x - 4.35916)(x^2 + \alpha x + 8.487873811)$ comparing x – coefficient on both sides.

x coeff: $8.487873811 - 4.35916\alpha = 20$

$\alpha = -2.640904713$

$(x^2 - 2.640904713x + 8.487873811)$

$2x - 2.640904713 = \pm\sqrt{6.974677703 - 33.95149524} = \pm 5.19395009i$

$x = 1.320452357 \pm 2.596975045i$

$E = (x - 4.35916)(x - 1.320452357 + 2.596975045i)(x - 1.32045235 - 2.596975045i)$

<u>Applying Gunita Samuccaya</u>

– 23 = – 230000647 ~ 23.00

3) $E = 2x^3 + 9x^2 - 18x + 35 = 0$, $f(x) = 2x^3 + 9x^2 - 18x = -35$

Solution x = a.bcde.....

	f(x) LHS	RHS	RHS – LHS
x = 1	–7	–35	–28
x = 2	16		–51
x = 3	81	–35	–116
x = –1	25		–60
x = –2	56		–91
x = –3	81		–116
x = –4	88		–113
x = –5	65		–100
x = –6	0		–35
x = –7	–119	–35	84

From Vilokanam a = – 6

Let x = – 6 ⇒ RHS – LHS = $\overline{35}$

CD = $3a^2$ Representation at x = – 6 is $[2(3x^2) + 9(2x) - 18]$ At x = – 6, it is 90

3a Representation at x = – 6 is = $\frac{1}{2}$ $[2(6x) + 9(2)]$ at x = – 6 it is $\overline{27}$

$3a^2 = 90 = CD$

$3a = \overline{27}$

	0	0	0	0	0	0	0
90	$\overline{35}$	$\overline{80}$	$\overline{17}$	46	28	$\overline{80}$	55
		243	54 + 972 = 1026	$324+\overline{1458}+972 = \overline{162}$	$\overline{486}+648+\overline{486}+\overline{2916} = \overline{3240}$	$432+\overline{162}+\overline{1944}+2187 +5184+\overline{972}=4725$	
		$3ab^2$	$2(b^3)+3a.2bc$	$2(3b^2c)+3a.2bd+3a.c^2$	$2(3b^2d)+2(3bc^2)+3a.2be+3a.2cd$	$2(c^3)+2(3b^2e)+2(6bcd) +3a.2bf+3a.d^2+3a.2ce$	Subtract ionTerms
$-6.$	$\bar{3}$	$\bar{6}$	9	3	$\overline{32}$	43	
a	b	c	d	e	f	g	

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD

2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms

3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a. For example $3ab^2\ \overline{27} \times 9 = 243$

Upto g (6 decimals) x $= \bar{6}.\bar{3}\,\bar{6}\,9\,3\,\overline{32}\,43 = \bar{6}.\bar{3}\,\bar{6}\,9\,\bar{0}\,2\,3 = \bar{6}.\bar{3}\,\bar{5}\,0\,\bar{9}\,\bar{7}\,\bar{7}$

Upto 5 decimals $x = \overline{6.35102}$

<u>Considering 5 decimals</u>

$(x+6.35102)$ is a factor of $E \Rightarrow (x+6.35102)A$. A should have x^2, x and constant terms

Applying Adyamadyena Antyamantyena and Argumentation

$\therefore E = (x+6.35102)(2x^2+\alpha x+5.510925804)$, Comparing coefficients of like terms

$5.510925804 + 6.35102\,\alpha = -18$

$\alpha = -3.701913363$

$E = (x+6.35102)(2x^2-3.701913363+5.510925804) = 2(x+6.35102)(x^2-1.850956682x+2.755462902)$

The QE is further solved by using differential relation

$2x - 1.850956682 = \pm\sqrt{3.426040637 - 11.02185161} = \pm 2.756049885i$

$x = 0.925478341 \pm 1.378024943i$

$E = 2(x + 6.35102)(x - 0.925478341 + 1.378024943i)(x - 0.925478341 - 1.378024943i)$

$28 = 28.00012664 \sim 28$

<u>Considering 6 decimals</u>

$(x + 6.350977)$ is a factor of E.

$\therefore$ $E = (x + 6.350977)A$. A should have x^2, x and constant terms

By Adyamadyena, Antyamantyena

$E = (x + 6.350977)(2x^2 + \alpha x + 5.510963116)$

Comparing x coefficient on both sides

$5.510963116 + 6.350977\,\alpha = -18$

$\alpha = -3.701944302$

$E = (x + 6.350977)(2x^2 - 3.701944302x + 5.510963116) = 2(x + 6.350977)(x^2 - 1.850972x + 2.755$ 558

The QE is factorised by differential relation

$2x - 1.850972 = \pm\sqrt{3.426097345 - 11.02192623} = \pm 2.756053136i$

$x = 0.925486 \pm 1.378026568i$

$\therefore$ $E = (x + 6.350977)(x - 0.925486 + 1.378027i)(x - 0.925486 - 1.378027i)$

<u>Applying Gunita Samuccayah Samuccaya Gunitaha Sutram for final verification</u>

$S_c = 28 = 28.00002942$

4) $E = 2x^3 - 15x^2 - 8x + 166 = 0$ $f(x) = 2x^3 - 15x^2 - 8x = -166$

Solution $x = a.bcde....$

	f(x) L.H.S	R.H.S.	RHS – LHS	
x = 1	– 21	– 166	– 145	
x = 2	– 60		– 106	
x = 3	– 105		– 61	
x = 4	– 144		– 22	
x = 5	– 165		– 1	Probable
x = 6	– 156		– 10	
x = 7	– 105		– 61	
x = 8	0		– 166	
x = – 1	– 9		– 157	
x = – 2	– 60		– 106	
x = – 3	– 165		– 1	Probable
x = – 4	– 336		170	
x = – 5	– 585		419	

From Vilokanam a = 5

Let x = 5 $\Rightarrow$ RHS – LHS = $\bar{1}$

CD = $3a^2$ Representation at x = 5 is $[2(3x^2) - 15(2x) - 8]$ At x = 5 it is – 8

3a Representation at x = 5 is $\frac{1}{2}$ $[2(6x) - 15(2)]$ At x = 5 it is 15

	0	0	0	0	0	0	Subtraction Terms
CD = $\bar{8}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	0	$\bar{2}$	$\bar{6}$	
		$\overline{15}$	$\bar{2}+\overline{120}=\overline{122}$	$\overline{24}+\overline{570}+\overline{240}=\overline{834}$	$\overline{114}+\overline{96}+\overline{3120}+\overline{2280}=\overline{5610}$		
		$3ab^2$	$2(b^3)+3a.2bc$	$2(3b^2c)+3a.2bd+3a.c^2$	$2(3b^2d)+2(3bc^2)+3a(2be)+3a(2cd)$		
5	1	4	19	104	703		
a	b	c	d	e	f		

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD

2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction te

3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a. For example $3ab^2 = 15 \times 1 = 1$

Upto f (5 decimals) x = 5. 1 4 19 104 703 = 5.17643

For values of x a reduction process is adopted by considering another variable z as $x = \frac{z}{2}, \frac{z}{5}, \frac{z}{10}, \frac{z}{100}$ at a time. The value of x determined and finally the value obtained for x with $x = \frac{z}{100}$ is considered for finding out all the roots. The working details are shown under Taylors Series Section. However Swamiji's method also can be adopted for this substitution method as well.

$E = 2x^3 - 15x^2 - 8x + 166 = 0$

Let $x = \frac{z}{100} \Rightarrow E = \frac{2z^3}{10^6} - \frac{15z^2}{10^4} - \frac{8z}{10^2} + 166 = 0$

$\Rightarrow 2z^3 - 1500z^2 - 80000z = -166000000$

$f(z) = z^3 - 750z^2 - 40000z = -83000000$

	f(z)	RHS – LHS
z = 528	– 83010048	10048
z = 529	– 83004861	4861
z = 530	– 82998000	– 2000

From Vilokanam a = 530

$3a^2$ representation z = 530 = $(3z^2 - 1500z - 40000)$ at z = 530 ⇒ 7700

3a representation at z = 530 = $\frac{1}{2}(6z-1500)$ at z = 530 ⇒ 840

	0	0	0	0	0	
CD = 7700	$\overline{2000}$	$\overline{4600}$	$\overline{3160}$	$\overline{5552}$	$\overline{5748}$	Subtraction Terms
		$\overline{3360}$	$8 + \overline{20160} = \overline{20152}$	$72 + \overline{20160} + \overline{30240} = \overline{50328}$		
		$3ab^2$	$b^3 + 3a.2bc$	$3b^2d + 3a(2bd) + 3a.c^2$		
530	$\bar{2}$	$\bar{6}$	$\bar{6}$	$\overline{13}$		
a	b	c	d	e		

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto e (4 decimals), z = $530.\ \bar{2}\ \bar{6}\ \bar{6}\ \overline{13} = 530.\ \bar{2}\ \bar{6}\ \bar{7}\ \bar{3} = 529.7327$

$\Rightarrow x = \frac{z}{100} = 5.297327$

<u>For 2nd factor of E</u> = $2x^3 - 15x^2 - 8x = -166 \Rightarrow$

From Vilokanam a = – 3

$3a^2$ representation at x = – 3 = $[2(3x^2) - 15(2x) - 8]$ at x = – 3 ⇒ 136

3a representation at $x = -3 = \frac{1}{2}[2(6x) - 30]$ at $x = -3 \Rightarrow -33$

	0	0	0	0	0	0	0	0
CD = 136	$\bar{1}$	$\overline{10}$	$\overline{100}$	$\overline{48}$	$\overline{72}$	$\overline{40}$	129	92
		0	0 − 10 = 0	0 + 0 + 0 = 0	0 + 0 + 0 +0 = 0	0 + 0 + 0 + 1617 + 0 + 0 = 1617	0 + 0 + 0 + 0 + 0 + 0 + 1386 = 1386	0 + 0 + 0 + 0 + 0 + 0 + 0 + 2310 + 297 = 2607
		$3ab^2$	$b^3 + 3a.2bc$	$3b^2c + 3a(2bd) + 3a.c^2$	$3b^2d + 3bc^2 + 3a.2be + 3a.2cd$	$c^3 + 3b^2e + 6bcd + 3a.d^2 + 3a.2bf + 3a.2ce$	$3b^2f + 3bd^2 + 3c^2d + 6bce + 3a.2bg + 3a.2cf + 3a.2de$	$3b^2g + 3c^2e + 3cd^2 + 6bcf + 6bde + 3a.2bh + 3a.2cg + 3a.2df + 3a.e^2$
a = −3	0	0	$\bar{7}$	$\bar{3}$	$\bar{5}$	8	19	25
	b	c	d	e	f	g	h	i

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD

2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID − corresponding subtraction terms

3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto i (8 decimals), $x = -3.00\bar{7}\,\bar{3}\,\bar{5}\,8\,19\,25 = \bar{3}.00\bar{7}\,\bar{3}\,\bar{4}\,0\,1\,5 = \bar{3}.00\bar{7}\,\bar{3}\,\bar{3}\,\bar{9}\,\bar{8}\,\bar{5}$

$E = (x - 5.2977327)(x + 3.00733985)A$

A should have x term and constant. Applying Adyamadyena Antyamantyena and Argumentation

$E = (x^2 - 2.29039285x - 15.93086259)(2x + \alpha)$ $\therefore\ \alpha = -10.42002585$

$\therefore$ The third factor is $(2x - 10.42002585)$

$E = (x^2 - 2.29039285x - 15.93086259)(2x - 10.42002585)$

Applying Gunita Samuccaya Sutram for final verification

$S_c(E) = (1 - 2.29039285 - 15.93086259)(2 - 10.42002585)$

$145 = (-17.22125544)(-8.42002585) = 145.003416 \simeq 145$ $\therefore E =$ 5.2977327 x + 3 733985) (2x 0.42002585

5) $E = x^3 - 3x + 1 = 0,\ f(x) = x^3 - 3x = -1$

Solution x = a.bcde....

	LHS $f(x) = x^3 - 3x$ f(x)	RHS	S − LHS	
x = 1	−2	−1	1	ve
x = 2	2	−1	−3	
x = −1	2	−1	−3	−ve
x = −2	−2	−1		

From Vilokanam a = −2

Let x = −2 ⇒ RHS − LHS = 1

CD = $3a^2$ Representation at x = −2 ⇒ $[(3x^2) - 3]$ at x = −2 = 9

3a Representation at x = −2 = $\frac{1}{2}$ [(6x)] at x = −2 = −6

	0	0	0	0	0	0	0
CD = 9	1	1	7	0	3	0	2
		6	$\bar{1} + 12 = 11$	$\bar{3} + 108 + 6 = 111$	$\overline{27} + \bar{3} + 144 + 108 = 222$	$\bar{1} + \overline{36} + \overline{54} + 336 + 144 + 486 = 875$	
		$3ab^2$	$b^3 + 3a.2bc$	$3b^2c + 3a.2bd + 3a.c^2$	$3b^2d + 3bc^2 + 3a.2be + 3a.2cd$	$c^3 + 3b^2e + 6bcd + 3a.2bf + 3a.2ce + 3a.d^2$	
−2.	1	1	9	12	$\overline{28}$	$\bar{9}$	
a	b	c	d	e	f	g	

Subtract ionTerms

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto g (6 decimals) x $= \bar{2}.11912\ 2897 = \bar{2}.120577 = \bar{1}.\bar{8}\bar{7}\bar{9}\bar{4}\bar{2}\bar{3}$

$x = -1.879423 \Rightarrow (x + 1.879423)$ is a factor of E

$\therefore\ E = (x + 1.879423)A$

A should contain x^2, x and constant terms.

Applying Adyamadyena. Antyamantyena and Argumentation

$E = (x + 1.879423)(x^2 + \alpha x + 0.532078196)$

Comparing x coefficients on both sides

$\Rightarrow 0.532078196 + 1.879423\alpha = -3$

$$\alpha = \frac{-3.532078196}{1.879423} \doteq -1.879341796$$

$\therefore\ E = (x + 1.879423)(x^2 - 1.879341796x + 0.532078196)$

The Quadratic Expression $(x^2 - 1.879341796x + 0.532078196)$ is further factorized by using the differential relation

$2x - 1.879341796 = \pm\sqrt{3.531925586 - 2.128312784} = \pm\ 1.18474166 \Rightarrow x = 1.532041728, 0.347300068$

$\Rightarrow E = (x + 1.879423)(x - 1.532041728)(x - 0.347300068)$

Applying Gunita Samuccaya Sutram for final verification

$S_c = -1 = (2.879423)(-0.532041728)(0.652699932)$

$= -0.999918796 \simeq -1$

6) $x^3 + 6x^2 - 11x - 8 = 0$ $f(x) = x^3 + 6x^2 - 11x = 8$

Solution $x = a.bcde...$

If $x = 1$ $f(x) = -4$

If $x = 2$ $f(x) = 10$ This is nearer RHS value

$\therefore$ $a = 2$ can be considered

Let $x = 2 \Rightarrow$ RHS – LHS $= \bar{2}$

From Vilokanam $a = 2$

CD $= 3a^2$ Representation at $x = 2 \Rightarrow [3x^2 + 6(2x) - 11]$ at $x = 2 \Rightarrow 25$

3a Representation at $x = 2 = \frac{1}{2}[6x + 12]$ at $x = 2 \Rightarrow 12$

	0	0	0	0	0	0	0	0	0	0
CD = 25	$\bar{2}$.	$\overline{20}$	0	0	$\overline{18}$	$\bar{5}$	$\overline{23}$	$\overline{24}$	$\overline{17}$	$\overline{20}$
		0	0	$0+\overline{768}+0$ $=\overline{768}$	0+0+0+0 =0	512+0+0+0+ $\overline{5760}+0$ $=\overline{5248}$	0+0+0+0+0 $+0+\overline{1344}+0$ $=\overline{1344}$	0+5760+0++0 $0+0+\overline{40512}+$ $0+\overline{10800}$ $=\overline{45552}$	0+0+0+1344+0 $+0+0+\overline{11904}+0$ $+\overline{5040}=\overline{15600}$	0+40512+21600 +0+0+0+0+0+0 $+\overline{351552}+0$ $\overline{151920}+\overline{588}$ $=\overline{441948}$
		$3ab^2$	$b^3+3a.2bc$	$3b^2c+3a.c^2$ $+3a.2bd$	$3b^2d+3bc^2$ $+3a.2be+$ $3a.2cd$	c^3+3b^2e $+6bcd+$ $3a.2bf$ $+3a.2ce+3a.d^2$	$3b^2f+3bd^2$ $+3c^2d+6bce$ $+3a.2bg+3a.$ $2cf+3a.2de$	$3ae^2+3b^2g$ $+3c^2e+3cd^2$ $+3a.2bh$ $3a.2cg+6bcf$ $+3a.2df+6bde$	$d^3+3b^2h+3be^2$ $+3c^2f+6bcg$ $+6bdf+6cde$ $+3a.2bi+3a.2ch$ $+3a.2dg+3a.2ef$	$3af^2+3b^2i$ $+3c^2g+3ce^2$ $+3d^2e+3a.2bj$ $+3a.2ci+6bch$ $+3a.2dh+6bdg$ $+6cdf+3a.2eg$ $+6bef$
2	0	$\bar{8}$	0	$\overline{30}$	$\bar{7}$	$\overline{211}$	$\overline{62}$	$\overline{1831}$	$\overline{630}$	$\overline{17685}$
a	b	c	d	e	f	g	h	i	j	k

Subtraction Terms

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto k (10 decimals) $x = 2.0\ \bar{8}\ 0\ \overline{30}\ \bar{7}\ \overline{211}\ \overline{62}\ \overline{1831}\ \overline{630}\ \overline{17685} = 2.0\ \bar{8}\ \bar{3}\ \bar{3}\ 0\ \bar{7}\ \bar{9}\ 0\ \bar{8}\ \bar{5} = 1.9166920915$

E can be factorized as.

$E = (x - 1.916692015)A$, A should contain x^2, x and constant terms

Applying Adyamadyena Antyamantyena and Argumentation

$= (x - 1.9166920915)(x^2 + \alpha x + 4.173857678)$

α can be determined by comparing like terms on both sides.

$4.173857678 - 1.9166920915\alpha = -11 \quad \therefore \alpha = 7.916690292$

$\therefore E = (x - 1.9166920915)(x^2 + 7.916690292x + 4.173857678)$

The Quadratic Expression is further factorised by using differential relation.

$2x + 7.916690292 = \pm\sqrt{62.67398518 - 16.69543071} = \pm\sqrt{45.97855447} = \pm 6.780748813$

$x = -7.348719553, -0.567970739$

$\therefore E = (x - 1.9166920915)(x + 0.567970739)(x + 7.348719553)$

Applying Gunita Samuccaya Sutram

$S_c = (-0.9166920915)(1.567970739)(8.348719553) = -12.00000179 \sim -12$

7) $E = 42x^3 + 8x^2 - 257x + 400 = 0 \qquad f(x) = 42x^3 + 8x^2 - 257x = -400$

Solution x = a.bcde...

	LHS	RHS	Diff
	f(x)	– 400	RHS – LHS
f(1)	– 207		– 193
f(2)	– 146		– 254
f(– 1)	223		– 623
f(– 2)	210		– 610
f(– 3)	– 291		– 109
f(– 4)	– 1532		1132

From Vilokanam a = – 3

Let x = – 3 $\Rightarrow$ RHS – LHS = $\overline{109}$

CD = $3a^2$ Representation at x = – 3 $\Rightarrow$ $[42(3x^2) + 8(2x) - 257]$ at x = – 3 $\Rightarrow$ 829 $\qquad 3a^2 = 829 = CD$

3a Representation at x = – 3 = $\frac{1}{2}[42(6x) + 8(2)] \Rightarrow \overline{370}$ $\qquad 3a = \overline{370}$

	0	0	0	0	0	0	0	
CD=829	$\overline{109}$	$\overline{261}$	$\overline{582}$	$\overline{153}$	586	657	145	$\overline{463}$
		370	42+1480 =1522	252+3700 +1480 =5432	$630+504+\overline{2960}$ +7400=5574	$336+\overline{504}+2520+9250$ $+\overline{9620}+\overline{5920}=\overline{3938}$	$\overline{1638}+3150+2520$ $+\overline{2016}+\overline{2220}+\overline{19240}$ $+\overline{14800}=\overline{34244}$	
Subtraction Terms		$3ab^2$	$42(b^3) + 3a.2bc$	$42(3b^2c)+$ $3a.2bd + 3a.c^2$	$42(3b^2d)+$ $42(3bc^2)+3a.2be$ $+3a.2cd$	$42(c^3)+42(3b^2e)$ $+42(6bcd)+3a.d^2$ $+3a.2bf + 3a.2ce$	$42(3b^2f)+42(3bd^2)$ $+42(3c^2d)+42(6bce)$ $+3a.2bg+3a.2cf+3a.2de$	
– 3	$\bar{1}$	$\bar{2}$	$\bar{5}$	4	13	3	$\overline{39}$	
a	b	c	d	e	f	g	h	

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto h (7 decimals) x $= \bar{3}.\bar{1}\,\bar{2}\,\bar{5}\,4\,13\,3\,\overline{39} = \bar{3}.\bar{1}\,\bar{2}\,\bar{5}\,5\,3\,0\,\bar{9} = \bar{3}.\bar{1}\,\bar{2}\,\bar{4}\,\bar{4}\,\bar{7}\,0\,\bar{9}$

$\therefore E = (x + 3.1244709)A$. A will contain x^2, x and constant terms.

Applying Adyamadyena Antyamantyena and Argumentation

$\therefore\ E = (x + 3.1244709)(42x^2 + \alpha x + 128.0216756)$

Comparing the coefficients of like terms on both sides.

x Coeff $\Rightarrow 1280216756 + 3.1244709\alpha = -257$

$\alpha = -123.2278001$

$E = (x + 3.1244709)(42x^2 - 123.2278001x + 128.0216756)$

Verification of x^2 coeff $= (3.1244709)42 - 123.2278001 = 7.999977 \simeq 8$

$D_1 = 84x - 123.2278001 \pm \sqrt{15185.09072 - 21507.6415}$

$x = 1.46699762 \pm 0.946600821\,i$

$E = 42(x + 3.1244709)(x - 1.46699762 - 0.946600821i)(x - 1.46699762 + 0.946600821i)$

Applying Gunita Samuccaya Sutram for final verification

$E = 42(4.1244709)[(-0.46699762)^2 - (0.946600821i)^2]$

$= 42(4.595237561) = 192.9999775 \sim 193 = S_c = 193$

8) $E = x^3 + 29x - 97 = 0$ $\quad f(x) = x^3 + 29x = 97$

Solution x = a.bcde .....

f(x)	LHS	RHS	RHS – LHS
x = 1	30	97	– 67
x = 2	66	97	31
x = 3	114	97	– 17

From Vilokanam a = – 3

Let x = 3 $\Rightarrow$ RHS – LHS = $\overline{17}$

CD = $3a^2$ Representation at x = 3 $\Rightarrow$ $[3x^2 + 29]$ at x = 3 $\Rightarrow$ 56 $\qquad 3a^2 = 56 = CD$

3a Representation at x = 3 = $\frac{1}{2}$ (6x) at x = 3 $\Rightarrow$ 9 $\qquad 3a = 9$

	0	0	0	0	0	0	0	0	0
CD=56	$\overline{17}$	$\bar{2}$	$\overline{45}$	$\overline{29}$	$\overline{32}$	$\overline{47}$	$\overline{35}$	$\bar{7}$	$\bar{2}$
		$\overline{81}$	$27+\overline{54}=\overline{27}$	$27+\overline{432}+\bar{9}=\overline{414}$	$216+9+\overline{648}+\overline{144}=\overline{567}$	$1+324+144+\overline{810}+\overline{216}+\overline{576}=\overline{1133}$	$405+576+24+216+\overline{1512}+\overline{270}+\overline{1728}=\overline{2289}$	$756+36+192+270+1728+\overline{2538}+\overline{504}+\overline{2160}+\overline{1296}=\overline{3516}$	Subtraction Terms
		$3ab^2$	$b^3+3a.2bc$	$3b^2c+3a.2bd+3a.c^2$	$3b^2d+3bc^2+3a.2be+3a.2cd$	$c^3+3b^2e+6bcd+3a.2bf+3a.2ce+3a.d^2$	$3b^2f+3bd^2+3c^2d+6bce+3a.2bg+3a.2cf+3a.2de$	$3b^2g+3c^2e+3cd^2+6bcf+6bde+3a.2bh+3a.2cg+3a.2df+3a.e^2$	
3	$\bar{3}$	$\bar{1}$	$\bar{8}$	$\overline{12}$	$\overline{15}$	$\overline{28}$	$\overline{47}$	$\overline{64}$	
a	b	c	d	e	f	g	h	i	

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD

2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms

3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto i (8 decimals) $x = 3.\bar{3}\,\bar{1}\,\bar{8}\,\overline{12}\,\overline{15}\,\overline{28}\,\overline{47}\,\overline{64} = 3.\bar{3}\,\bar{1}\,\bar{9}\,\bar{3}\,\bar{8}\,\bar{3}\,\bar{3}\,\bar{4} = 2.68061666$

$(x - 2.68061666)$ is a factor of E.

$E = (x - 2.68061666)A$. A should contain x^2, x and constant terms.

Applying Adyamadyena Antyamantyena and Argumentation.

$E = (x - 2.68061666)(x^2 + \alpha x + 36.18570363)$

Comparing x – coefficient on both sides

$36.18570363 - 2.68061666\alpha = 29$

$$\alpha = \frac{-7.18570363}{-2.68061666} = 2.680615896$$

$E = (x - 2.68061666)(x^2 + 2.680615896x + 36.18570363)$

The Quadratic Expression $(x^2 + 2.680615896x + 36.18570363)$ is further factorised by using differential relation

$2x + 2.680615896 = \pm\sqrt{7.185701582 - 144.7428145} = \pm 11.72847445i$

$x = -1.340307948 \pm 5.864237225i$

$\therefore\ E = (x - 2.68061666)(x + 1.340307948 - 5.864237225i)(x + 1.340307948 + 5.864237225i)$

Applying Gunita Samuccaya Sutram for final verification

$S_c = -67 = (-1.68061666)(2.340307948 - 5.864237225i)(2.340307948 + 5.864237225i) = 67.00000076 \simeq 67$

9) $E = 2x^3 + 9x^2 + 18x + 20 = 0$; $f(x) = 2x^3 + 9x^2 + 18x = -20$

Solution x = a.bcde .....

	LHS	RHS	RHS – LHS Diff
x = 1	29	– 20	– 49
x = 2	88		– 108
x = – 1	– 11		– 9
x = – 2	– 16		– 4
x = – 3	– 27		7

Try – 2.25, – 2.5, – 2.75

From Vilokanam a = – 3

Let x = – 3 ⇒ RHS – LHS = 7

CD = $3a^2$ Representation at x = – 3 ⇒ $[2(3x^2) + 9(2x) + 18]$ at x = – 3 ⇒ 18 Effective $3a^2 = 18$ = CD

3a Representation at x = – 3 = $\frac{1}{2}$ [2(6x) + 9(2)] at x = – 3 ⇒ $\bar{9}$ 3a = $\bar{9}$

	0	0	0	0	0	
CD = 18	7	16	7	16	7	
		81	$\overline{54}$+702=648	$\overline{702}$+2106++1521=2925		Subtraction Terms
		$3ab^2$	$2(b^3)$+3a.2bc	$2(3b^2c)$+3a.2bd+3a.c^2		
$\bar{3}$	3	13	39	171		
a	b	c	d	e		

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto e (4 decimals) x $= \bar{3}.3\ 13\ 39\ 171 = \bar{3}.4861 = \bar{2}.\bar{5}\bar{1}\bar{3}\bar{9}$

For the value of x, a reduction process is adopted by considering another variable 'z' as $x = \frac{z}{2}$ substituted. The value of x is determined and finally the value obtained for x with $x = \frac{z}{2}$ is considered for finding out the roots. The working details are shown under Taylor's series section. However Swamiji's method also can be adopted for this substitution method as well.

On Substitution of $x = \frac{z}{2}$, The given equation becomes $z^3 + 9z^2 + 36z + 80 = 0$

	LHS f(z)	RHS – LHS
z = 1	46	– 126
z = 2	116	– 196
z = – 1	– 28	– 52
z = – 2	– 44	– 36
z = – 3	– 54	– 26
z = – 4	– 64	– 16
z = – 5	– 80	0

Z = – 5 is a solution; $x = \frac{z}{2} = -2.5$ is a solution

$\therefore$ $E = (x + 2.5)$ A. A should have x^2, x and constant terms.

Applying Adyamadyena Antyamantyena and Argumentation

$E = (x + 2.5)(2x^2 + \alpha x + 8) = 0$ Comparing the coefficients on both sides $\alpha = 4$

$\therefore E = (x + 2.5)(2x^2 + 4x + 8) = 0$, Solving the Quadratic Equation by Differential relation

$(x + 2.5)\, 2(x^2 + 2x + 4)$

$2x + 2 = \pm\sqrt{4 - 16}$ $x = -1 \pm \sqrt{3}\,i$

$E = 2(x + 2.5)(x + 1 + \sqrt{3}\,i)(x + 1 - \sqrt{3}\,i)$

$S_c = 49 = 2(24.5) = 49$

Applying Gunita Samuccaya Sutram for final verification

$S_c = 49 = 2(3.5)(2+\sqrt{3}\,i)(2-\sqrt{3}\,i) = 49$

10) $x^3 - 2x^2 - 51x - 110 = 0 \Rightarrow x^3 - 2x^2 - 51x = 110$

Solution x = a.bcde .....

	L	R	R – L
x = 1	– 52	110	162
x = 2	– 102		212
x = 3	– 144		254
x = – 1	48		62
x = – 2	86		24
x = – 3	108		2
x = – 4	108		2
x = – 5	80		30
x = – 6	18		92

From Vilokanam a = – 3

Let x = – 3 $\Rightarrow$ RHS – LHS = 2

effective $3a^2 = \overline{12} = CD$

$CD = 3a^2$ Representation at x = – 3 $\Rightarrow [3x^2 - 4x - 51]$ at x = – 3 $\Rightarrow \overline{12}$

effective $3a = \overline{11}$

3a Representation at x = – 3 = $\frac{1}{2}[6x - 4]$ at x = – 3 $\Rightarrow \overline{11}$

CD = $\overline{12}$	0	0	0	0	
	2	8	7	9	Subtraction Terms
		11	1 + 154 = 155		
		$3ab^2$	$b^3 + 3a.2bc$		
– 3	$\bar{1}$	$\bar{7}$	$\overline{18}$		
a	b	c	d		

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto d (3 decimals) x = $\bar{3}.\bar{1}\,\bar{7}\,\overline{18}$ = – 3.188

For values of x, a reduction process is adopted by considering another variable 'z' as $x = \frac{z}{2}, \frac{z}{10}$ at a time. The value of x is determined and finally the value obtained for x with $x = \frac{z}{10}$ is considered for finding out all the roots. The working details are shown under Taylor's series section. However Swamiji's method also can be adopted for this substitution method as well.

$E = x^3 - 2x^2 - 51x - 110 = 0$

Let $x = \frac{z}{10} \Rightarrow E = \frac{z^3}{1000} - \frac{2z^2}{100} - \frac{51z}{10} - 110 = 0$

$\Rightarrow f(z) = z^3 - 20z^2 - 5100z = 110000$

	f(z)	RHS – LHS
z = – 32	109952	48
z = – 33	110583	– 583

From Vilokanam a = – 32

$3a^2$ representation at $z = -32 = (3z^2 - 40z - 5100)$ at $z = -32 \Rightarrow -748$

3a representation at $z = -32 = \frac{1}{2}(6z - 40) = -116$

	0	0	0	0	0	0	
CD = $\overline{748}$	48	480	312	128	220	288	
		0	0	0 + 0 + 4176 = 4176	0 + 0 + 0 + 5568 = 5568	Subtraction Terms	
		$3ab^2$	$b^3 + 3a.2bc$	$3b^2c + 3a.2bd + 3a.c^2$	$3b^2d + 3bc^2 + 3a.2be + 3a.2cd$		
a – 32	0	$\bar{6}$	$\bar{4}$	$\bar{7}$	$\overline{10}$		
	b	c	d	e	f		

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD

2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction

3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto f (5 decimals), $z = -32\,.\,0\;\bar{6}\;\bar{4}\;\bar{7}\;\overline{10} = -32.06480$

$$x = \frac{z}{10} = -3.206480$$

E = (x + 3.20648)A. A will contain x^2, x and constant terms

Applying Argumentation, Adyamadyena and comparing like terms on both sides.

$E = (x + 3.20648)(x^2 + \alpha x - 34.3055313)$

$(x^2 - 5.20647835x - 34.3055313)$ Solving the quadratic equation using

$f'(x) = \pm\sqrt{Discriminant}$

$2x - 5.20674835 = \pm\sqrt{27.10741681 + 137.2221252} = \pm 12.81910847$

$x = +9.012793411, -3.80631506$ $\quad\therefore E = (x + 3.20648)(x - 9.012793411)(x + 3.80631506)$

Applying Gunitha Samuccayah Sutram for final verification

$S_c = 162 = -161.9999983 \sim 162$

11) $x^3 - 20x^2 - 31x + 1609 = 0 \Rightarrow x^3 - 20x^2 - 31x = -1609$

Solution x = a.bcde .....

x value	LHS	RHS	RHS – LHS
1	– 50	– 1609	– 1509
2	– 134		– 1475
3	– 246		– 1363
4	– 380		– 1229
5	– 530		– 1079
6	– 690		– 919
7	– 854		– 755
– 1	10		– 1619
– 2	– 26		– 1583
– 3	– 114		– 1495
– 4	– 260		– 1349
– 5	– 470		– 1139
– 6	– 750		– –859
– 7	– 1106		– 503
– 8	– 1544		– 65
– 9	– 2070		461

From Vilokanam a = – 8

Let x = – 8 ⇒ RHS – LHS = $\overline{65}$

$CD = 3a^2$ Representation at x = – 8 ⇒ $[3x^2 - 40x - 31]$ at x = – 8 ⇒ 48

3a Representation at x = – 8 = $\frac{1}{2}$ [6x – 40] at x = – 8 ⇒ $\overline{44}$

effective 481 = CD

$3a^2$ 48 : CD

3a $\overline{44}$

	0	0	0	0	0	0	
CD = 481	$\overline{65}$	$\overline{169}$	$\overline{203}$	$\overline{322}$	$\overline{146}$	$\overline{192}$	Subtract ion Terms
		44	1+264=265	9+264+396=669	9+27+440+792=1268		
		$3ab^2$	$b^3+3a.2bc$	$3b^2c+3a.2bd+3a.c^2$	$3b^2d+3bc^2+3a.2be+3a.2cd$		
−8	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{5}$	0		
a	b	c	d	e	f		

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD

2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms

3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto f (5 decimals) $x = -8.\bar{1}\bar{3}\bar{3}\bar{5}0 = -8.13350$

$E = (x + 8.1335)A$, A will contain x^2, x and constant terms

Applying Adyamadyena Antyamantyena and Argumentation

$\therefore\ E = (x + 8.1335)(x^2 + \alpha x + 197.8238151)$ comparing the coefficients of like terms.

x coeff: $197.82815 + 8.1335\alpha = -31$ $\qquad \alpha = -28.13349911$

$\therefore\ E = (x + 8.1335)(x^2 - 28.13349911x + 197.8238151)$ Quadrate expression is further factorized using differential relation.

$f'(x) = \pm\sqrt{Discriminant}$

$2x - 28.13349911 = \pm\sqrt{791.4937721 - 791.2952604} = \pm 0.445546599$

$\therefore\ x = 14.28952285,\quad 13.84397626$

$E = (x + 8.1335)(x - 14.28952285)(x - 13.84397626)$

Applying Gunita Samuccaya for final verification

$1559 = 1559.000001 \sim 1559$

12) $E = x^3 - 6x^2 + 11x - 10 = 0$ $f(x) = x^3 - 6x^2 + 11x = 10$
Solution x = a.bcde .....

x value	LHS	RHS	RHS – LHS
x = 1	6	10	4
x = 2	6		4
x = 3	6		4
x = 4	12		– 2

From Vilokanam a = 4

Let x = 4 $\Rightarrow$ RHS – LHS = $\bar{2}$

CD = $3a^2$ Representation at x = 4 $\Rightarrow$ $(3x^2 - 12x + 11)$ at x = 4 $\Rightarrow$ 11

3a Representation at x = 4 = $\frac{1}{2}$ [6x – 12] at x = 4 $\Rightarrow$ 6

effective
$3a^2 = 11 = CD$
3a = 6

	0	0	0	0	0	0	0	0
CD=11	$\bar{2}$	$\bar{9}$	$\bar{8}$	$\overline{10}$	$\bar{2}$	$\bar{5}$	$\bar{5}$	$\bar{7}$
		$\bar{6}$	$1+\overline{96}$ $=\overline{95}$	$24+\overline{180}+\overline{384}$ =540	$45+192+\overline{696}$ $+\overline{1440}$ $=\overline{1899}$	512+174+720 $+\overline{1350}+\overline{2088}$ $+\overline{5568}$ $=\overline{7600}$	522+675+ 2880+2784 $+\overline{8340}+\overline{16704}$ $+\overline{10440}$ $=\overline{28623}$	2085+11136 +5400+8352 $+5220+\overline{31272}$ $+\overline{66720}+\overline{31320}$ $+\overline{20184}$ $=\overline{117303}$
		$3ab^2$	$b^3+3a.2bc$	$3b^2c+$ $3a.2bd+3a.c^2$	$3b^2d+3bc^2+3a.2be$ $+3a.2cd$	$c^3+3b^2e+6bcd$ $+3a.d^2+3a.2bf$ $+3a.2ce$	$3b^2f+3bd^2$ $+3c^2d+6bce$ $+3a.2bg+3a.2cf$ $+3a.2de$	$3b^2g+3c^2e+3cd^2$ $+6bcf+6bde$ $+3a.2bh+3a.2cg$ $+3a.2df+3a.e^2$
4	$\bar{1}$	$\bar{8}$	$\overline{15}$	$\overline{58}$	$\overline{174}$	$\overline{695}$	$\overline{2606}$	$\overline{10670}$
a	b	c	d	e	f	g	h	i

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto i (8 decimals) $x = 4.\dot{1}\ \bar{8}\ \overline{15}\ \overline{58}\ \overline{174}\ \overline{695}\ \overline{2606}\ \overline{10670} = 4.\ \bar{2}\ \bar{0}\ \bar{3}\ \bar{6}\ \bar{0}\ \bar{2}\ \bar{3}\ \bar{0} = 3.79639770$

$\therefore\ E = (x - 3.7963977)A$, A should contain x^2, x and constant terms

Applying Adyamadyena Antyamantyena and Argumentation

$E = (x - 3.79639770)(x^2 + \alpha x + 2.634075982)$ comparing the like terms

$2.634075982 - 3.79639770\alpha = 11$

$\alpha = -2.203647952$ $\therefore$ Second factor of $E = (x^2 - 2.203647952x + 2.634075982)$ is factorized using differential relation.

$f'(x) = \pm\sqrt{Discriminant}$

$2x - 2.203647952 = \pm\sqrt{4.856064296 - 10.53630393} = \pm 2.383325331i$

$x = 1.101823976 \pm 1.91662665i$

$\therefore\ E = (x - 3.796397770)(x - 1.10101826796 + 1.191662665i)(x - 1.101823976 - 1.191662665i)$

Applying Gunita Samucayah Sutram for verification

$-4 = -4.00004565 \sim -4$

3. $E = 2x^3 + 3x^2 + 3x + 1 = 0 \quad f(x) = 2x^3 + 3x^2 + 3x = -1$

Solution x = a.bcde .....

	LHS	RHS	RHS – LHS
x = 1	8	– 1	– 9
x = 2	34		– 35
x = – 1	– 2		1
x = – 3	– 36		35
x = – 0.5			0

From Vilokanam a = – 1

Let x = – 1 $\Rightarrow$ RHS – LHS = 1

CD = $3a^2$ Representation at x = – 1 $\Rightarrow$ $[2(3x^2) + 6x + 3]$ at x – 1 $\Rightarrow$ 3   effective $3a^2 = 3 = CD$

3a Representation at x = – 1 = $\frac{1}{2}$ [2(6x) + 6] at x = – 1 $\Rightarrow \bar{3}$ 3a = 3

	0	0	0	0	0
3	1	1	1	1	1
		27	$\bar{54}$+216=162	$\bar{648}$+1026+ 432=810	$\bar{3078}$ + $\bar{2592}$ + 4914+4104 =3348
		$3ab^2$	$2(b^3)$+3a.2bc	$2(3b^2c)$+3a.2bd +3a.c^2	$2(3b^2d)$+3bc^2 +3a.2be+3a.2c
– 1.	3	12	57	273	1119
a	b	c	d	e	f

Subtraction Terms

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto f (5 decimals) $x = \bar{1}.3\ 12\ 57\ 273\ 1119 = \bar{1}.51549 = 0.\bar{4}\bar{8}\bar{8}\bar{4}\bar{5}\bar{1}$

For the value of x, a reduction process is adopted by considering another variable 'z' as $x = \frac{z}{2}$ substituted. The value of x is determined and finally, the value obtained for x with $x = \frac{z}{2}$ is considered for finding out the roots. The working details are shown under Taylor's series section. However Swamiji's method also can be adopted for this substitution method as well.

By Substitution of $x = \frac{z}{2}$ the equation becomes

$E = \frac{2z^3}{8} + \frac{3z^2}{4} + \frac{3z}{2} + 1 = 0 \Rightarrow z^3 + 3z^2 + 6z + 4 = 0 \Rightarrow f(z) = z^3 + 3z^2 + 6z = 4 \qquad \Rightarrow z^3 + 3z^2 + 6z =$

z value	LHS	RHS – LHS
1	10	–14
2	32	–36
–1	–4	0

$\therefore z = -1 \Rightarrow x = \frac{z}{2} = -0.5$ is one solution $\qquad \Rightarrow (x + 0.5)$ is a factor of E

$\therefore E = (x + 0.5)A$, A should contain x^2, x and constant terms

By applying Adyamadyena Antyamantyena and Argumentation

$\therefore \quad E = (x + 0.5)(2x^2 + \alpha x + 2)$ comparing the coefficients of like terms

x coeff $= 2 + \frac{\alpha}{2} = 3; \quad \alpha = 2$

$E = (x + 0.5)(2x^2 + 2x + 2)$

Using $f(x) = \pm\sqrt{Discriminant}$

$2x+1 = \pm\sqrt{1-4} = \pm\sqrt{3}\,i \qquad x = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2} \qquad \therefore\ E = 2\left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\left(x+\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)$

Applying Gunita Samuccayah for final verification

$$9 = 2\left(\frac{3}{2}\right)\left(\frac{9}{4}+\frac{3}{4}\right) = 2\left(\frac{3}{2}\right)(3) = 9$$

$E = x^3 + 7x^2 + 9x + 11 = 0; \quad f(x) = x^3 + 7x^2 + 9x = -11$

Solution x = a.bcde .....

	LHS	RHS	RHS – LHS
x = 1	17	– 11	– 28
x = 2	54		65
x = 3	117		·128
x = – 1	– 3		– 8
x = – 2	2		– 13
x = – 3	9		– 20
x = – 4	12		– 23
x = – 5	5		– 16
x = – 6	– 18		7
x = – 7	– 63		52

From Vilokanam $a = -6$

Let $x = -6 \Rightarrow$ RHS – LHS = 7

CD = $3a^2$ Representation at $x = -6 \Rightarrow [3x^2 + 14x + 9]$ at $x = -6 \Rightarrow 33$

effective $3a^2 = 33 =$ CD

$3a = -11$

3a Representation at $x = -6 = \frac{1}{2}[6x + 14]$ at $x = -6 \Rightarrow -11$

	0	0	0	0	0	0	0	0
CD=33	7	4	18	29	24	11	4	18
		44	$\bar{8}+88=80$	$\overline{24}+308+44$ =328	$\overline{84}+\overline{24}+792$ +308=992	$\bar{8}+\overline{216}+\overline{168}+1628+792$ +539=2567	$\overline{444}+\overline{294}+\overline{84}+\overline{432}+3564+1628$ +2772=6710	
		$3ab^2$	$b^3+3a.2bc$	$3b^2c+3a.2bd$ $+3a.c^2$	$3b^2d+3bc^2$ $+3a.2be+3a.2cd$	$c^3+3b^2e+6bcd$ $+3a.2bf+3a.2ce+3a.d^2$	$3b^2f+3bd^2+3c^2d+6bce+3a.2bg$ $+3a.2cf+3a.2de$	
–6	2	2	7	18	37	81	204	
a	b	c	d	e	f	g	h	

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto h (7 decimals) $x = -6.2\ 27\ 18\ 37\ 81\ 204 = -6.2292714 = \bar{5}.\bar{7}\bar{7}\bar{0}\bar{7}\bar{2}\bar{8}\bar{6}$

$E = (x + 5.7707286)A$, A should contain x^2, x and constant terms.

Applying Adyamadyena Antyamantyena and Argumentation

$E = (x + 5.7707286)(x^2 + \alpha x + 1.906171779)$ comparing the like terms on both sides.

x coeff: $1.906171779 + 5.7707286\alpha = 9$

$\alpha = 1.229277742$

Considering factorisation of Quadratic expression

$(x^2 + 1.229277742x + 1.906171779)$

$f'(x) = \pm\sqrt{Discriminant}$

$2x + 1.229277742 = \pm\sqrt{1.51112376660 - 7.624687116} = \pm 2.4724562102i$

$x = -0.6146368871 \pm 1.236281051i$ $\quad \therefore E = (x + 5.7707286)(x + 0.614638871 + 1.236281051i)(x + 0.614638871 - 1.23628105$

By Gunita Samuccaya for final verification

$28 = 28.00000635 \sim 28$

15) $E = 2x^3 + 5x^2 + 6x - 164 = 0;$ $\quad f(x) = 2x^3 + 5x^2 + 6x = 164$

Solution x = a.bcde .....

	LHS	RHS	RHS – LHS
x = 1	13	164	151
x = 2	48		116
x = 3	117		47
x = 4	232		– 68

Frcm Vilokanam a = 4

Let $x = 4 \Rightarrow RHS - LHS = \overline{68}$

$CD = 3a^2$ Representation at $x = 4 \Rightarrow [2(3x^2) + 10x + 6]$ at $x = 4 \Rightarrow 142$

effective $= 3a^2 = 142 = CD$

$3a$ Representation at $x = 4 = \frac{1}{2}[2(6x) + 10]$ at $x = 4 \Rightarrow 29$

$3a = 29$

	b	c	d	e	f	g	h	
	0	0	0	0	0	0	0	0
142	$\overline{68}$	$\overline{112}$	$\overline{22}$	$\overline{88}$	$\overline{125}$	$\overline{2}$	$\overline{16}$	$\overline{112}$
		$\overline{464}$	$128+\overline{2552}$ $=\overline{2424}$	$1056+\overline{4176}$ $+\overline{3509}=\overline{6629}$	$1728+2904$ $+\overline{12064}+\overline{11484}$ $=\overline{18916}$	$2662+4992+9504$ $+\overline{32944}+\overline{33176}+9396$ $=\overline{58358}$	$13632+7776+13068$ $+27456+\overline{95352}+\overline{90596}$ $+\overline{54288}=\overline{178304}$	
		$3ab^2$	$2(b^3)+3a.2bc$	$2(3b^2c)+3a.2bd$ $+3a.c^2$	$2[3b^2d+3bc^2]$ $+3a.2be+3a.2cd$	$2(c^3)+2(3b^2e)+2(6bcd)$ $+3a.2bf+3a.2ce+3a.d^2$	$2(3b^2f)+2(3bd^2)+2(3c^2d)$ $+2(6bce)+3a.2bg+3a.2cf$ $+3a.2de$	Subtraction Terms
							$\overline{1256}$	
4	$\overline{4}$	$\overline{11}$	$\overline{18}$	$\overline{52}$	$\overline{142}$	$\overline{411}$	h	
a	b	c	d	e	f	g		

Upto h (7 decimals) $x = 4.\overline{4}\ \overline{11}\ \overline{18}\ \overline{52}\ \overline{142}\ \overline{411}\ \overline{1256} = 4.\overline{5}\ \overline{3}\ \overline{5}\ \overline{1}\ \overline{5}\ \overline{6}\ \overline{6} = 3.4648434$

For the value of x, a reduction process is adopted by considering another variable 'z' as $x = \frac{z}{2}$ substituted. The value of x is determined.

Finally the value obtained for x with $x = \frac{z}{2}$ is considered for finding out the roots. The working details are shown under Taylors series section. However Swamiji's method can also be adopted for this substitution method as well.

$E = 2x^3 + 5x^2 + 6x - 164 = 0$

Let $x = \frac{z}{2} \Rightarrow E = \frac{2z^3}{8} + \frac{5z^2}{4} + \frac{6z}{2} - 164 = 0 \quad \Rightarrow f(z) = z^3 + 5z^2 + 12z = 656$

z value	f(z)	RHS – LHS
6	468	188
7	672	– 16

From Vilokanam a = 7

CD = $3a^2$ representation at z = 7 = $(3z^2 + 10z + 12)$ at z = 7 $\Rightarrow$ 229

3a representation at z = 7 = $\frac{1}{2}(6z + 10)$ at z = 7 $\Rightarrow$ 26

	0	0	0	0	0	0	0
CD = 229	$\overline{16}$	$\overline{160}$	$\overline{226}$	$\overline{199}$	$\overline{178}$	$\bar{8}$	$\overline{218}$
		0	0	$0+0+\overline{936} = \overline{936}$	$0+0+0+\overline{2808} = \overline{2808}$	$216+0+0+0 + \overline{3744} + \overline{2106} = \overline{5634}$	$0+0+972+0+0+\overline{6240} + \overline{5616} = \overline{10884}$
		$3ab^2$	$b^3 + 3a.2bc$	$3b^2c + 3a.2bd + 3a.c^2$	$3b^2d + 3bc^2 + 3a.2be + 3a.2cd$	$c^3 + 3b^2e + 6bcd + 3a.2bf + 3a.2ce + 3a.d^2$	$3b^2f + 3bd^2 + 3c^2d + 6bce + 3a.2bg + 3a.2cf + 3a.2de$
7	0	$\bar{6}$	$\bar{9}$	$\overline{12}$	$\overline{20}$	$\overline{24}$	$\overline{57}$
a	b	c	d	e	f	g	g

Subtraction Terms

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD

2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms

3) Values of subtraction terms containing 3a a;e to be worked out in terms of representation value 3a.

Upto h (7 decimals), z = $7.0\,\bar{6}\,\bar{9}\,\overline{12}\,\overline{20}\,\overline{24}\,\overline{57} = 7.0\,\bar{7}\,0\,\bar{4}\,\bar{2}\,\bar{9}\,\bar{7} = 6.9295703$

$x = \frac{z}{2} = 3.46478515$

$\therefore$ E = (x – 3.46478515)A. A should contain x^2, x and constant terms.

By Adyamadyena Antyamantyena and Argumentation

$E = (x - 3.46478515)(2x^2 + \alpha x + 47.33338227)$ comparing the like terms

$47.33338227 - 3.46478515\alpha = 6$

$\alpha = 11.92956575$

$E = (x - 3.46478515)\, 2(x^2 + 5.964782877x + 23.66669114)$

The Quadratic Equation is factorized

$2x + 5.964782877 = \pm\sqrt{35.57863477 - 94.66676454}$

$x = -2.982391439 \pm 3.843440184i$

$\therefore E = 2(x - 3.46478515)(x + 2.982391439 + 3.843440184i)$

$(x + 2.982391439 - 3.843440184i)$

Applying Gunita Samuccaya for final verification

$-151 \doteq -151.0000046 \simeq 151$

6) $x^3 + 3.78x^2 - 0.01x + 2.26 = 0$

Solution x = a.bcde .....

$\Rightarrow 100x^3 + 378x^2 - x + 226 = 0 \Rightarrow 100x^3 + 378x^2 - x = -226$

	LHS	RHS	RHS – LHS
x = – 1	279	– 226	– 505
x = – 2	714	"	– 940
x = – 3	705	"	– 931
x = – 4	– 348	"	122

From Vilokanam a = – 4

Let x = – 4 $\Rightarrow$ RHS – LHS 22

CD = $3a^2$ Representation at x : 4 $\Rightarrow$ $3x^2) + 756x$ at $x = -4 \Rightarrow 1775$

effective = $3a^2$ 775 CD

3a Representation at $x = -4 = \frac{1}{2}\,[100(6x) + 756]$ at $x = -4 \Rightarrow \overline{822}$ $\qquad 3a = \overline{822}$

	0	0	0	0	0	0	
CD=1775	122	1220	1550	1300	1767	732	
		0	0	0+0+29592=29592	0+0+0+78912=78912		Subtraction Terms
		$3ab^2$	$100(b^3)+3a.2bc$	$100(3b^2c)+3a.2bd+3a.c^2$	$100(3b^2d)+100(3bc^2)$ $+3a.2be+3a.2cd$		
−4	0	6	8	23	54		
a	b	c	d	e	f		

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto f (5 decimals) $\quad x = \bar{4}.0682354 = \bar{4}.07084 = \bar{3}.\bar{9}\bar{2}\bar{9}\bar{1}\bar{6}$

For the values of x, a reduction process is adopted by considering another variable 'z' as $x = \frac{z}{2}, \frac{z}{10}$ at a time. The value of x is determined and finally the value obtained for x with $x = \frac{z}{10}$ is considered for finding out the roots. The working details are shown under Taylor's series section However Swamiji's method also can be adopted for this substitution method as well

$E = 100x^3 + 378x^2 - x + 226 = 0$

Let $x = \frac{z}{10} \Rightarrow \frac{100z^3}{1000} + \frac{378z^2}{100} - \frac{z}{10} + 226 = 0 \Rightarrow 100z^3 + 3780z^2 - 100z = -226000$

$\Rightarrow 10z^3 + 378z^2 - 10z = -22600$

Solution x = a.bcde ...

Z value	f(z)	RHS – LHS
– 39	– 17862	– 4738
– 40	– 34800	12200

From Vilokanam a = – 39

CD = $3a^2$ representation at z = – 39 = $(30z^2 + 756z - 10)$ at z = – 39 ⇒ 16136

3a representation at z = – 39 = $\frac{1}{2}[60z + 756]$ at x = – 39 ⇒ 792

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing 3a are to be worked out in terms of representation value 3a.

Upto e (4 decimals), $z = -39.\bar{2}\,\bar{9}\,0\,5 = \overline{39}.\bar{2}\,\bar{8}\,\bar{9}\,\bar{5}$ $\quad x = \frac{z}{10} = -3.92895$ ∴ $(x + 3.92895)$ is a factor of E

∴ $E = (x + 3.92895)A$. A should have x^2, x and constant terms

Applying Adyamadyena Antyamantyena and Argumentation

∴ $E = (x + 3.92895)(100x^2 + \alpha x + 57.52172972)$ comparing the like terms

By comparing x coeff $57.52172972 + 3.92895\alpha = -1$ $\alpha = -14.89500496$

$E = (x + 3.92895)(100x^2 - 14.89500496x + 57.52172972)$

$2x - 0.1489500496 = \pm\sqrt{0.022186117 - 2.300869189} = \pm 1.509530746i$

$x = 0.074475024 \pm 0.754765372i$

$\therefore E = 100(x+3.92895)(x-0.074475024+0.754765372i)(x-0.074475024-0.754765372i)$

Applying Gunita Samuccayah Sutram for final verification

$703 = 702.9999951 \sim 703$

Note: Similar procedure is workable for any higher order equation provided the corresponding Common Divisors from first differential of the respective equations.

1) In case of 4^{th} degree equation the respective subtraction terms wherein $6a^2$ representation and $4a$ representation as derived from second and third derivatives respectively
2) In case of 5^{th} degree equation the respective subtraction terms wherein $10a^3$ representation $10a^2$ representation and $5a$ representation as derived from second, third, fourth derivatives respectively.
3) In case of 7^{th} degree equation the respective subtraction terms wherein $21a^5$ representation $35a^4$ representation $35a^3$ representation $21a^2$ representation and $7a$ representation as derived from second, third, fourth, fifth and sixth derivatives respectively.

For full working details refer Part II, Section J

Section – D Solution of Cubic and Higher Order Equations Using Taylor's Expansion Method

Cubic Equations

One can obtain the roots of a cubic equation by adopting the procedure for finding out the cube roots of numbers which is described in the Section B as the procedure adopted is direct digit by digit solution.

Let us consider a cubic equation

$E = x^3 - 3x^2 + 17x - 52 = 0 \Rightarrow f(x) = x^3 - 3x^2 + 17x = 52$

One of the solutions can be similarly written as x = a.bcde.........

Step 1: a can be identified as equal to 3 since f(a) = 51 which is the nearest value.

Difference between RHS – LHS = 1 ⇒ first dividend is 10

Step 2: $f^1(a) = 3a^2 - 6a + 17 = 26$ (common Divisor CD)

$\frac{1}{2}f''(a) = \frac{1}{2}(6a-6) = 6$

$\frac{1}{6}f'''(a) = \frac{6}{6} = 1$

the working details are

$f^1(a) =$	26	52.	1^0	10^0	22^0	12^0	14^0
$\frac{1}{2}f^{11}(a) =$	6			$\bar{0}$	$\bar{0}$	$\overline{54}$	$\overline{288}$
$\frac{1}{6}f^{111}(a) =$	1				$\bar{0}$	$\bar{0}$	$\bar{0}$
				3	8	2	$\bar{5}$
		a	b	c	d	e	f

Step 3: Dividing the first dividend i.e. 10 by 26, the quotient b = 0 and remainder is 10, leading to the intermediate dividend 100.

Step 4: Subtracting $\frac{1}{2}f^{11}(a)b^2$ from 100, one gets 100 as new dividend

Step 5: Dividing 100 by the divisor 26, quotient c = 3 and the remainder is 22, leading to the intermediate dividend 220.

Step 6: Subtracting $\frac{1}{2}$ $f^{11}(a)$ 2bc + $\frac{1}{6}f^{111}(a)$. b^3 from 220, we get 220 as new dividend.

Step 7: Dividing 220 by 26, we get the quotient d = 8 and the remainder 12, leading to the intermediate dividend 120.

Step 8: Subtracting $\frac{1}{2}$ $f^{11}(a)$ $(2bd+c^2)$ + $\frac{1}{6}$ $f^{111}(a)$. $3b^2c$ from 120, we get 66 as the new dividend.

Step 9: Dividing 66 by 26, the quotient e = 2 and 14 is the remainder, leading to intermediate dividend as 140.

Step 10: Subtracting $\frac{1}{2}$ $f^{11}(a)$ (2be+2cd) + $\frac{1}{6}f^{111}(a)$ $(3bc^2+3b^2d)$ from 140, we get −148 as new dividend.

Step 11: Dividing − 148 by 26 ; we get the quotient $f = \bar{5}$ and remainder as $\overline{18}$

The solution can be simply written as

$$x = 3.0382\bar{5} = 3.03815$$

After getting the first solution one obtains as the other two solutions by dividing the given cubic equation by the first factor i.e. (x − 3.03815) one can follow the straight division, method also obtain the quadratic expression as

$x^2 + 0.03815x + 17.11591$

writing the factor as

$(x-3.03815)(x^2+0.03815x+17.11591) = 0$

is the given cubic equation

Solving the quadratic equation, one gets $x = \pm 4.1375i - 0.019075$

Now the three solutions of given cubic equation satisfy the given cubic equation,

(2) When duplex and triplex corrections (subtraction terms) become large when compared with the common divisor, one can transform the given equations suitably into another variable z. so that z = nx. The derived equation will have a constant larger than that of the given equation. The procedure adopted is the same as in previous working. After obtaining the value of z, one can obtain the value of x. This is called refinement.

(3) One can extend this method for directly solving, in general higher−order equations as well. The method applied to cubic equation can be generalized for this also, where one has to go for the corresponding derivative, derivable for the function, (Refer Table M).

The problems worked – out using the Swamiji's method are attempted using Taylor's Expansion Method

1) $E = 2.3x^3 + 8.7x^2 - 0.01x = 5.2$

$230x^3 + 870x^2 - x = 520$

Solution x = a.bcde .....

$f(x) = 230x^3 + 870x^2 - x = 520$

The solutions lie between 0.5 and 1; – 0.8 and – 1; –3 and – 4.

Refer details of Vilokanam Method, shown in section

From this it is clear that one can try for the three values, for the ranges shown above.

$f(x) = 230x^3 + 870x^2 - x = 520$

$f'(x) = 690x^2 + 1740x - 1$

$f''(x) = 1380x + 1740$

$f'''(x) = 1380$

Taking x = – 1

From Vilokanam a = – 1

$f'(-1) \quad = -1051$

$\frac{1}{2}f''(-1) = 180$

$\frac{1}{6}f'''(-1) = 230$

x = a. b c d e f ....

		0	0	0	0	0	0	0	0
$CD = f'(-1)$	$\overline{1051}$	$\overline{121}$	$\overline{159}$	$\overline{719}$	$\overline{423}$	$\overline{263}$	$\overline{578}$	$\overline{426}$	$\overline{450}$
$\frac{1}{2}f''(-1)$	180		$\overline{180}$	$\overline{360}$	$\overline{2700}$	$\overline{5040}$	$\overline{15660}$	$\overline{34200}$	
$\frac{1}{6}f'''(-1)$	230			$\overline{230}$	$\overline{690}$	$\overline{5520}$	$\overline{14720}$	$\overline{56580}$	
	– 1	1	1	7	7	12	34	90	
	a	b	c	d	e	f	g	h	

Step 1 : $R = \overline{121}$

$10R = \overline{1210}$

$\overline{1051}$) $\overline{1210}$ (1

$\overline{1051}$

$\overline{159}$

$Q = b = 1, \quad R = \overline{159}$

Step 2 : $180(b^2) = 180(1^2) = 180 \rightarrow \overline{180} \rightarrow$ (i)

$10R + (i) = \overline{1770}$

$(\overline{1770}) \div \overline{1051}$

$q = c = 1 \quad R = \overline{719}$

Step 3 : $180(2bc) = 360 \longrightarrow \overline{360}$ (i)

(d) $(230b^3) = 230 \longrightarrow \overline{230}$ (ii)

$10R + (i) + (ii) \longrightarrow \overline{7780}$

$\overline{7780} \div \overline{1051}$

$q = d = 7, \quad R = \overline{423}$

Step 4 : $180(2bd + c^2) = 2700 \longrightarrow \overline{2700}$ (i)

(e) $230(3b^2c) = 690 \longrightarrow \overline{690}$ (ii)

$10R + (i) + (ii) \longrightarrow \overline{7620}$

$\overline{7620} \div \overline{1051}$

$q = e = 7, \quad R = \overline{263}$

Step 5 : $180(2be + 2cd) = 5040 \longrightarrow \overline{5040}$ (i)

(f) $230(3b^2d + 3bc^2) = 5520 \longrightarrow \overline{5520}$ (ii)

$10R + (i) + (ii) \longrightarrow \overline{13190}$

$\overline{13190} \div \overline{1051}$

$q = f = 12, \quad R = \overline{578}$

Step 6 : $180(2bf + 2ce + d^2) = 15660$

(g) $\therefore \overline{15660} \longrightarrow$ (i)

$230(3b^2e + 6bcd + c^3) = 14720$

$\overline{14720} \longrightarrow$ (ii)

$10R + (i) + (ii) \longrightarrow \overline{36160}$

$\overline{36160} \div \overline{1051}$

$q = g = 34, \quad R = \overline{426}$

Step 7 : $180(2bg + 2cf + 2de) = 34200$

(h) $\therefore \overline{34200} \longrightarrow$ (i)

$230(3b^2f + 6bce + 3bd^2 + 3c^2d) = 56580$

$\overline{56580} \longrightarrow$ (ii)

$10R + (i) + (ii) \longrightarrow \overline{95040}$

$\overline{95040} \div \overline{1051}$

$q = h = 90, \quad R = \overline{450}$

a	b	c	d	e	f	g	h
− 1.	1	1	7	7	12	34	90
− 1.	1	1	7	8	6	3	0
0.	$\overline{8}$	$\overline{8}$	$\overline{2}$	$\overline{1}$	$\overline{3}$	$\overline{7}$	0

$x = \overline{0.8821370}$ $\qquad E \simeq -0.0000307327$

<u>Trial With x = − 4</u> to confirm the value in the interval (− 3, − 4)

$E = 2.3x^3 + 87x^2 - 0.01x = 5.2$

$= 230x^3 + 870x^2 - x = 520$

$f(x) = 230x^3 + 870x^2 - x$

From Vilokanam a = − 4

$f'(-4) = 690x^2 + 1740x - 1 \longrightarrow 4079$

$\frac{1}{2}f''(-4) = 1380x + 1740 \longrightarrow -1890$

$\frac{1}{6}f'''(-4) = 1380 \longrightarrow 230$

	0	0	0	0	0	0	0
CD = $f'(a)$ 4079	1316	923	1766	1989	995	2421	3167
$\frac{1}{2}f''(a) = -1890$		17010	68040	283500	1168020	4889430	
$\frac{1}{6}f'''(a) = 230$			$\overline{6210}$	$\overline{37260}$	$\overline{192510}$	$\overline{925290}$	
– 4.	3	6	19	65	241	977	
a	b	c	d	e	f	g	

Upto 'f' –3.612209 E = – 5.0093775

Upto 'g' – 3.611113 E = – 2.0335888

Applying substitution method, another solution is attempted $x = \frac{z}{10}$

z = – 36 f(z) = 547920 Dif – 27920

z = – 37 f(z) = 263810 Dif + 256190

$\frac{230z^3}{1000} + \frac{870z^2}{100} + \frac{z}{10} = 520$

$230z^3 + 8700z^2 - 100z = 52000$

From Vilokanam a = – 36

$f'(-36) = 690z^2 + 17400z - 100 = 267740$

$\frac{1}{2}f''(x) = 1380z + 17400 = -32280 = -16140$

$\frac{1}{6}f'''(x) = \frac{\quad}{6} = 230$

	0	0	0	0	0
CD=$f'(a)$ 267740	– 29920	– 11460	– 18460	– 181150	– 108220
$\frac{1}{2}f''(a) = -16140$		16140	0	96840	193680
$\frac{1}{6}f'''(a) = 230$			230	0	2070
– 36	– 1	0	– 3	– 6	– 3
a.	b	c	d	e	f

z= – 36.10363 ; x = – 3.610363 E = 0.0005176

if x = – 1 – 3.610344202 is one solution if $x = -4$ -3.611113 which is further refined

with substitution of $x = \frac{z}{10}$

$x = -3.610363$

2) $E = x^3 - 7x^2 + 20x - 37 = 0$

$f(x) = x^3 - 7x^2 + 20x = 37$

Solution x = a.bcde .....

The solution lies between 4 and 5;

Refer details of Vilokanam Method, shown in section

From Vilokanam a = 4

$f'(x) = 3x^2 - 14x + 20$ $\quad f'(4) = 12$

$f''(x) = 6x - 14$ $\quad \frac{1}{2}f''(4) = 5$

$f'''(x) = 6$ $\quad \frac{1}{6}f'''(4) = 1$

		0	0	0	0	0	0	0
$CD = f'(4)$	12	5	2	0	4	$\bar{9}$	0	$\bar{2}$
$\frac{1}{2}f''(4) = 5$	5		$\overline{80}$	200	$\overline{565}$	1470	$\overline{3595}$	
$\frac{1}{6}f'''(4) = 1$	1			$\overline{64}$	240	$\overline{828}$	2549	
	4.	4	$\bar{5}$	11	$\overline{23}$	46	$\overline{87}$	
	a	b	c	d	e	f	g	

$x = 4.44\bar{4}\bar{1}16$ $\quad \Rightarrow x = 4.35916$ (upto 5 decimals) $\quad E = 0.0012296977$

The value lies between x_1 and x_2 comparable with respect to x_1 and x_2

$x_1 = 4.35916$ $\quad E = 0.00122$

$x_2 = 4.358899$ $\quad E = -0.002$

The value lies between x_1 and x_2 comparatable w.r.t. x_1 or x_2

$\therefore$ $x = 4.35916$

3) $E = 2x^3 + 9x^2 - 18x + 35 = 0$

$f(x) = 2x^3 + 9x^2 - 18x = -35$

Solution x = a.bcde .....

The solution lies between -6 and -7

From Vilokanam $a = -6$

$f'(x) = 6x^2 + 18x - 18$

$f''(x) = 12x + 18$

$f'''(x) = 12$

	0	0	0	0	0	0	0
CD = $f'(a)$ 90	$\overline{35}$	$\overline{80}$	$\overline{17}$	46	28	$\overline{80}$	55
$\frac{1}{2}f''(a) = \overline{27}$		243	972	$\overline{486}$	$\overline{3402}$	6399	
$\frac{1}{6}f'''(a) = 2$			54	324	162	$\overline{1674}$	
$-6.$	$\overline{3}$	$\overline{6}$	9	3	$\overline{32}$	43	
a	b	c	d	e	f	g	

x = 6 . 3 6 9 0 2   x = 6 . 3 5 1 0 2

$\therefore\ x_1 = -6.35102$ (upto decimals) $\quad E = -0.0051079$

$x_2 = 6.350977$ (upto 6 decimal) $\quad E = -0.000391173$

4) $E = 2x^3 - 15x^2 - 8x + 166 = 0$

$2x^3 - 15x^2 - 8x = -166$

Solution x = a.bcde .....

The solutions lie between 5 and 6; -3 and -4

Refer details of Vilokanam Method, shown in section

Multiple substitutions are attempted

From Vilokanam a = 5

$f(x) \quad = 2x^3 - 15x^2 - 8x$

$f'(x) \quad = 6x^2 - 30x - 8$

$f'(5) \quad = 6 \times 25 - 150 - 8 = -8$

$\frac{1}{2}f''(x) \quad = 12x - 30 = \frac{1}{2}(60 - 30) = 15 \qquad \frac{1}{6}f'''(x) = 12 = \frac{1}{6}(12) = 2$

	0	0	0	0	0	0
CD = $f'(5)$ -8	-1	-2	$\overline{3}$	0	$\overline{2}$	$\overline{6}$
$\frac{1}{2}f''(5)$ 15		$\overline{15}$	$\overline{120}$	$\overline{810}$	$\overline{5400}$	
$\frac{1}{6}f'''(5)$ 2			$\overline{2}$	$\overline{24}$	$\overline{210}$	
5	1	4	19	104	703	
a	b	c	d	c	f	

x = 5.17643   E = 0.0664568

Let $x = \frac{z}{2}$

Then the equation is

$$\frac{2z^3}{8} - \frac{15z^2}{4} - \frac{8z}{2} + 166 = 0$$

$2z^3 - 30z^2 - 32z + 1328 = 0$

$z^3 - 15z^2 - 16z + 664 = 0$

$f'(z) \quad = 3z^2 - 30z - 16$

$f'(10) \quad = 300 - 300 - 16 = -16$

$f''(z) \quad = 6z - 30 \qquad f''(10) = \frac{1}{2}(60 - 30) = 15$

$f'''(z) \quad = 6 \qquad f'''(10) = \frac{1}{6}(1) = 1$

From Vilokanam a = 10 (a)

$(10)^3 - 15(10^2) - 160 = -664$

$1000 - 1500 - 160 \qquad = -664$

variation is – 4	RHS	RHS – LHS
f(a) = – 660	– 664	– 4

	0	0	0	0	0	0
CD = f′(10) – 16	– 4	$\bar{8}$	$\overline{12}$	0	$\bar{8}$	$\bar{8}$
$\frac{1}{2}f''(10)$ 15		$\overline{60}$	$\overline{480}$	$\overline{3240}$	$\overline{21600}$	
$\frac{1}{6}f'''(10)$ 1			$\bar{8}$	$\overline{96}$	$\overline{840}$	
10	2	8	38	208	1407	
a	b	c	d	e	f	g

z = 10.35287

⇒ x = 5.176435

5.176435 E = 0.066444 = 38

Let $x = \frac{z}{5}$

$$2\frac{z^3}{125} - 15\frac{z^2}{25} - \frac{8z}{5} + 166 = 0 \qquad f(x) = 2z^3 - 75z^2 - 200z = -20750$$

	LHS f(x)	RHS	RHS – LHS
z = 25	– 20625	– 20750	– 125
z = 26	– 20748		– 2
z = 27	– 20709		41

$f(z) = 2z^3 - 75z^2 - 200z + 20750 = 0$ $\qquad f'(z) = 6z^2 - 150z - 200$

Let a = 26

$\frac{f'(26)}{1!} = -44, \quad f''(z) = 12z - 150, \quad \frac{f''(26)}{2!} = 81, \quad f'''(z) = 12, \quad \frac{f'''(26)}{3!} = 2$

		0	0	0	0	0	0
CD $f'(a) = -44$		$\overline{2}$	$\overline{20}$	$\overline{24}$	$\overline{20}$	0	$\overline{28}$
$\frac{1}{2}f''(a) = 81$		–	0	0	$\overline{1296}$	$\overline{3240}$	
$\frac{1}{6}f'''(a) = 2$				0	0	0	
	26.	0	4	5	34	73	
	a	b	c	d	e	f	

z = 26.04913 $\qquad$ x = 5.209826

x = 5.209826 $\qquad$ E = 0.00027225

Let $x = \frac{}{10}$

$\frac{2z^3}{1000} - \frac{15z^2}{100} - \frac{8z}{10} = -166$ $\qquad f' = 3z^2 - 150z - 400$

$2z^3 - 150z^2 - 800z = -166000$ $\qquad f'' = 6z - 150$

$z^3 - 75z^2 - 400z = -83000$ $\qquad f''' = 6$

	LHS	RHS f(x)	RHS – LHS
z = 52	– 82992	– 83000	– 8
z = 53	– 8.2998		– 2
z = 54	– 82836		– 164

		0	0	0	0	0	0
CD = $f'(a)$ 77		$\overline{2}$	$\overline{20}$	$\overline{46}$	$\overline{75}$	$\overline{8}$	$\overline{66}$
$\frac{1}{2}f''(a) = 84$		–	–	–	$\overline{336}$	$\overline{1690}$	
$\frac{1}{6}f''(a) = 1$		–	–	–	–	–	
	53.	0	$\overline{2}$	$\overline{5}$	$\overline{14}$	$\overline{22}$	
	a	b	c	d	e	f	

$53.0\overline{2}\,\overline{6}\,\overline{6}\,\overline{2}$

z = 52.97338

x = 5.297338 $\qquad$ E = 0.00001953

Let $x = \frac{}{100}$

$$\frac{2z^3}{1000000} \quad \frac{15z^2}{10000} \quad \frac{8z}{100} = -166$$

$2z^3 - 1500z^2 - 80000z = -166000000$

$z^3 - 750z^2 - 40000z = -83000000$

	f(z)	Diff
-z = 528	– 83010048	+ 10048
z = 529	– 83004861	+ 4861 –
-z = 530	– 82998000	– 2000 –

$f'(z) = 3z^2 - 1500z - 40000$

$f''(z) = 6z - 1500$

$f'''(z) = 6$

		0	0	0	0	0
$f'(a)$ 7700		$\overline{2000}$	$\overline{4600}$	$\overline{3160}$	$\overline{5552}$	$\overline{5748}$
$\frac{1}{2}f''(a) = 840$			$\overline{3360}$	$\overline{20160}$	$\overline{50400}$	
$\frac{1}{6}f''' = (a) = 1$				8	72	
	530.	$\overline{2}$	$\overline{6}$	$\overline{6}$	$\overline{13}$	
	a	b	c	d	e	

$\Rightarrow x = 5.297327$ E = 0.000003576

2[nd] factor $2x^3 - 15x - 8x = -166$

		0	0	0	0	0	0	0	0	0
CD= $f'(a)$ 136		$\overline{1}$	$\overline{10}$	$\overline{100}$	$\overline{48}$	$\overline{72}$	$\overline{40}$	129	92	127
$\frac{1}{2}f''(a) = -33$			0	–	–	–	1617	1386	2607	
$\frac{1}{6}f'''(a) = 2$			–	–	–	–	–	0	0	
	– 3	0	0	$\overline{7}$	$\overline{3}$	$\overline{5}$	8	19	25	
	a.	b	c	d	E	f	g	h	I	

$x = -3.00733985 = 0.000001787$

5) $E = x^3 - 3x + 1 = 0$

$f(x) = x^3 - 3x = -1$

Solution x = a.bcde .....

The solutions lie between 1 and 2; – 1 and – 2

Refer details of Vilokanam Method, shown in section

From Vilokanam a = – 2

$f'(x) = 3x^2 - 3$, $f''(x) = 6x$, $f'''(x) = 6$

Let a = – 2

			0	0	0	0	0	0	0
CD= $f'(a)$ 9		1	1	7	0	3	0	2
$\frac{1}{2}f''(a) = -6$			6	12	114	252	966	
$\frac{1}{6}f'''(a) = 1$				$\overline{1}$	$\overline{3}$	$\overline{30}$	$\overline{91}$	
	– 2	1	1	9	12	28	97	
	a	b	c	d	e	f	g	

2.120577

$\overline{1}.\overline{8}\,\overline{7}\,\overline{9}\,\overline{4}\,\overline{2}\,\overline{3}$

x = – 1.879423 ⇒ E = – 0.00028683113

for second factor

a = 2

		0	0	0	0	0	0	0	0
		$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$
CD = $f'(2) = 9$									
$\frac{1}{2}f''(2) = 6$			$\overline{54}$	$\overline{324}$	$\overline{1782}$	$\overline{10152}$	$\overline{69400}$	$\overline{373096}$	
$\frac{1}{6}f'''(2) = 1$				27	243	1701	11259	74034	
	2.	$\overline{3}$	$\overline{9}$	$\overline{36}$	$\overline{174}$	$\overline{942}$	$\overline{5472}$	$\overline{33321}$	
	a	b	c	d	e	f	g	h	

1.5566 0.115985966

1.541708 0.039305561

upto h: 1.5383759 ⇒ E = 0.02559335

$E = x^3 - 3x + 1$

Let $x = \frac{z}{2}$

$\frac{z^3}{8} - \frac{-3z}{2} + 1 = 0$

$z^3 - 12z + 8 = 0$

$f(z) = z^3 - 12z = -8$

$a = 3 \quad \Rightarrow \quad RHS - LHS = 1$

From Vilokanam a = 3

$f'(z) = 3z^2 - 12 \qquad f'(3) = 15$

$f''(z) = 6z \qquad \frac{1}{2}f''(3) = 9$

$f'''(z) = 6 \qquad \frac{1}{6}f'''(3) = 1$

	1^0	10^0	10^0	10^0	$\overline{14}^0$	$\bar{8}^0$
CD=f′=15						
$\frac{1}{2}f''(a)=9$		0	0	$\overline{324}$	$\overline{648}$	
$\frac{1}{6}f'''(a)=1$			0	0	0	
3.	0	6	6	$\overline{14}$	$\overline{52}$	
a	b	c	d	e	f	

$3.065\bar{9}\,\bar{2}$

$3.06408 \qquad x = \frac{z}{2} = \frac{3.06408}{2}$

$x = 1.53204 \quad E = -0.00019758177$

6) $x^3 + 6x^2 - 11x - 8 = 0$

$f(x) = x^3 + 6x^2 - 11x = 8$

Solution x = a.bcde .....

The solution lies between 1 and 2

Refer details of Vilokanam Method, shown in section

From Vilokanam a = 2

$\therefore$ a = 2 can be considered

$f'(x) = 3x^2 + 12x - 11 \qquad f'(2) = 25$, the common divisor

$f''(x) = 6x + 12$; $\frac{1}{2} f''(2) = 12$

$f'''(x) = 6$ $\frac{1}{6} f'''(2) = 1$

	0	0	0	0	0	0	0	0	0	0
CD=$f'(a)$=25	$\overline{2}$	$\overline{20}$	0	0	$\overline{18}$	$\overline{5}$	$\overline{23}$	$\overline{24}$	$\overline{17}$	$\overline{20}$
$\frac{1}{2}f''(a)$=12		0	0	$\overline{768}$	0	$\overline{5760}$	$\overline{1344}$	$\overline{51312}$	$\overline{16944}$	$\overline{504060}$
$\frac{1}{6}f''(a)$=1			0	0	0	512	0	5760	1344	62112
a=2.	0	$\overline{8}$	0	$\overline{30}$	$\overline{7}$	$\overline{211}$	$\overline{62}$	$\overline{1831}$	$\overline{630}$	$\overline{17685}$
	b	c	d	e	f	g	h	i	j	k

RHS – f(a) = 8 – f(2) = $\overline{2}$ is the first remainder

Step 2 : To find 'b'

$\overline{20} + 25 \Rightarrow b = 0;\ R = \overline{20}$

Step 3 : To find 'c'

$\overline{200} - 12(b^2) = \overline{200} - 0 = \overline{200}$

$\overline{200} \div 25 \Rightarrow c = \overline{8};\ R = 0$

Step 4 : To find 'd'

$0 - 12(2bc) - 1(b^3) = 0$

$0 \div 25 \Rightarrow d = 0;\ R = 0$

Step 5 : To find 'e'

$0 - 12(2bd + c^2) - 1(3b^2c) =$

$0 - 12\,(64) - 0 = -768$

$\overline{768} \div 25 \Rightarrow e = \overline{30};\ R = \overline{18}$

Step 6: To find 'f'

$\overline{180} - 12(2be + 2cd) -$
$1(3b^2d + 3bc^2)$

$= \overline{180} - 12(0+0) - 1(0) = \overline{180}$

$\overline{180} \div 25 \Rightarrow f = \overline{7};\ R = \overline{5}$

Step 7 : To find 'g'

$\overline{50} - 12(2bf+2ce+d^2) -$
$1(3b^2e+6bcd+c^3)$

$= \overline{50} - 12(2\times\overline{8}\times\overline{30}) - 1((\overline{8})^3)$

$= \overline{5298}$

$= \overline{50} - 5760 + 512 = \overline{5298}$

$\overline{5298} \div 25 \Rightarrow g = \overline{211};\ R = \overline{23}$

Step 8 : To find 'h'

$\overline{230} - 12(2bg + 2cf + 2de) -$
$1(3b^2f + 6bce + 3bd^2 + 3c^2d)$

$= \overline{230} - 12\,(2\times\overline{8}\times\overline{7}) - 1(0)$

$= \overline{230} - 1344 = \overline{1574}$

$\overline{1574} \div 25 \Rightarrow h = \overline{62};\ R = \overline{24}$

Step 9 : To find 'i'

$\overline{240} - 12(2bh + 2cg + 2df + e^2)$
$- 1(3b^2g + 6bcf + 6bde +$
$3c^2e + 3cd^2)$

$= 45792 = 45792 \div 25$

$i = 1831;\quad R = 17$

Step 10 To find 'j'

$\overline{170} - 12(2bi + 2ch + 2dg + 2cf)$
$- 1(3b^2h + 6bcg + 6bdf +$
$3be^2 + 3c^2f + 6cde + d^3)$

$= \overline{170} - 12$
$(2\times\overline{8}\times\overline{62} + 2\times\overline{30}\times\overline{7})$
$- 1\,(3\times(\overline{8})^2\times\overline{7} + 0)$

$= \overline{15770}$

$= \overline{15770} \div 25 \Rightarrow j = \overline{630};\ R = \overline{20}$

Step 11 : To find 'k'

$\overline{200}$ - 12 (2bj + 2ci + 2dh + 2eg + f^2)

- 1 ($3b^2i$ + 6bch + 6bdg + 6bef + $3c^2g$ + 6cdg + $3ce^2$ + $3d^2e$)

$\overline{200}$ -12(0+2x$\bar{8}$x$\overline{1831}$+0+2x$\overline{30}$x$\overline{211}$+49)

–(0+0+0+ 0+3x64x$\overline{211}$+3x$\bar{8}$x900+0)

- 1[(3x$\bar{8}$x$\bar{8}$x$\overline{211}$)+(6x$\bar{8}$x$\bar{8}$x$\overline{30}$)+(3x0x$\overline{30}$)] = $\overline{442148}$

$\overline{442148}$ ÷ 25; R = $\overline{17685}$; R = $\overline{23}$

a	b	c	d	e	f	g	h		j	
2.	0	$\bar{8}$	0	$\overline{30}$	$\bar{7}$	211	$\overline{62}$	1831	$\overline{630}$	17685

x = 1.9166920915 is a solution of E

∴ (x – 1.9166920915) is a factor of E.

Variation of f(x) with variation in number of decimal points

Considering x = 2.0

f(x) = 10 ⟶ (1)

Considering two decimals i.e.

x = 2. 0$\bar{8}$ = 1.92, f(x) = 8.0762 ⟶ (2

Considering three or four decimals

x = 2.0 $\bar{8}$ 0 $\overline{30}$ = 1.9170

f(x) = 8.007 ⟶ (3

Considering five decimals

x = 2.0 $\bar{8}$ 0 $\overline{30}$ $\bar{7}$ = 1.91693

f(x) = 8.00548

Considering six decimals

x = 2.0 $\bar{8}$ 0 $\overline{30}$ $\bar{7}$ $\overline{211}$ = 1.916719

f(x) = 8.000626086

Considering seven decimals

x = 2.0 $\bar{8}$ 0 $\overline{30}$ $\bar{7}$ $\overline{211}$ $\overline{62}$ = 1.9167128

f(x) = 8.000483353 ⟶ (6)

Considering eight decimals

x = 2.0 $\bar{8}$ 0 $\overline{30}$ $\bar{7}$ $\overline{211}$ $\overline{62}$ $\overline{1831}$

= 1.91669449

f(x) = 8.000061824 ⟶ (7)

Considering nine decimals

x = 2.0 8 0 30 7 211 62 1831 630

= 1.916693860

f(x) = 8.00004732 ⟶ (8)

Considering ten decimals

x = 2.0 $\bar{8}$ 0 $\overline{30}$ $\bar{7}$ $\overline{211}$ $\overline{62}$ $\overline{1831}$ $\overline{630}$

$\overline{17685}$ = 1.9166920915

f(x) = 8.000006593 ⟶ (9)

7) E = $42x^3 + 8x^2 - 257x + 400 = 0$

f(x) = $42x^3 + 8x^2 - 257x = -400$

Solution x = a.bcde .....

The solution lies between – 3 and – 4

Refer details of Vilokanam Method, shown in section

From Vilokanam a = – 3

$f'(x) = 126x^2 + 16x - 257 = 829$

$f'(-3) = 829$

$f''(x) = 252x + 16$

$\frac{1}{2}f''(-3) = -\frac{740}{2} = -370$

$f'''(x) = 252$

$\therefore \frac{1}{6}f'''(-3) = 42$

		0	0	0	0	0	0	0
$CD = f'(x) = 829$		$\overline{109}$	$\overline{261}$	$\overline{582}$	$\overline{153}$	586	657	145
$\frac{1}{2}f''(x) = -370$			370	1480	5180	4440	$\overline{6290}$	$\overline{3626}\bar{0}$
$\frac{1}{6}f'''(x) = 42$				42	252	1134	2352	2016
	a.	b	c	d	e	f	g	h
	−3.	$\bar{1}$	$\bar{2}$	$\bar{5}$	4	13	3	$\overline{39}$

$= 3.\bar{1}\bar{2}\bar{5}530\bar{9} = \bar{3}.\bar{1}\bar{2}\bar{4}\bar{4}\bar{7}0\bar{9}$ (seven decimals)

$E = 0.00021808$

8) $x^3 + 29x - 97 = 0$

$f(x) = x^3 + 29x = 97$

Solution x = a.bcde .....

The solution lies between 2 and 3

Refer details of Vilokanam Method, shown in section

From Vilokanam $a = -3$

$f'(x) = 3x^2 + 29 \quad f'(-3) = 56$

$\frac{1}{2}f''(x) = 3x \qquad x = -3 \qquad -9$

$f'''(x) = 6 =$

	0	0	0	0	0	0	0	0
CD=f′=56	$\bar{17}$	$\bar{2}$	$\bar{45}$	$\bar{29}$	$\bar{32}$	$\bar{47}$	$\bar{35}$	$\bar{7}$
$\frac{1}{2}f''(a)=9$		$\bar{81}$	$\bar{54}$	$\bar{441}$	$\bar{792}$	$\bar{1602}$	$\bar{3510}$	$\bar{6498}$
$\frac{1}{6}f'''(a)=1$			27	27	225	469	1221	2982
3.	$\bar{3}$	$\bar{1}$	$\bar{8}$	$\bar{12}$	$\bar{15}$	$\bar{28}$	$\bar{47}$	$\bar{64}$
a.	b	c	d	e	f	g	h	i

x = 2.68061666 E = 0.000005494

The value of x and the errors considering different decimals.

Upto e: $3.3 1 9 2 = 2.6808$

$E = 9.274906 \times 10^{-3}$

Upto f: $3.\bar{3}\bar{1}\bar{9}\bar{3}\bar{5} = 2.68065$

$E = 1.6910772 \times 10^{-3}$

Upto g: $3.\ \dot{3} 1 9 3 \bar{7} \bar{8} = 2.680631$

$E = 2.754692 \times 10^{-4}$

Upto h: $3.\ \bar{3}\bar{1}\bar{9}\bar{3}\bar{8}\bar{2}\bar{7} = 2.6806173$

$E = 3.78505 \times 10^{-4}$

When 7 decimals are considered x = 5.6806173

When 8 decimals are considered x = 2.68061666

On substitution of $x = \frac{z}{2}$

Let $x = \frac{z}{2}$ $E = \frac{z^3}{8} + \frac{29z}{2} = 97$

$f(z) = z^3 + 116z = 776$

	LHS	RHS	LHS – RHS
z = 5	705	776	71 —
z = 6	912		– 136 —

From Vilokanam a = 5

$f'(z) = 3z^2 + 116$ $f'(5) = 75 + 116 = 191$

$f''(z) = 6z$ $\frac{1}{2}f''(5) = 15$

$f'''(z) = 6$ $\frac{1}{6}f'''(5) = 1$

	0	0	0	0	0	0	0
191	71	137	89	132	146	176	164
15		$\bar{135}$	$\bar{540}$	$\bar{630}$	$\bar{360}$	$\bar{645}$	
1			$\bar{27}$	$\bar{162}$	$\bar{351}$	$\bar{378}$	
5.	3	6	1	2	3	3	
a	b	C	d	e	f	g	

Upto g: z = 5.361233 x = 2.6806165

$E = -2.5952 \times 10^{-6} = -0.0000025952$

9) $2x^3 + 9x^2 + 18x + 20 = 0$

The solution lies between – 2 and – 3

Solution x = a.bcde .....

Refer details of Vilokanam Method, shown in section

$f'(x) = 6x^2 + 18x + 18 \qquad f'(-3) = 18$

$f''(x) = 12x + 18 \qquad \frac{1}{2} f''(-3) = -9$

$f'''(x) = 12 \qquad \frac{1}{6} f'''(-3) = 2$

	0	0	0	0	0
CD=f′=18	7	16	7	16	7
$\frac{1}{2}f''(a) = -9$		81	702	3627	
$\frac{1}{6}f'''(a) = 2$			$\overline{54}$	$\overline{702}$	
– 3.	3	13	39	171	
a	b	c	d	e	

(four decimals)

x = – 2.5139 E = – 0.14711

Let $x = \frac{z}{2}$ $\qquad \frac{2z^3}{8} + \frac{9z^2}{4} + \frac{18z}{2} = -20$

$z^3 + 9z^2 + 36z = -80$

z = – 5 is one solution

∴ x = – 2.5 is one solution

10) $x^3 - 2x^2 - 51x - 110 = 0$

Solution x = a.bcde .....

The solution lies between – 3 and – 4

Refer details of Vilokanam Method, shown in section

	0	0	0	0
CD=f′(a)=– 12	2	8	7	9
$\frac{1}{2}f'''(a) = -11$		11	154	
$\frac{1}{6}f'''(a) = 1$			1	
– 3.	$\bar{1}$	$\bar{7}$	$\overline{18}$	
a	b	c	d	

$x = -3.188$ $E = -0.139428672$

$x = \frac{z}{2}$

$\frac{z^3}{8} - \frac{2z^2}{4} \quad \frac{51z}{2} = 110$

$z^3 - 4z^2 - 204z = 880$, $f(-6) = 16$

	f(z)	Diff
z = – 4	688	112
z = – 6	864	16
z = – 7	889	– 9
z = – 8	– 1376	2256

From Vilokanam a = – 6

$f'(z) = 3z^2 - 8z - 204 \quad \therefore \frac{f'}{1}(-6) = -48$

$f''(z) = -6z - 8 \quad \therefore \frac{1}{2}''(-6) = -\frac{44}{2} = -22$

$f'''(z) = 6 \quad \therefore \frac{1}{6}f'''(-6) = 1$

	0	0	0	0	0		0
CD= f'(a)=–48	16	16	22	19	17	11	21
$\frac{1}{2}f''(a)$=– 22		198	924	4246	20064	100848	532092
$\frac{1}{6}f'''(a)$=1			27	189	1089	5959	32796
– 6.	$\overline{3}$	$\overline{7}$	$\overline{24}$	$\overline{96}$	$\overline{444}$	$\overline{2227}$	$\overline{11772}$
	b	c	d	e	f	g	h

z = – 6. 4 1 1 4 4 4 2 $x = -3.2057221$ $E = -0.005$

$x = \frac{z}{10}$ $E = \frac{\ }{1000} \frac{2z^2}{100} \frac{51z}{10} = 110$

$f(z) = z^3 - 20z^2 - 5100z = 110000$

	L	R	R – L
z = – 32	109952	110000	48-
z = – 33	110583		– 583-

	0	0	0	0	0	0
CD=f′(a)= – 748	48	480	312	128	220	288
$\frac{1}{2}f''(a)$= – 116		0	0	4176	5568	
$\frac{1}{6}f'''(a)$=1			0	0		
– 32.	0	$\bar{6}$	$\bar{4}$	$\bar{7}$	$\overline{10}$	
a	b	c	d	e	f	

$z = -32.0\ \bar{6}\ \bar{4}\ \bar{8}\ 0$

$x = -3.206480 \quad E = 0.00001696$

11) $x^3 - 20x^2 - 31x + 1609 = 0$

Solution x = a.bcde .....

The solution lies between – 8 and – 9

Refer details of Vilokanam Method, shown in section

From Vilokanam a = – 8

$f(x) = x^3 - 20x^2 - 31x = -1609$

$f'(x) = 3x^2 - 40x - 31 = 4 \qquad \frac{f'(-8)}{1} = 481 = 481$

$f''(x) = 6x - 40 \qquad \frac{f''(-8)}{2} = \frac{-88}{2} = -44$

$f'''(x) = 6 \qquad \frac{f''(-8)}{6} = \frac{6}{6} = 1$

		0	0	0	0	0	
CD=f′(a)=481		$\overline{65}$	$\overline{169}$	$\overline{203}$	$\overline{322}$	$\overline{146}$	$\overline{192}$
$\frac{1}{2}f''(a)$= – 44			44	264	660	1232	
$\frac{1}{6}f'''(a)$=1				1	9	36	
	– 8.	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{5}$	0	
	a	b	c	d	e	f	

$x = -8.13350 \qquad E = 0.000058277$

12) $E = x^3 - 6x^2 + 11x - 10 = 0$

Solution x = a.bcde .....

The solution lies between 3 and 4

Refer details of Vilokanam Method, shown in section

$f(x) = x^3 - 6x^2 + 11x = 10$

From Vilokanam a = 4

$f'(x) = 3x^2 - 12x + 11$ $\quad f'(4) =$

$f''(x) = 6x - 12$ $\quad \frac{1}{2}f''(4) = 6$

$f'''(x) = 6$ $\quad \frac{1}{6}f'''(4) = 1$

		0	0	0	0	0	0	0	0	0
		$\bar{2}$	$\bar{9}$	$\bar{8}$	$\overline{10}$	$\bar{2}$	$\bar{5}$	$\bar{5}$	$\bar{7}$	$\bar{3}$
$CD = f'(a) = 11$										
$\frac{1}{2}f''(a) = 6$			$\bar{6}$	$\overline{96}$	$\overline{564}$	$\overline{2136}$	$\overline{9006}$	$\overline{35484}$	$\overline{149496}$	
$\frac{1}{6}f'''(a) = 1$				1	24	237	1406	6861	32193	
	4.	$\bar{1}$	$\bar{8}$	$\overline{15}$	$\overline{58}$	$\overline{174}$	$\overline{695}$	$\overline{2606}$	$\overline{10670}$	
	a	b	c	d	e	f	g	h	i	

$4.\bar{2}\,\bar{0}\,\bar{3}\,\bar{6}\,\bar{0}\,\bar{2}\,\bar{3}\,\bar{0}$ $\quad \therefore x = 3.79639770$ $\quad E = 0.0006579$

13) $E = 2x^3 + 3x^2 + 3x + 1 = 0$

Solution x = a.bcde .....

The solution lies between 1 and – 1

Refer details of Vilokanam Method, shown in section

From Vilokanam a = – 1

$f(x) = 2x^3 + 3x^2 + 3x = -1$

$f'(x) = 6x^2 + 6x + 3$ $\quad \frac{1}{1}f'(-1) = 3$

$f''(x) = 12x + 6$ $\quad \frac{1}{2}f''(-1) = -3$ $\quad$ Try – 0.5, – 0.75

$f'''(x) = 12$ $\quad \frac{1}{6}f'''(-1) = 2$

	0	0	0	0	0	0
CD=f′(a)=3	1	1	1	1	1	1
$\frac{1}{2}f''(a)=-3$		27	216	1458	9018	
$\frac{1}{6}f'''(a)=2$			$\overline{54}$	$\overline{648}$	$\overline{5670}$	
– 1.	3	12	57	273	1119	
a.	b	c	d	e	f	

$-1.51549 = 0.\bar{4}\ \bar{8}\ \bar{4}\ \bar{5}\ \bar{1}$ $E = 0.0232$

Let $x = \frac{z}{2}$

$E = \frac{2z^3}{8} + \frac{3z^2}{4} + \frac{3z}{2} + 1 = 0$

$z^3 + 3z^2 + 6z + 4 = 0$

$f(z) = z^3 + 3z^2 + 6z = -4$

The solution is – 1. Refer details of Vilokanam

$\therefore\ z = -1 \Rightarrow x = -0.5$

14) $x^3 + 7x^2 + 9x + 11 = 0$

Solution x = a.bcde .....

The solution lies between – 5 and – 6

Refer details of Vilokanam Method, shown in section

From Vilokanam a = – 11

$f(x) = x^3 + 7x^2 + 9x = -11$

$f'(x) = 3x^2 + 14x + 9$ $\frac{1}{1}f'(-6) = 33$

$f''(x) = 6x + 14$ $\frac{1}{2}f''(-6) = -11$

$f'''(x) = 6$ $\frac{1}{6}f'''(-6) = 1$

	0	0	0	0	0	0	0	0
CD=f′(a)=33	7	4	18	29	24	11	4	18
$\frac{1}{2}f''(a)=-11$		44	88	352	1100	2959	7964	
$\frac{1}{6}f'''(a)=1$			$\bar{8}$	$\overline{24}$	$\overline{108}$	$\overline{392}$	$\overline{1254}$	
– 6	2	2	7	. 18	37	81	204	
a.	b	c	d	e	f	g	h	

$-6.22917 = 5.77083$ upto f, $E = 0.003$

$-6.2292714 = \bar{5}.\bar{7}\bar{7}0\bar{7}\bar{2}\bar{8}\bar{6}$ upto h, $E = 0.000211$

$2x^3 + 5x^2 + 6x - 164 = 0$

Solution x = a.bcde .....

The solution lies between 3 and 4

Refer details of Vilokanam Method, shown in section

From Vilokanam a = 4

$f(x) = 2x^3 + 5x^2 + 6x = 164$

$f'(x) = 6x^2 + 10x + 6$ $\quad \therefore \dfrac{f'(4)}{1} = 142$

$f''(x) = 12x + 10$ $\quad \therefore \dfrac{f''(4)}{2} = 29$

$f'''(x) = 12$ $\quad \therefore \dfrac{f'''(4)}{6} = 2$

	0	0	0	0	0	0	0
CD=$f'(a)$=142	$\overline{68}$	$\overline{112}$	$\overline{22}$	$\overline{88}$	$\overline{125}$	$\bar{2}$	$\overline{16}$
$\frac{1}{2}f''(a)$=29		$\overline{464}$	$\overline{2552}$	$\overline{7685}$	$\overline{23548}$	$\overline{75516}$	$\overline{240236}$
$\frac{1}{6}f'''(a)$=2			128	1056	4632	17158	61932
4.	$\bar{4}$	$\overline{11}$	$\overline{18}$	$\overline{52}$	$\overline{142}$	$\overline{411}$	$\overline{1256}$
a	b	c	d	e	f	g	h

x = 3.4648434 $\quad E = 0.00661$

Let us consider $x = \dfrac{z}{2} \Rightarrow \dfrac{2z^3}{8} + \dfrac{5z^2}{4} + \dfrac{6z}{2} = 164$

$\Rightarrow z^3 + 5z^2 + 12z = 656$

From Vilokanam a = 7

$f'(z) = 3z^2 + 10z + 12$ $\quad \frac{1}{1}f'(7) = 229$

$f''(z) = 6z + 10$ $\quad \frac{1}{2}f''(7) = 26$

$f'''(z) = 6$ $\quad \frac{1}{6}f'''(7) = 1$

	0	0	0	0	0	0	0	0
$CD=f'(a)=229$	$\overline{16}$	$\overline{160}$	$\overline{226}$	$\overline{199}$	$\overline{178}$	$\overline{8}$	$\overline{218}$	$\overline{11}$
$\frac{1}{2}f''(a)=26$		0	0	$\overline{936}$	$\overline{2808}$	$\overline{5850}$	$\overline{11856}$	
$\frac{1}{6}f'''(a)=1$			0	0	0	216	972	
7.	0	$\overline{6}$	$\overline{9}$	$\overline{12}$	$\overline{20}$	$\overline{24}$	$\overline{57}$	
	b	c	d	e	f	g	h	

$z = 7.0704297 = 6.9295703$

$x = 3.46478515 \quad E = 0.000054564$

16) $x^3 + 3.78x^2 - 0.01x + 2.26 = 0$

$100x^3 + 378x^2 - x + 226 = 0$

Solution x = a.bcde .....

The solution lies between – 3 and – 4

Refer details of Vilokanam Method, shown in section

From Vilokanam a = – 4

$f(x) = 100x^3 + 378x^2 - x = -226$

$f'(x) = 300x^2 + 756x - 1 \qquad \frac{1}{1}f'(-4) = 1775$

$f''(x) = 600x + 756 \qquad \frac{1}{2}f''(-4) = -822$

$f'''(x) = 600 \qquad \frac{1}{6}f'''(-4) = 100$

	0	0	0	0	0	0
$CD=f'(a)=1775$	122	1220	1550	1300	1767	732
$\frac{1}{2}f''(a)=-822$		0	0	29592	78912	
$\frac{1}{6}f'''(a)=100$			0	0	0	
– 4.	0	6	8	23	54	
a	b	c	d	e	f	

$x = -4.07084 = \overline{3}.\overline{9}\overline{2}\overline{9}\overline{1}\overline{6} = -3.92916 \qquad E = -0.0034849752$

Let $x = \frac{z}{2}$

$$4\left(\frac{25z^3}{8} + \frac{189z^2}{2} - \frac{z}{2}\right) = 226 \qquad f'(z) = 75z^2 + 378z - 1$$

$$25z^3 + 189z^2 - z = -452 \qquad f''(z) = 150z + 378$$

$$\underset{f(x)}{25z^3 + 189z^2 - z} = -452 \qquad f'''(z) = 150$$

	LHS	RHS	RHS – LHS
z = – 8	– 696	– 452	244
z = – 7	693		– 1145

	0	0	0	0
CD=f′(a)1775	244	665	1736	276
$\frac{1}{2}f''(a)$=411		411	2466	
$\frac{1}{6}f'''(a)$=25			$\overline{25}$	
– 8.	1	3	1	
a	b	c	d	

z = 7 . 8 5 9

∴ x = – 3.9295 E = – 0.0091300524

Let $x = \frac{z}{10}$

$100x^3 + 378x^2 - x = -226$

$$\frac{100z^3}{1000} + \frac{378z^2}{100} - \frac{z}{10} = 226$$

$100z^3 + 3780z^2 - 100z = -226000$

$10z^3 + 378z^2 - 10z = -22600$

	LHS	RHS	RHS – LHS
z = – 39	– 17862	– 22600	– 4738
z = – 40	– 34800		12200

	0	0	0	0	0
CD=f′(a)=16136	$\overline{4738}$	$\overline{15108}$	$\overline{2688}$	1712	1672
$\frac{1}{2}f''(a)=\overline{792}$		3168	28512	64152	
$\frac{1}{6}f'''(a)$=10			80	1080	
– 39.	$\overline{2}$	$\overline{9}$	0	5	

Verified $\Rightarrow$ $z = 39.2895 \Rightarrow x - 3.92895$ $E = 0.00007652$

$f'(z) = 30z^2 + 756z - 10$

$f''(z) = 60z + 756$

$f'''(z) = 60$

The same procedure can be adopted to the evaluation of the roots of any degree equation provided the corresponding differentials as multipliers can be worked out and together with the expansion of decimal contributions to that degree

In case of 4th degree equations the subtraction terms are duplex, triplex and quadruplex and their multipliers are second, third and fourth differentials respectively

In case of 5th degree equations the subtraction terms are duplex, triplex and quintet and their multipliers are second, third, fourth and fifth differentials respectively.

In case of 7th degree equations the subtraction terms are duplex, triplex, quadruplex, quintet, sextet and hept and their multipliers are second, third, fourth, fifth, sixth and seventh differentials respectively. For full working details refer Part – II Section J

Section – J Higher Order Equations

4^{th} Degree Equations

In solving the higher order equations, we have adopted both the methods as shown in the following working.

1) $E = x^4 - 4x^3 - 3x + 23 = 0$
$f(x) = x^4 - 4x^3 - 3x = -23$
Solution x = a.bcde .....

Value of x	L.HS.	R.H.S.	R.H.S. – L.H.S.
1	– 6	– 23	– 17
2	– 22		– 1
3	– 36		13
4	– 12		– 11
5	110		– 133
6	414		– 437
– 1	8		– 31
– 2	54		– 77
– 3	198		– 221
– 4	524		– 547

From Vilokanam a = 2

Swamiji's Method

Let $x = 2 \Rightarrow RHS - LHS = \bar{1}$

CD = $4a^3$ Representation at $x = 2 = [4x^3 - 12x^2 - 3]$ at $x = 2 = \overline{19}$

$6a^2$ Representation at $x = 2 = \frac{1}{2}[12x^2 - 24x]$ at $x = 2 = 0$

4a Representation at $x = 2 = \frac{1}{6}[24x - 24]$ at $x = 2 = 4$

	0	0	0	0	0	0	0
CD= $\overline{19}$	$\bar{1}$	$\overline{10}$	$\bar{5}$	$\overline{12}$	$\bar{6}$	$\bar{3}$	$\overline{17}$
		0	0	0+0+0+0=0	0	$\overline{500}+0=\overline{500}$	
		$6a^2b^2$	$4a.b^3+6a^2.2bc$	$b^4+6a^2.c^2+$ $6a^2 2bd+$ $4a.3b^2c$	$4b^3c+6a^2.2be$ $+6a^2.2cd+$ $4a.3b^2d+$ $4a.?bc^2$	$4ac^3+4b^3d+$ $6b^2c^2+6a^2.2bf$ $+6a^2.2ce+6a^2d^2$ $+4a6bcd+4a.3b^2c$	
2.	0	5	2	6	3	27	
a	b	c	d	e	f	g	

1) The first decimal b is obtained as coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the useful manner ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing $4a^3$, $6a^2$, 4a are to be worked out in terms of representation values $4a^3$, $6a^2$, 4a. For example $6a^2b^2 = 0 \times 0 = 0$, $4ac^3 = 4 \times 125 = 5000$ and so on

Upto g (6 decimals) x = 2.0 52 63 27 = 2.052657

Taylor's Method

$f(x) = x^4 - 4x^3 - 3x$

$f'(x) = 4x^3 - 12x^2 - 3$

$f''(x) = 12x^2 - 24x$

$f'''(x) = 24x - 24$

$f^{iv}(x) = 24$

Considering x = 2

$f'(2) = -19 = \overline{19}$

$\frac{1}{2}f''(2) = \frac{1}{2}(0) = 0$

$\frac{1}{6}f'''(2) = \frac{1}{6}(24) = 4$

$\frac{1}{24}f^{iv}(2) = 1$

x = a. b c de f g h

			0	0	0	0	0	0	0
$f'(2)$	$\overline{19}$		$\overline{1}$	$\overline{10}$	$\overline{5}$	$\overline{12}$	$\overline{6}$	$\overline{3}$	$\overline{17}$
$\frac{1}{2}f''(2)$	0			0	0	0	0	0	
$\frac{1}{6}f'''(2)$	4				0	0	0	$\overline{500}$	
$\frac{1}{24}f'''(2)$	1					0	0	0	
		2.	0	5	2	6	3	27	
		a.	b	c	d	e	f	g	

Step1 : $\overline{19})\ \overline{10}(0$

$\overline{0}$

$\overline{10}$

$q = b = 0, \quad R = \overline{10}$

Step2 : $\overline{0}(b^2) =$ (i)

$10R + (i) = \overline{100} + 0 = \overline{100}$

$\overline{100} \div \overline{19}$

$q = (c) = 5, \quad R = \overline{5}$

Step3 : $0(2b)c = 0 \longrightarrow$ (i)

$4(b^3) = 0 \longrightarrow$ (ii)

$10R + (i) + (ii) = \overline{50} + 0 + 0 = \overline{50}$

$\overline{50} \div \overline{19}$

$q = d = 2, \quad R = \overline{12}$

Step4 : $0(2bd + c^2) = 0$ (i)

$4(3b^2c) = 0 \longrightarrow$ (ii)

$1(b^4) = 0 \longrightarrow$ (iii)

$10R + (i) + (ii) + (iii) = \overline{120}$

$\overline{120} \div \overline{19}$

$q = e = 6, \quad R = \overline{6}$

Step5 : $0(2be + 2cd) = 0 \longrightarrow$ (i)

$4(3b^2d + 3bc^2) = 0 \longrightarrow$ (ii)

$1(4b^3c) = 0 \longrightarrow$ (iii)

$10R + (i) + (ii) + (iii) = \overline{60}$

$\overline{60} \div \overline{19}$

$q = f = 3, R = \overline{3}$

Step6 : $0(2bf + 2ce + d^2) = 0 \longrightarrow$ (i)

$4(3b^2e + 6bcd + c^3) = 500 \longrightarrow$ (ii)

$1(4b^3d + 6b^2c^2) = 0 \longrightarrow$ (iii)

$10R + (i) + (ii) + (iii)$

$= \overline{30} + 0 + \overline{500} + 0 = \overline{530}$

$\overline{530} \div \overline{19}$

$q = g = 27, \quad R = \overline{17}$

$\therefore x = 2.052657 \qquad E = 0.000108709, \sim 0$

As both the methods give the same result for the 1st root, we have continued for the second root by applying Taylors method with refinement with the substitution for x as $x = \frac{z}{4}, \frac{z}{100}$. The refined value is found to be x = 3.785317 as the second solution

From Vilokanam a = 4

Considering x = 4

$f'(x) = 61$

$\frac{1}{2}f''(x) = \frac{1}{2}(96) = 48$

$\frac{1}{6}f'''(x) = \frac{1}{6}(72) = 12$

$\frac{1}{24}f^{iv}(x) = \frac{1}{24}(24) = 1$

x = a. b c d e f g h ....

5		0	0	0	0	0	0	0	0
CD=$f'(a)$=61		$\overline{11}$	$\overline{49}$	$\overline{50}$	$\overline{36}$	$\overline{2}$	$\overline{43}$	$\overline{55}$	$\overline{16}$
$\frac{1}{2}f''(a)$= 48			$\overline{48}$	$\overline{768}$	$\overline{4992}$	$\overline{23,328}$	$\overline{114912}$	$\overline{564768}$	$\overline{2874480}$
$\frac{1}{6}f'''(a)$=12				12	288	3024	20652	120276	673884
$\frac{1}{24}f^{iv}(a)$=1					$\overline{1}$	$\overline{32}$	$\overline{464}$	$\overline{4300}$	$\overline{31156}$
	4	$\overline{1}$	$\overline{8}$	$\overline{20}$	$\overline{83}$	$\overline{333}$	$\overline{1559}$	$\overline{7366}$	$\overline{36588}$
	a .	b	c	d	e	f	g	h	i

Step 1 : $\overline{110} \div 61$

$61)\ \overline{110}\ (\overline{1}$

$\overline{61}$

$\overline{49}$

$q = b = \overline{1}$ $R = \overline{49}$

Step 2 : $48(b^2) = 48$ (i) 48

$10R + (i) = \overline{538}$

$\overline{538} \div 61$

$q = c = \overline{8}$ $R = \overline{50}$

Step 3 : $48(2bc) = 768 \rightarrow$ (i) $\overline{768}$

$12(b^3) = \overline{12} \rightarrow$ (ii) 12

$10R + (i) + (ii) = \overline{1256}$

$1256 \div 61$

$q = d = \overline{20}$ $R = \overline{36}$

Step 4 : $48(2bd+c^2)=4992 \rightarrow (i) \rightarrow \overline{4992}$

$\overline{12}(3b^2c) = \overline{288} \rightarrow (ii) \rightarrow 288$

$1(b^4) = 1 \rightarrow (iii) \rightarrow \overline{1}$

$10R + (i) + (ii) + (iii) = \overline{5065}$

$5065 \div 61$

$q = e = \overline{83}$, $R = \overline{2}$

Step 5 : $48(2be+2cd)=23328 \rightarrow (i) \rightarrow \overline{23,328}$

$12(3b^2d+3bc^2)=\overline{3024} \rightarrow (ii) \rightarrow 3024$

$1(4b^3c) = 32 \rightarrow (iii) \rightarrow \overline{32}$

$10R + (i) + (ii) + (iii) = \overline{20,356}$

$\overline{20356}$ 61

$q = f = \overline{333}$, $R = 43$

Step 6: $48(2bf + 2ce + d^2)$

$= 114912 \rightarrow (i) \rightarrow \overline{114912}$

$12(3b^2e + 6bcd + c^3)$

$= \overline{20652} \rightarrow (ii) \rightarrow 20652$

$1(4b^3d + 6b^2c^2)$

$= 464 \rightarrow (iii) \rightarrow \overline{464}$

$10R + (i) + (ii) + (iii) = 95154$

$\overline{95154} \div 61$

$q = g = \overline{1559}$, $R = \overline{55}$

Step7: $48(2bg+2cf+2de)$

$=564768 \rightarrow (i) \rightarrow \overline{564768}$

$12(3b^2f + 6bce + 3bd^2 + 3c^2d)$

$= 120,276 \rightarrow (ii) \rightarrow 120,276$

$1(4b^3e+12b^2cd+4bc^3)$

$=4300 \rightarrow (iii) \rightarrow \overline{4300}$

$10R + (i) + (ii) + (iii) = \overline{449342}$

$\overline{449342} \div 61$

$Q = h = 7366$, $R = 16$

Step 8 : $48(2bh + 2cg + 2df + e^2)$

$= 2874480 \longrightarrow (i) \longrightarrow \overline{2874480}$

$12(3b^2g + 6bcf + 6bde + 3c^2e + 3cd^2)$

$= \overline{673884} \longrightarrow (ii) \longrightarrow 673884$

$1\,(4b^3f + 12b^2ce + 6b^2d^2 + 12bc^2d + c^4)$

$= 31156 \longrightarrow (iii) \longrightarrow \overline{31{,}156}$

$10R + (i) + (ii) + (iii) = 223\overline{1912}$

$\overline{2231912} \div 61$

$q = i = 36588, \quad R = 44$

$\therefore\ x = 3.78570852$ $\qquad E = 0.016443352$

This can be further refined with $x = \frac{z}{2}$ substituted

The given equation E is

$\left(\frac{z}{2}\right)^4 - 4\left(\frac{z}{2}\right)^3 - 3\left(\frac{z}{2}\right) + 23 = 0$ $\qquad \cdot\, z^4 + - 8z^3 - 24z + 368 = 0$

$f(z) = z^4 - 8z^3 - 24z = -368$

if $z = 8$ $\quad f(z) = -192 \quad \Rightarrow \quad$ RHS – LHS = – 176

From Vilokanam a = 8

$f'(8) = 488$ $\qquad f'(z) = 4z^3 - 24z^2 - 24$

$\frac{1}{2}f''(8) = 192$ $\qquad f''(z) = 12z^2 - 48z$

$\frac{1}{6}f'''(8) = 24$ $\qquad f'''(z) = 24z - 48$

$\frac{1}{24}f^{iv}(8) = 1$ $\qquad$ if $f^{iv}(z) = 24$

	0	0	0	0	0	0	0	0
CD= $f'(8)$ = 488	$\overline{176}$	$\overline{296}$	$\overline{296}$	$\overline{480}$	$\overline{457}$	$\overline{46}$	$\overline{294}$	$\overline{356}$
$\frac{1}{2}f''(8) = 192$		$\overline{1728}$	$\overline{10368}$	$\overline{44352}$	$\overline{187776}$	$\overline{800832}$	$\overline{3477120}$	
$\frac{1}{6}f'''(8) = 24$			648	5832	33696	171720	834840	
$\frac{1}{24}f^{iv}(8) = 1$				$\overline{81}$	$\overline{972}$	$\overline{7074}$	$\overline{42552}$	
8.	$\overline{3}$	$\overline{9}$	$\overline{25}$	$\overline{88}$	$\overline{327}$	$\overline{1304}$	$\overline{5507}$	
a.	b	c	d	e	f	g	h	

z = 7.5710753 $x = \frac{z}{2} = 3.78553765$ E = 0.009260

A further fine method is attempted with $x = \frac{\;}{100}$

$E = x^4 - 4x^3 - 3x = -23$

$$\frac{z^4}{10^8} - \frac{4z^3}{10^6} - \frac{3z}{10^2} = -23$$

$z^4 - 400z^3 - 3000000z = -23 \times 10^8$

	LHS f(x)	RHS -23×10^8	RHS – LHS
x = 377	– 2363400559		63400559
x = 378	– 2322223344		22223344
x = 379	– 2280238719		– 19761281
x = 380	– 2237440000		– 62560000

From Vilokanam a = 378

$f(z) = z^4 - 400z^3 - 3000000z = -23 \times 10^8$

$f'(z) = 4z^3 - 1200z^2 - 3000000$ $f'(378) = 41579808$

$f''(z) = 12z^2 - 2400z$ $\frac{1}{2} f''(378) = 403704$

$f'''(z) = 24z - 2400$ $\frac{1}{6} f'''(378) = 1112$

$f^{iv}(z) = 24$ $\frac{1}{24} f^{iv}(378) = 1$

	0	0	0	0
CD=$f'(a)$=41579808	22223344	14334400	8511976	31289832
$\frac{1}{2} f''(a)$ =403704		$\overline{10092600}$	$\overline{12111120}$	$\overline{7670376}$
$\frac{1}{6} f'''(a)$ =1112			$\overline{139000}$	$\overline{250200}$
$\frac{1}{24} f^{iv}(8)$ =1				$\overline{625}$
378	5	3	1	7
a	b	c	d	e

Upto e: z = 378.5317 x = 3.785317 E = – 0.000010639

We have adopted Swamiji's Argumentation method to solve the remaining two solutions making use of 1st and 2nd solutions so obtained.

∴ E = (x – 2.052657) (x – 3.785317) A, A should have x^2, x and constant terms Applying Argumentation, Adyamadyena and comparing the coefficients of like terms

$= (x^2 - 5.837974x + 7.769957)(x^2 + \alpha x + 2.960119)$

Comparing x coeff on both sides :

$-17.28109776 + 7.769957\alpha = -3$ $\alpha = 1.837989$

$E = (x - 2.052657)(x - 3.785317)(x^2 + 1.837989x + 2.960119)$

The QE can be factorized by using differential relation.

$2x + 1.837989 = \pm\sqrt{3.378204 - 11.840476} = \pm 2.908998$

$x = -0.918995 \pm 1.454499i$

$E = (x^2 - 2.052657)(x - 3.785317)(x + 0.918995 + 1.454499i)(x + 0.918995 - 1.454499i)$

Applying Gunita Samuccaya Sutram for final verification

$S_c = 17$ $= (1 - 2.052657)(1 - 3.785317)(1.918995 + 1.454499i)(1.918995 - 1.454499i)$

$= 16.99996 \sim 17$

2) $E = x^4 - 2x^3 - 13x^2 - 11x + 133 = 0$

$f(x) = x^4 - 2x^3 - 13x^2 - 11x$

Solution x = a.bcde .....

	L	R	Diff
x = 1	– 25	– 133	– 108
x = 2	– 74		– 59
x = 3	– 123		– 10
x = 4	– 124		– 9
x = 5	– 5		– 128
x = 6	330		– 463
x = 7	1001		– 1134
x = – 1	1		– 134
x = – 2	2		– 135
x = – 3	51		– 184
x = – 4	220		– 353
x = – 5	605		– 738

Refined values are tried

3.1	– 126.2599	– 6.7401
3.2	– 128.9984	– 4.0016
3.3	– 131.1519	– 1.8421
3.4	– 132.6544	– 0.3456
3.5	– 133.4375	0.4375
3.6	– 133.4304	0.43
3.7	– 132.5599	– 0.4401
3.8	– 130.7504	– 2.2496
3.9	– 127.9239	– 5.0761
4	– 124	– 9

We have considered the decrease followed by increase in the difference as one of the criterian for a trial for one of the solutions eg x = 3 and x = 4 . It is noticed that two roots lie between

1) 3.4 and 3.5
2) 3.6 and 3.7

Swamiji's Method

From Vilokanam a = 3

Let $x = 3 \Rightarrow$ RHS – LHS = $\overline{10}$

CD = $4a^3$ Representation at x = 3 $\Rightarrow [4x^3 - 6x^2 - 26x - 11]$ at x = 3 $\Rightarrow \overline{35}$

$6a^2$ Representation at x = 3 $\Rightarrow \frac{1}{2}[12x^2 - 12x - 26]$ at x = 3 $\Rightarrow$ 23

4a Representation at x = 3 $\Rightarrow \frac{1}{6}[24x - 12]$ at x = 3 = 10

	0	0	0	0	0	0
CD= $\overline{35}$	$\overline{10}$	$\overline{30}$	$\overline{7}$	$\overline{7}$	$\overline{15}$	$\overline{17}$
		$\overline{92}$	$\overline{80}+\overline{1012}=\overline{1092}$	$\overline{16}+\overline{2783}+\overline{3036}+\overline{1320}=\overline{7155}$	$\overline{352}+\overline{18952}+\overline{16698}+\overline{3960}+\overline{7260}=\overline{47222}$	
		$6a^2b^2$	$4a.b^3+6a^2.2bc$	$b^4+6a^2.c^2+6a^2 2bd+4a.3b^2c$	$4b^3c+6a^2.2bc+6a^2.2cd+4a.3b^2d+4a.3bc^2$	
3	2	11	33	206	1353	
a	b	c	d	e	f	

Subtraction Terms

1) The first decimal b is obtained as coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the useful manner ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing $4a^3$, $6a^2$, 4a are to be worked out in terms of representation values $4a^3$, $6a^2$, 4a.

Upto f (5 decimals) x = 3. 2 11 33 206 1353 = 3.3 7 713

Taylor's Method

From Vilokanam a = 4

at x = 3 $f'(x) = 4x^3 - 6x^2 - 26x - 11 = -35$ common divisor

$f''(x) = 12x^2 - 12x - 26$ $\frac{1}{2} f''(3) = 23$

$f'''(x) = 24x - 12$ $\frac{1}{6} f'''(3) = 10$

$f^{iv}(x) = 24$ $\frac{1}{24} f^{iv}(3) = 1$

	0	0	0	0	0
CD=$f'(a)$=− 35	$\overline{10}$	$\overline{30}$	$\overline{7}$	$\overline{7}$	$\overline{15}$
$\frac{1}{2}f''(a)$=23		$\overline{92}$	$\overline{1012}$	$\overline{5819}$	$\overline{35650}$
$\frac{1}{6}f'''(a)$=10			$\overline{80}$	$\overline{1320}$	$\overline{11220}$
$\frac{1}{24}f^{iv}(a)$=1				$\overline{16}$	$\overline{352}$
3.	2	11	33	206	1353
A	b	c	d	c	f

If one consider x = 3.343, (3^{rd} decimal point)

Upto d = 3.343 E = 1.118303457

Or if one considers x = 3.3636 upto 4[th] decimal then E = 0.812889465

Upto f, x = 3.37713 E = 0.6282812

Both the methods have given the same solution as the 1[st] one. It appeared one has to still extend for further decimal positions. Hence substitution is adopted as $x = \frac{z}{2}, \frac{z}{10}, \frac{z}{100}$ evaluating at each stage, the error in the result.

Let $x = \frac{z}{2}$

$$\frac{z^4}{16} - \frac{2z^3}{8} - \frac{13z^2}{4} - \frac{11z}{2} = -133$$

$f(z) = z^4 - 4z^3 - 52z^2 - 88z = -2128$

at z = 6 LHS = – 1968 Diff = – 160

From Vilokanam a = 6

$f'(z) = 4z^3 - 12z^2 - 104z - 88 = -280$

$\frac{1}{2} f''(z) = 12x^2 - 24z - 104 = \frac{184}{2} = 92$

$\frac{1}{6}\ f'''(z) = 24z - 24 = \frac{1}{6} = 20$

$\frac{1}{24} f^{iv}(z) = \frac{1}{24} 24 = 1$

z = 6

	0	0	0	0
CD=$f'(a)$=–280	– 160	$\overline{200}$	$\overline{100}$	$\overline{220}$
$\frac{1}{2} f''(a)$=92		$\overline{2300}$	$\overline{13800}$	$\overline{76820}$
$\frac{1}{6} f'''(a)$=20			$\overline{2500}$	$\overline{22500}$
$\frac{1}{24} f^{iv}(a)$=1				$\overline{625}$
6.	5	15	61	364
a	b	c	d	e

z = 6.711 x = 3.3555 E = 0.9294

Let $x = \frac{z}{10}$

$$\frac{z^4}{10000} - \frac{2z^3}{1000} - \frac{13z^2}{100} - \frac{11z}{10} = -133$$

$z^4 - 20z^3 - 1300z^2 - 11000z = -1330000$

RHS = – 1330000

	LHS	RHS – LHS
z = 30	– 1230000	– 100000
z = 31	– 1262599	– 67401
z = 32	– 1289984	– 40016
z = 33	– 1311519	– 18481
z = 34	– 1326544	– 3456
z = 35	– 1334375	4375
z = 36	– 1334304	4304
z = 37	– 1325599	– 4401

From Vilokanam a = 34

$f'(z) = 4z^3 - 60z^2 - 2600z - 11000$ at z = 34, $f'(z) = -11544$

$\frac{1}{2}f''(z) = 12z^2 - 120z - 2600 = 3596$

At z = 34

$\frac{1}{6}f'''(z) = 24z - 120 = 116$

$\frac{1}{24}f^{iv}(z) = 1$

	0	0	0	0	0
CD=$f'(a)$= – 11544	$\overline{3456}$	$\overline{11472}$	$\overline{2120}$	$\overline{7192}$	10852
$\frac{1}{2}f''(a)$=3596		$\overline{14384}$	$\overline{158224}$	$\overline{650876}$	
$\frac{1}{6}f'''(a)$=116			$\overline{928}$	$\overline{15312}$	
$\frac{1}{24}f^{iv}(a)$=1				$\overline{16}$	
34.	2	11	15	63	
	b	c	d	e	

Upto 'd':

$x = \frac{34.325}{10} = 3.4325$ E = 0.008802072

upto e: E = 0.003039882

$x = \frac{z}{100}$

$x^4 - 2x^3 - 13x^2 - 11x + 133 = 0$

f(x) = – 133

$\frac{z^4}{10^8} - \frac{2z^3}{10^6} - \frac{13z^2}{10^4} - \frac{11z}{10^2} = -133$

$z^4 - 200z^3 - 130000z^2 - 11000000z = -13300000000$

	LHS	RHS	RHS – LHS
		– 13300000000	
z = 341	– 13276623240		– 23376760
z = 342	– 13287080300		– 12919700
z = 343	– 13296804200		– 3195800
z = 344	– 13305787900		5787900
z = 360	– 13343040000		43040000
z = 361	– 13338343160		38343160
z = 362	– 13332775660		32775660
z = 363	– 13326330040		26330040
z = 364	– 13318998780		18998780
z = 365	– 13310774380		+ 10774380
z = 366	– 13301649260		1649260
z = 367	– 13291615880		– 8384120

From Vilokanam a = 343

$f'(z) = 4z^3 - 600z^2 - 260000z - 11000000$

when z = 343 $\quad f'(z) = -9354972$

$f''(z) = 12z^2 - 1200z - 260000$

when z = 343 $\quad \frac{1}{2}f''(z) = 370094$

when z = 343 $\quad \frac{1}{6}f'''(z) = 1172$

$\frac{1}{24}f''''(z) = = 1$

	0	0	0	0
CD=$f'(a)$=–9354972	$\overline{3195800}$	$\overline{3893084}$	$\overline{4841798}$	$\overline{202048}$
$\frac{1}{2}f''(a)$=370094		$\overline{3330846}$	$\overline{8882256}$	
$\frac{1}{6}f'''(a)$=1172			$\overline{31644}$	
$\frac{1}{24}f^{iv}(a)$=1				
343	3	4	6	

z = 343.346

From Vilokanam a = 360

$x = \frac{z}{100} = 3.43346$ $\qquad E = 3.3 \times 10^{-5}$

Second Solution:using $x = \frac{z}{100}$

	0	0	0	0
CD=$f'(a)$=9577984	1649260	6914616	1646136	524208
$\frac{1}{2}f''(a)$=454136		$\overline{454136}$	$\overline{6357904}$	
$\frac{1}{6}f'''(a)$=1264			$\overline{1264}$	
$\frac{1}{24}f^{iv}(a)$=1				
.366.	1	7	1	
	b	c	d	

z = 366.171 x = 3.66171 $E = 1.857 \times 10^{-5}$

$f'(z) = 4z^3 - 600z^2 - 260000z - 11000000$ at z = 366 f(z) = 9577984

$f''(z) = 12z^2 - 1200z - 260000$ $\frac{1}{2}f''(z)$ (at z = 366) = 454136

$f'''(z) = 24z - 1200 = 7584$ $\frac{1}{6}f'''(z)$ (at z = 366) = 1264

(x – 3.43346) and (x – 3.66171) are two factors

Finally the refined solutions are considered as x = 3.43346 and x =3.66171 are considered for final evaluation of remaining two solutions. At this stage Swamijis Argumentation Method is adopted.

$(x - 3.43346)(x - 3.66171) \Rightarrow x^2 - 7.09517x + 12.57233482 = 0$

$E = (x^2 - 7.09517x + 12.57233482)A$, Where A should have x^2, x and constant terms

By Argumentation By Adyamadyena

$(x^2 - 7.09517x + 12.57233482)(x^2 + \alpha x + 10.57878285)$

Comparing the coefficients of like terms

$\alpha\ 12.57233482 + - 10.57878285 \times 7.09517 = -11$

(x term) $\alpha = \frac{64.05826271}{12.57233482} = 5.095176324$

$x^2 + 5.09516324x + 10.57878285 = 0$

$2x + 5.09516324 = \pm\sqrt{25.96068844 - 42.3151314}$

$x = \frac{-5.09516324}{2} \pm \frac{4.04406268i}{2}$

$(x_3, x_4) - 254758162 \pm 2.02203134i$

$x_1 = 3.43346$ $x_2 = 3.66171$

Applying Gunitha Samuccayah Sutram for final verification

$S_c = 108 = (1 - 3.43346)(1 - 3.66171)(1 + 2.54758162 - 2.02203134i)(1 + 2.54758162 + 2.02203134i) \simeq 107.999897$

3) $E = x^4 - 6x^3 + 3x^2 + 22x = 6$

Solution x = a.bcde .....

	LHS f(x)	RHS	Diff RHS – LHS
x = 1	20	6	– 14
x = 2	24		– 18
x = 3	12		– 6
x = 4	8		– 2
x = 5	60		– 54
x = – 1	– 12		18
x = – 2	– 2		– 26

One root lies between x = 1 and x = – 1 and worked as Set B

By way of Vilokanam, (By way of the rule of usual procedure by scanning the difference between RHS and LHS of the given equation.) We couldn't exactly locate the possibility of at least one solution. However, in this case it could be seen that between x = 3 and x = 4 there is a likelihood of trial for one solution. As the difference at x = 4 being lower an attempt is made with that value for x.

Set A

Swamiji's Method

Let $x = 4 \Rightarrow$ RHS – LHS = 2

CD = $4a^3$ Representation at x = 4 = $[4x^3 - 18x^2 + 6x + 22]$ at $x = 4 \Rightarrow 14$

$6a^2$ Representation at x = 4 = $\frac{1}{2}[12x^2 - 36x + 6]$ at $x = 4 \Rightarrow 27$

$4a$ Representation at x = 4 = $\frac{1}{6}[24x - 36]$ at x = 4 = 10

	0	0	0	0	0	0	
CD=14	$\overline{2}$	$\overline{6}$	$\overline{3}$	$\overline{8}$	$\overline{13}$	$\overline{12}$	
		$\overline{27}$	$10+\overline{324}$ $=\overline{314}$	$\overline{1}+\overline{972}+\overline{1296}$ $+180=\overline{2087}$	$\overline{24}+\overline{8316}+\overline{7776}$ $+720+1080=\overline{14316}$	$2160+\overline{96}+\overline{216}+\overline{55674}$ $+\overline{49896}+\overline{15552}+8640+$ $4620=\overline{106014}$	Subtraction Terms
		$6a^2b^2$	$4a.b^3+6a^2.2bc$	$b^4+6a^2.c^2+$ $6a^2 2bd+$ $4a.3b^2c$	$4b^3c+6a^2.2be$ $+6a^2.2cd+$ $4a.3b^2df+4a.3bc^2$	$4a.c^3+4b^3d+6b^2c^2+6a^2.2bf$ $+6a^2.2ce+6a^2.d^2+4a.6bcd$ $+4a.3b^2e$	
4	$\overline{1}$	$\overline{6}$	$\overline{24}$	$\overline{154}$	$\overline{1031}$	$\overline{7581}$	
a	b	c	d	e	f	g	

1) The first decimal b is obtained as coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the useful manner ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing $4a^3$, $6a^2$, 4a are to be worked out in terms of representation values $4a^3$, $6a^2$, 4a.

Upto g (6 decimals) x $= 4.\ \bar{1}\ \bar{6}\ \overline{24}\ \overline{154}\ \overline{1031}\ \overline{7581}$

$= 4.\ \bar{2}\ \bar{1}\ \bar{7}\ \bar{2}\ \bar{9}\ \bar{1} = 3.782709$

Taylor's Method

Let x = 4

$f' = 4x^3 - 18x^2 + 6x + 22 = 14$

$f'' = 12x^2 + 36x + 6 = 54$

$f''' = 24x - 36$

$96 - 36 = 60$

$f^{iv}\ 24 = 24$

		0	0	0	0	0	0	
CD=$f'(a)$=14		$\bar{2}$	$\bar{6}$	$\bar{3}$	$\bar{8}$	$\overline{13}$	$\overline{12}$	Subtraction Terms
$\frac{1}{2}f''(a)$=27			$\overline{27}$	$\overline{324}$	$\overline{2268}$	$\overline{16092}$	$\overline{121122}$	
$\frac{1}{6}f'''(a)$=10				10	180	1800	15420	
$\frac{1}{24}f^{iv}(a)$=10					$\bar{1}$	$\overline{24}$	$\overline{312}$	
	4.	$\bar{1}$	$\bar{6}$	$\overline{24}$	$\overline{154}$	$\overline{1031}$	$\overline{7581}$	
		b	c	d	e	f	g	

x = 3.782709 E = 0.132375747

As both the methods give the same result as the 1st root (3.782709) the error is found to be more. Hence a substitution is necessitated. For the substitution onwards we have continued Taylor's Expansion method.

Let 2x = z

$$x = \frac{z}{2}$$

$$\frac{z^4}{16} - \frac{6z^3}{8} + \frac{3z^2}{4} + \frac{22z}{2} - 6 = 0$$

$$z^4 - 12z^3 + 12z^2 + 176z - 96 = 0$$

$z^4 - 12z^3 + 12z^2 + 176z = 96$

			b	c	d	e	f
			0	0	0	0	0
CD=$f'(a)$– 48			– 9	$\overline{42}$	$\overline{42}$	$\overline{16}$	$\overline{35}$
$\frac{1}{2}f''(a)$=54				$\overline{54}$	$\overline{972}$	$\overline{7506}$	$\overline{46352}$
$\frac{1}{6}f'''(a)$=16					$\overline{16}$	$\overline{432}$	$\overline{5280}$
$\frac{1}{24}f^{iv}(a)$=1						$\overline{1}$	$\overline{36}$
	7.		1	9	29	168	1083

x = 3.623315 E = 0.043131799

$f' = 4z^3 - 36z^2 + 24z + 176$

$f'' = 12z^2 - 72z + 24$

$f''' = 24z - 72 = 96$

$f^{iv} = 24$

x value upto 6[th] decimal point shows that x = 3.782709. At this stage a substitution is attempted for $x = \frac{z}{2}$. This gives a value x = 3.623315. This suggested that there is a likelihood of two values for x lying between 3 and 4. Hence further attempt is made to workout the details with $x=\frac{z}{10}$ which clearly shows that there are two solutions for z between 36 and 37, and between 37 and 38 (refer Vilokanam)

$E_1 = z^4 - 60z^3 + 300z^2 + 22000z = 60000$

	LHS	
z = 33	82401	37599
z = 34	72896	59681
z = 35	65625	– 5625
z = 36	61056	– 1056
z = 37	59681	319
z = 38	62016	– 2016

From Vilokanam a = 36

$f' = 4z^3 - 180z^2 + 600z + 22000$

$f'' = 12z^2 - 360z + 600$

$f''' = 24z - 360$

$f^{iv} = 24$

z = 36

		0	0	0	0
CD=f′(a)=– 3056		$\overline{7056}$	$\overline{1392}$	$\overline{780}$	$\overline{1516}$
$\frac{1}{2}f''(a)=1596$			$\overline{14364}$	$\overline{86184}$	$\overline{426132}$
$\frac{1}{6}f'''(a)=84$				$\overline{2268}$	20412
$\frac{1}{24}f^{iv}(a)=1$					$\overline{81}$
36.		3	9	31	151
a		b	c	d	e

z = 36.4361 x = 3.64361

z = 38

	0	0	0	0	0
CD=f′(a)=4368	$\overline{2016}$	$\overline{2688}$	$\overline{4080}$	$\overline{2464}$	$\overline{1580}$
$\frac{1}{2}f''(a)=2124$		$\overline{33984}$	$\overline{220896}$	$\overline{1344492}$	
$\frac{1}{6}f'''(a)=92$			5888	57408	
$\frac{1}{24}f^{iv}(a)=1$				$\overline{256}$	
38.	$\overline{4}$	$\overline{13}$	$\overline{58}$	$\overline{300}$	
a	b	c	d	e	

z = 38. $\overline{6}\,\overline{1}\,\overline{8}\,\overline{0}$ z = 3.73820 x = 3.73820

It is observed that the variation is slow. And hence, an attempt is made for further substitution for x as $\frac{z}{100}$

$$E_2 = \frac{z^4}{10^8} - \frac{6z^3}{10^6} + \frac{3z^2}{10^4} + \frac{22z}{10^2} = 6$$

$$E_2 = z^4 - 600z^3 + 30000z^2 + 22000000z = 6\text{x } 10^8$$

Probability of two values are clearly indicated

value	LHS	RHS 6×10^8	RHS – LHS
361	607664441		– 7664441
362	605093136		– 5093136
363	602851161		– 2851161
364	600943616		– 943u16
365	599375625		624375
371	597388281		2611719

372	598342656
373	599678441
374	601400976

1657344
+ 321559
− 1400976

From Vilokanam a = 364

$4z^3 - 1800z^2 + 60000z + 22000000$

$12z^2 - 3600z + 60000$

$24z - 3600$

24

	0	0	0	0	0	0
CD=$f'(a)=\overline{1738624}$	$\overline{943616}$	$\overline{743040}$	$\overline{1243056}$	$\overline{122008}$	$\overline{232113}$	$\overline{1593354}$
$\frac{1}{2}f''(a)=169776$		$\overline{4244400}$	$\overline{10186560}$	$\overline{28182816}$	$\overline{55346976}$	
$\frac{1}{6}f'''(a)=856$			$\overline{107000}$	$\overline{385200}$	$\overline{1296840}$	
$\frac{1}{24}f^{iv}(a)=1$				$\overline{625}$	$\overline{3000}$	
364	5	6	13	17	33	
	b	c	d	e	f	

364.57503

From Vilokanam a = 373

	0	0	0	0	0	0
CD=$f'(a)=1528268$	321559	159054	817044	521964	1352144	1241776
$\frac{1}{2}f''(a)=193374$		$\overline{773496}$	0	$\overline{3867480}$	0	$\overline{11022318}$
$\frac{1}{6}f'''(a)=892$			$\overline{7136}$	0	$\overline{53520}$	0
$\frac{1}{24}f^{iv}(a)=1$				$\overline{16}$	0	$\overline{160}$
373	2	0	5	0	8	0
	b	c	d	e	f	g

z = 373.20508 x = 3.7320508

(x − 3.6457503) and (x − 3.7320508) are the two factors of E.

Applying Swamiji's method of Argumentation for evaluating the remaining roots.

$E = (x^2 - 7.3778011x + 13.60612532)A$

A should have x^2. x and constant terms.

$\therefore\ E = (x^2 - 7.3778011x + 13.60612532)(x^2 + \alpha x - .440977858)$

Comparing x coefficient on both sides

$13.60612532\alpha + 3.253446926 = 22$

$\Rightarrow \alpha = 1.37780247$

$\therefore$ E = $(x^2 - 7.3778011x + 13.60612532)(x^2 + 1.37780247 - 0.440977858)$

Second Quadratic expression is further factorised using differential relation.

$2x + 1.37780247 = \pm\sqrt{1.898339646 + 1.763911432} = \pm\ 1.913700885$

$\Rightarrow x = -1.6457516775,\ 0.26794920$

$\therefore$ E = $(x - 3.6457503)(x - 0.7320508)(x + 1.6457516775)(x - 0.26794920)$

Applying Gunita Samuccaya Sutram for final Verification.

$S_c = 14 = (-2.6457503)(-2.7320508)(2.6457516775)(0.7320508) = 13.9999964 \simeq 14$

To verify the remaining two values $-1.6457516775,\ 0.26794920$

Set B

$x = \frac{z}{4}$

One can expect two roots to lie between 1 and − 1; − 1 and − 2. For a finer step, a trial with x = 0.75, 0.5, 0.25, − 0.25, − 0.5, − 0.75, − 1.25, − 1.5, − 1.75, can be attempted. This results in that one root lies between 0.25 and 0.5 another root lies between − 1.5 and − 1.75. This is followed by a substitution for x as $x = \frac{z}{4}$ and $\frac{z}{2}$ respectively.

When $x = \frac{z}{4}$

$E = \frac{z^4}{256} - \frac{6z^3}{64} + \frac{3z^2}{16} + \frac{22z}{4} = 6 \quad = z^4 - 24z^3 + 48z^2 + 1408z = 1536$

	f(z)	RHS 1536	RHS − LHS
if z = 1	1433		103
if z = 2	2832		− 1296

From Vilokanam a = 1

$f'(z) = 4z^3 - 72z^2 + 96z + 1408$

$f''(z) = 12z^2 - 144z + 96$

$f'''(z) = 24z - 144$

$f^{iv}(z) = 24$

	0	0	0	0	0	0	0	0	0
CD = $f'(a)$ = 1436	103	1030	248	1044	1270	28	306	1340	253
$\frac{1}{2}f''(a) = -18$		0	0	882	252	1782	2520	2718	3744
$\frac{1}{6}f'''(a) = -20$			0	0	0	6860	2940	21000	32360
$\frac{1}{24}f^{iv}(a) = 1$				0	0	0	0	$\overline{2401}$	$\overline{1372}$
1.	0	7	1	7	9	6	5	24	25
a	b	c	d	e	f	g	h	i	j

Upto g = 0.267949 E = – 0.00000430

Upto j = z = 1.071796765 ⇒ x = 0.26794919 E = 0.0000000544

Ref working with $x = \frac{z}{100}$ wherein same accuracy to obtained when x is considered upto 'g' only

Let us substitute $x = \frac{z}{2}$ to get the second solution between – 1.5 and – 1.75.

When $x = \frac{z}{2}$

$$E = \frac{z^4}{16} - \frac{6z^3}{8} + \frac{3z^2}{4} + \frac{22z}{2} = 6$$

$z^4 - 12z^3 + 12z^2 + 176z = 96$

From Vilokanam a = – 3

$f'(z) = 4z^3 - 36z^2 + 24z + 176$

$f''(z) = 12z^2 - 72z + 24$

$f'''(z) = 24z - 72$

$f^{iv}(z) = 24$

	f(x)	RHS	RHS – LHS	
z = – 3	– 15	96	111	
z = – 4	512		96	– 416

	0	0	0	0	0	0	0	0
CD=f′(a)=–328	111	126	$\overline{306}$	$\overline{100}$	235	$\overline{254}$	22	140
$\frac{1}{2}f''(a)=174$		$\overline{1566}$	0	11484	$\overline{32364}$	50982	24708	$\overline{558714}$
$\frac{1}{6}f'''(a)=-24$			$\overline{648}$	0	7128	$\overline{20088}$	18576	88992
$\frac{1}{24}f^{iv}(a)=1$				$\overline{81}$	0	1188	$\overline{.3348}$	918
– 3.	$\overline{3}$	0	11	$\overline{31}$	69	$\overline{90}$	$\overline{122}$	1425
a	b	C	d	e	f	g	h	i

z = 3 . 312 50205 = 3 . 2 9 1 4 9 3 9 5

x = – 1.64574897 E = – 0.000127488

For Further refinement. Substitute $x = \frac{z}{100}$ for the finer value in the first solution

0.26794919

Resulting in +26
+27
−164
−165

In the second solution − 1.64574897

When $x = \frac{z}{100}$

$$E = \frac{z^4}{10^8} - \frac{6z^3}{10^6} + \frac{3z^2}{10^4} + \frac{22z}{10^2} = 6$$

$$z^4 - 600z^3 + 30000z^2 + 22000000z = 6 \times 10^8$$

[illegible])		RHS 6×10^8	RHS − LHS
z = 26	582191376		17808624
z = 27	604591641		−4591641
z = 28	626963456		
z = −163	515429961		
z = −164	568841216		+31158784
z = −165	623225625		−23225625

From Vilokanam a = 1

$f(z) = z^4 - 600z^3 + 30000z^2 + 22000000z = 6 \times 10^8$

$f'(z) = 4z^3 - 1800z^2 + 60000z + 22000000$

$f''(z) = 12z^2 - 3600z + 60000$

$f'''(z) = 24z - 3600$

$f^{iv}(z) = 24$

		0	0	0	0	0	0
CD = $f'(a)$ = 22,413,504		17808624	21191712	10820040	20322256	3905561	20313110
$\frac{1}{2}f''(a)$ = − 12744			624456	1605744	1745928	2523312	2446843
$\frac{1}{6}f'''(a)$ = − 496				170128	656208	1135344	1767744
$\frac{1}{24}f^{iv}(a)$ = 1					2401	12348	29302
	26	7	9	4	9	1	9
	a	b	c	d	e	f	g

Note : This is same as the value at $\frac{6}{4}$ upto 'j'

z = 26.794919

⇒ x = 0.26794919 E = − 0.00000005444

z = − 164

	0	0	0	0	0	0
∴ CD=$f'(a)$=– 53896576	31158784	42104960	31609168	12391480	21187175	9858282
$\frac{1}{2}f''(a)$=486576		$\overline{1216400}$	$\overline{34060320}$	$\overline{48171024}$	$\overline{38926080}$	$\overline{33573744}$
$\frac{1}{6}f'''(a)$=– 1256			$\overline{157000}$	$\overline{659400}$	$\overline{1394160}$	$\overline{1843808}$
$\frac{1}{24}f^{iv}(a)$=1				$\overline{625}$	$\overline{3500}$	$\overline{9850}$
– 164.	$\bar{5}$	$\bar{7}$	$\bar{5}$	$\bar{1}$	$\bar{3}$	$\bar{1}$
a	b	c	d	e	f	g

Note: When $x = \frac{z}{100}$, the value obtained is, more accurate at even 'g'.

z = – 164.575131 E =– 0.000000058

x = – 1.64575131

∴ (x + 1.6457513) & (x – 0.26794919) are the two factors of E

∴ E = (x^2 + 1.37780212x – 0.44097773) (x^2 + αx + 13.60612927)

Comparing x coefficients on both sides.

– 0.44097773α + 18.74655375 = 22

α = – 7.377801709

∴ E = (x^2 + 1.37780212x – 0.44097773) (x^2 – 7.377801709x + 13.60612927)

The second quadratic expression can be further factorized by using differential relation

2x – 7.377801709 $= \pm\sqrt{54.43195806 - 54.42451708}$

$= \pm\sqrt{0.00744098} = \pm\ 0.086261115$

⇒ x = 3.73203141, 3.64577030

E = (x + 1.64575131) (x – 0.26794919) (x – 3.73203141) (x– 3.64577030)

Applying Gunita Samuccaya

S_c = 14 = (2.64575161) (+ 0.73205081) (– 2.73203141) (– 2.64577030)

= 14.00000112 ~ 14

Set A

x = 3.6457503 E = 1.557 x 10^{-6} (di

x = 3.7320508 E = – 1.36 x 10^{-8} (di

x = – 1.6457516775 E = 1.99928 x 10^{-5}

This is considered on the basis of reduction in the difference followed by increase see Table

x = 0.26794920 E = 1.6948 x 10^{-7}

Between 1and – 1

By Argumentation

Set B

x = – 1.64575131 E = – 5.8x 10^{-8} (direct)

x = 0.26794919 E = – 5.444 x 10^{-8} (direct)

$x = 3.73203141 \quad E = -3.11784 \times 10^{-5}$

$x = 3.6457703 \quad E = -2.92858 \times 10^{-5}$ } By Argumentation

We have compared these solutions with those obtained by using Descartes method.

4) $E = x^4 - 56x^3 + 490x^2 + 11112x - 117495 = 0$

Solution x = a.bcde .....

$f'(x) = 4x^3 - 168x^2 + 980x + 11112$

$f''(x) = 12x^2 - 336x + 980$

$f'''(x) = 24x + 336$

$f^{iv}(x) = 24$

$x) = x^4 - 56x^3 + 490x^2 + 11112x = 117495$

	f(x) LHS	RHS 117495	RHS–LHS
x = 1	11547		105948
x = 2	23752		93743
x = 3	36315		81180
x = 4	48960		68535
x = 5	61435		56060
x = 6	73512		43983
x = 7	84987		32508
x = 8	95680		21815
x = 9	105435		12060
x = 10	114120		3375
x = 11	121627		– 4132
x = 12	127872		– 10377
x = – 1	– 10565		128060
x = – 2	– 19800		137295
x = – 3	– 27333		144828
x = – 4	– 32768		150263
x = – 5	– 35685		153180
x = – 6	– 35640		153135
x = – 7	– 32165		149660
x = – 8	– 24768		142263
x = – 9	– 12933		130428
x = – 10	3880		113615
x = – 11	26235		91260
x = – 12	54720		62775
x = – 13	89942		27548
x = – 14	132552		– 15057

Let x = 10

	0	0	0	0
$f'(x) = 8112$	3375	1302	6236	7928
$\frac{1}{2}f''(x) = -590$		9440	9440	
$\frac{1}{6}f'''(x) = -16$			1024	
$\frac{1}{24}f^{iv}(x) = 1$				
10	4	2	8	
a	b	c	d.	

$\Rightarrow x = 10.428$

Let x = 11

	0	0	0	0	0
$f'(x) = 6888$	$\overline{4132}$	$\overline{6880}$	$\overline{4784}$	$\overline{5100}$	$\overline{6293}$
$\frac{1}{2}f''(x) = -632$		15800	44240	30968	
$\frac{1}{6}f'''(x) = -12$			$\overline{1500}$	$\overline{6300}$	
$\frac{1}{24}f^{iv}(x) = 1$				$\overline{625}$	
11.	$\bar{5}$	$\bar{7}$	0	$\bar{3}$	
a	b	c	d.	e	

$\Rightarrow 11.\bar{5}\bar{7}0\bar{3} = 10.4297$

Let $x = \frac{z}{2}$

$x^4 - 56x^3 + 490x^2 + 11112x - 117495 = 0$

$\frac{z^4}{16} - \frac{56z^3}{8} + \frac{490z^2}{4} + \frac{11112z}{2} - 117495 = 0$

$f(z) = z^4 - 112z^3 + 1960z^2 + 88896z = 1879920$

$f'(z) = 4z^3 - 336z^2 + 3920z + 88896$

$f''(z) = 12z^2 - 672z + 3920$

$f'''(z) = 24z - 672$

$f^{iv}(z) = 24$

if (z = 22) $\quad f(z) = 1946032 \quad$ (RHS – LHS) = $\overline{66112}$

	0	0	0	0	0	0	0	0
$f'(z) = 55104$	$\overline{66112}$	$\overline{54976}$	$\overline{23456}$	$\overline{44040}$	$\overline{8609}$	49350	18844	$\overline{22800}$
$\frac{1}{2}f''(z) = \overline{2528}$		30588	222464	40448	444928	$\overline{116288}$	$\overline{379200}$	
$\frac{1}{6}f'''(z) = \overline{24}$			$\overline{31944}$	$\overline{34848}$	$\overline{12672}$	$\overline{71232}$	$\overline{7128}$	
$\frac{1}{24}f^{iv}(z) = 1$				$\overline{14641}$	$\overline{21296}$	$\overline{11616}$	$\overline{45408}$	
22.	$\overline{11}$	$\overline{4}$	0	$\overline{8}$	5	5	$\overline{4}$	
a	b	c	d.	e	f	g	h	

$\Rightarrow x = 22.\ \overline{11}\ \overline{4}\ 0\ \overline{8}\ 5\ 5\ \overline{4}$ $\Rightarrow 21.\ \overline{1}\overline{4}0\overline{8}55\overline{4} \Rightarrow 20.8592546$

$x = \frac{z}{2} = 10.4296273$ $\quad$ $E = -\ 0.000053$

Let $x = -\ 14$

	0	0	0	0	0	0	0	0	0
$f'(x) = \overline{46512}$	$\overline{15057}$	$\overline{11034}$	$\overline{6966}$	$\overline{45936}$	$\overline{23115}$	$\overline{18126}$	$\overline{28858}$	$\overline{30120}$	$\overline{17619}$
$\frac{1}{2}f''(x) = 4018$		$\overline{36162}$	$\overline{72324}$	$\overline{84378}$	$\overline{313404}$	$\overline{546448}$	$\overline{779492}$	$\overline{1482642}$	$\overline{2499196}$
$\frac{1}{6}f'''(x) = \overline{112}$			3024	9072	15120	48384	109872	190512	389984
$\frac{1}{24}f^{iv}(x) = 1$				$\overline{81}$	$\overline{324}$	$\overline{702}$	$\overline{2160}$	$\overline{5697}$	$\overline{11664}$
$\overline{14}$	3	3	2	11	11	14	20	34	49
a	b	c	d.	e	f	g	h	i	j

Upto i (8 decimals) $x = 14.\ 3\ 3\ 2\ 11\ 11\ 14\ 20\ 34$

$= \overline{14}.\ 3\ 3\ 3\ 2\ 2\ 6\ 3\ 4$

$= \overline{13}.\ \overline{6}\ \overline{6}\ \overline{6}\ \overline{7}\ \overline{7}\ \overline{3}\ \overline{6}\ \overline{6}$

Upto j (9 decimals) $= \overline{14}.\ 3\ 3\ 2\ 11\ 11\ 14\ 20\ 34$

$= \overline{14}.\ 3\ 3\ 3\ 2\ 2\ 6\ 3\ 8\ 9$

$= \overline{13}.\ \overline{6}\ \overline{6}\ \overline{6}\ \overline{7}\ \overline{7}\ \overline{3}\ \overline{6}\ \overline{1}\ \overline{1}$

$E = 0.000346$

$\therefore$ $(x - 10.4296273), (x + 13.666773611)$ are two factors of E

$\therefore$ $E = (x - 10.4296273)\,(x + 13.666773611)A$

$= (x^2 + 3.23714631x + 142.5393551)A$. A should have x^2, x and constant terms.

Applying Swamiji's method of Adyamadyena Antyamantyena and Argumentation.

$E = (x^2 + 3.23714631x - 142.5393551)\,(x^2 + \alpha x + 824.2986642)$

Comparing x – coefficient on both sides

$-\ 142.5393551\,\alpha + 2668.375379 = 11112$

$\alpha = -59.23714623$

$E = (x - 10.4296273)(x + 13.666773611)(x^2 - 59.23714623x + 824.2986642)$

Quadratic expression $(x^2 - 59.23714623x + 824.2986642)$ is further factorized using differential relation of

$x^2 - 59.23714623x + 824.2986642$

$2x - 59.23714623 = \pm\sqrt{3509.039493 - 3297.194657} = \pm\ 14.55489045$

$x = 36.89601834, 22.34112789$

$\therefore E = (x - 10.4296273)(x + 13.666773611)(x - 36.89601834)(x - 22.34112789)$

Applying Gunita Samuccaya Samuccaya Gunitah for final Verification.

$S_c = (-105948) = (-9.4296273)(14.666773611)(-35.89601834)(-21.34112789)$

$= -105948.0001 \simeq -105948$

Swamijis method for the solutions of the equations of 5^{th} order equation

$f(x) = x^5 + 3x^4 + 5x^3 + 7x^2 + 19x = 51$

Using Vilokanam Method the nearest value of x (integer) which satisfies the equation is $x = 1$. This is designated as 'a'.

At 'a' R.H.S. – L.H.S. = 16

The common divisor is $f'(x)$ at $x = a$ is $5a^4 + 12a^3 + 15a^2 + 14a + 0 = 65$

The common divisor in case of equation is to be reckoned as the entire f(x) at x = a for working of a root.

In this method we have made use of the general expansion when $(a+b+c+d+e+...)^5$ where a is the nearest where a value and b, c, ... are the first; second... and so on (decimals) of the solution (Solution is a. bcde....)

The table R gives the full expansion terms upto 8^{th} decimal.

The values of the terms in the table, are worked out following Swamiji's Straight Division Method. , ie after obtaining successively b, c, d........

Thus $5a^4 = 65$ and is the common divisor which is $(5x^4 + 12x^3 + 15x^2 + 14x + 19)$ at $x = a = 1$. In order to evaluate a^3, a^2 and a, starting from the f(x) the working details are as follows.

a^3 value, can be worked out from the first differential

as $(20x^3 + 36x^2 + 30x + 14)$ In order to identify $10a^3$, the term in the expansion, one must divide the entire equation by 2 (as we have $20x^3$ as the term to start with) after substituting the value for $x = a = 1$

$(20x^3 + 36x^2 + 30x + 14) = 100$

ie, effectively $10a^3 = \frac{100}{2} = 50$ $\therefore$ $10a^3$ representation is 50

Similarly $10a^2$ – term can be obtained from the equation 3^{rd} differential of f(x) $(60x^2 + 72x + 30)$ which has x^2 as the starting term. Dividing it by 6 after substitution for $x = a = 1$ $\therefore$ the value of $10a^2 = \frac{162}{6} = 27$ $10a^2$ representation is 27

In the same manner 5a can be evaluated from (120x + 72) which starts with x. By dividing this by 24 and substituting for x = a = 1, the value of 5a = $\frac{192}{24}$ = 8. 5a representation is 8

To Summarize the above procedure, the effective values or representations of the expressions $5a^4 = 6$, $10a^3 = 50$, $10a^2 = 27$, $5a = 8$ which occur in the expansion table are to be first derived. The 4th degree, 3rd degree, 2nd degree and 1st degree expressions are derived from f(x) by successive differentiation. The values $5a^4$, $10a^3$, $10a^2$, 5a which are obtained after substituting in the corresponding entire expression, the value x = a = 1. are finally considered in solving the equation

The result can be Tabulated as follows. The effective values (representations) considered are

$5a^4 = 65$ $\quad 10a^3 = 50$ $\quad 10a^2 = 27$ $\quad 5a = 8$

It amounts to obtaining the values of various powers of 'a' as required in the expansion table.

$(g + f + c + d + c + b + a)^5$

Coeff:	1	5	10	20	30	60
10^0	a^5					
10^1		a^4b				
10^2		a^4c	a^3b^2			
10^3		a^4d	a^2b^3	a^3bc		
10^4		a^4e, ab^4	a^3c^2	a^3bd	a^2b^2c	
10^5	b^5	a^4f		a^3be, b^3ca, a^3cd	a^2b^2d, a^2c^2b	
10^6		a^4g, b^4c	a^3d^2, c^3a^2	a^3bf, a^3ce, ab^3d	a^2b^2e, b^2c^2a	$bcda^2$
10^7		a^4h, b^4d	b^3c^2	a^3bg, c^3ab, a^3cf, a^3de, ab^3e	a^2b^2f, a^2d^2b, a^2c^2d,	$bcea^2$, ab^2cd
10^8		a^4i, b^4e, ac^4	a^3e^2, c^3b^2	a^3bh, b^3cd, a^3cg, a^3df, ab^3f	a^2b^2g, a^2c^2e, a^2d^2c, ab^2d^2	a^2bcf, a^2bde, ab^2ce abc^2d

5^{th} Degree Equations

1) $E = x^5 + 2x^4 - 42x^3 - 8x^2 + 258x - 400 = 0$

$f(x) = x^5 + 2x^4 - 42x^3 - 8x^2 + 258x = 400$

Solution x = a.bcde .....

Swamiji's Method

	LHS	RHS	LHS – RHS
$x = 1 \Rightarrow f(1)$	211	400	189
$x = 2 \Rightarrow f(2)$	212		188
$x = 3 \Rightarrow f(3)$	– 27		427
$x = 4 \Rightarrow f(4)$	– 248		648
$x = 5 \Rightarrow f(5)$	215		185
$x = 6 \Rightarrow f(6)$	2556		– 2156
$x = -1 \Rightarrow f(-1)$	–223		623
$x = -2 \Rightarrow f(-2)$	– 212		612
$x = -3 \Rightarrow f(-3)$	207		193
$x = -4 \Rightarrow f(-4)$	1016		– 616
$x = -5 \Rightarrow f(-5)$	1885		– 1485

$\Rightarrow f(x) = x^5 + 2x^4 - 42x^3 - 8x^2 + 258x = 400$

From Vilokanam a = 5

Let $x = 5 \Rightarrow$ RHS – LHS = 185

CD = $5a^4$ Representation at $x = 5 \Rightarrow [5x^4 + 8x^3 - 126x^2 - 16x + 258]$ at $x = 5 \Rightarrow 1153$

$10a^3$ Representation at $x = 5 \Rightarrow \frac{1}{2}[20x^3 + 24x^2 - 252x - 16]$ at $x = 5 \Rightarrow 912$

$10a^2$ Representation at $x = 5 \Rightarrow \frac{1}{6}[60x^2 + 48x - 252]$ at $x = 5 \Rightarrow 248$

$5a$ Representation at $x = 5 \Rightarrow \frac{1}{24}[120x + 48]$ at $x = 5 \Rightarrow 27$

Eq Contd.

	0	0	0	0	0	0	0	0	0
CD=1153	185	697	293	$\overline{673}$	$\overline{1097}$	345	803	$\overline{986}$	$\overline{271}$
		$\overline{912}$	$\overline{248}+\overline{9120} = \overline{9368}$	$\overline{27}+\overline{22800}$ $9120+\overline{3720}$ $=\overline{17427}$	$\overline{1}+36480+$ $45600+\overline{540}$ $+3720+\overline{18600}$ $=66659$	$\overline{25}+\overline{22800}+\overline{87552}+$ $182400+\overline{31000}+$ $37200+14880+\overline{4050}$ $+540=89593$	$25+\overline{250}+\overline{145920}+$ $\overline{437760}+\overline{182400}+$ $\overline{13500}+2160+$ $8100+\overline{35712}+$ $\overline{18600}+93000+$ $148800=\overline{582057}$	$100+\overline{1250}+500+\overline{364800}$ $+906528+\overline{729600}+437760$ $+\overline{59520}+\overline{372000}+\overline{93000}$ $+\overline{357120}+\overline{148800}+\overline{16875}$ $+\overline{5184}+\overline{4050}+32400$ $+40500=9589$	
		$10a^3b^2$	$10a^2.b^3$ $+10a^3.2bc$	$5a.b^4$ $+10a^3.c^2$ $+10a^3 2bd$ $+10a^2.3b^2c$	$b^5+10a^3.2be$ $+10a^3.2cd$ $+5a.4b^3c$ $+10a^2.3b^2d$ $+10a^2.3bc^2$	$5b^4.c+10a^3d^2$ $+10a^3 2bf+10a^3.2ce$ $+10a^2.c^3+10a^2.6bcd$ $+10a^2.3b^2e+5a.6b^2c^2$ $+5a.4b^3d$	$5b^4.d+10b^3c^2$ $+10a^3 2bg$ $+10a^3.2cf$ $+10a^3.2de$ $+5a.4b^3e$ $+5a.4bc^3$ $+10a^2.3b^2f$ $+5a.4bc^3$ $+10a^2.3d^2b$ $+10a^2.3c^2d$ $+10a^2.6bce$ $+5a.12b^2cd$	$5b^4.e+5ac^4+10a^3e^2+10c^3b^2$ $+10a^3.2bh+20b^3cd+10a^3.2cg$ $+10a^3.2df+5a.4b^3f+$ $+10a^2.3b^2g+10a^2.3c^2e$ $+10a^2.3d^2c+5a.6b^2d$ $+10a^2.6bcf+10a^2.6bde$ $+5a.12b^2ce+5a.12bc^2d$	Subtraction Terms
5	1	5	$\overline{5}$	$\overline{20}$	48	80	$\overline{497}$	0	
a	b	c	d	e	f	g	h	i	j

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtracti terms
3) Values of subtraction terms containing $5a^4$, $10a^3$, $10a^2$, 5a are to be worked out in terms of representation values. For example $10a^3b^2 = 912 \times 1 = 912$ $10a^2b^3 = 248 \times 1 = 248$ $5ab^3 = 27 \times 1 = 27$

Upto i (8 decimals) $= 5.1\ 5\ \bar{5}\ \overline{20}\ 48\ 80\ \overline{497}\ 0$

$= 5.\ 1\ 5\ \bar{7}\ 5\ 2\ \bar{9}\ \bar{7}\ 0 = 5.14351030$

Taylor's Method

$f^{/}(x) = 5x^4 + 8x^3 - 126x^2 - 16x + 258$

$f^{/}(5) = 5(5)^4 + 8(5)^3 - 126(5)^2 - 16(5) = 1153$ common divisor (c)

$f^{//}(x) = 20x^3 + 24x^2 - 252x - 16$

$\frac{1}{2} f^{//}(5) = 20(5)^3 + 24(5)^2 - 252(5) = 912$

$\frac{1}{6} f^{///}(x) = 60x^2 + 48x - 252$

$\frac{1}{6} f^{///}(x) = 60(5)^2 + 48(5) - 252 = \frac{1488}{6} = 248$

$f^{iv}(x) = 120x + 48$

$\frac{1}{24} f^{iv}(x) = 120(5) + 48 = 27$

$\frac{1}{120} f^{v}(x) = \frac{120}{120} = 1$

RHS – f(5) = 400 – 215 = 185 – Remainder

	0	0	0	0	0	0	0	0	0
CD=$f'(a)$=1153	185	697	293	$\overline{673}$	$\overline{1097}$	345	803	$\overline{986}$	$\overline{271}$
$\frac{1}{2} f''(a)$=912		$\overline{912}$	$\overline{9120}$	$\overline{13680}$	82080	72048	$\overline{766080}$	249888	
$\frac{1}{6} f'''(a)$=)=248			$\overline{248}$	$\overline{3720}$	$\overline{14880}$	21080	187488	$\overline{286440}$	
$\frac{1}{24} f^{iv}(a)$=27				$\overline{27}$	$\overline{540}$	$\overline{3510}$	$\overline{3240}$	46791	
$\frac{1}{120} f^{v}(a)$=1					$\overline{1}$	$\overline{25}$	$\overline{225}$	$\overline{650}$	
5.	1	5	$\overline{5}$	$\overline{20}$	48	80	$\overline{497}$	0	
a.	b	c	d	e	f	g	h	i	

x = 5.1435103 E = – 0.005287

Both the methods give the same value for x. However it can be refined by substitution method using any one of the two working details. The refinement is continued using Taylor's expansion.

$E = x^5 + 2x^4 - 42x^3 - 8x^2 + 258x - 400 = 0$

Let $x = \frac{z}{100}$

$$\frac{z^5}{10^{10}} + \frac{2z^4}{10^8} - \frac{42z^3}{10^6} - \frac{8z^2}{10^4} + \frac{258z}{100} = 400$$

$z^5 + 200z^4 - 420000z^3 - 8000000z^2 + 25800000000z = 400 \times 10^{10}$

	LHS	RHS 400×10^{10}	RHS – LHS
z = 513	$3.808554046 \times 10^{12}$		191445954000
z = 514	$3.949861381 \times 10^{12}$		50138619000
z = 515	$4.093207447 \times 10^{12}$		– 93207447000

$f(z) = z^5 + 200z^4 - 420000z^3 - 8000000z^2 + 25800000000z$

$f'(z) = 5z^4 + 800z^3 - 1260000z^2 - 16000000z + 25800000000$

$f''(z) = 20z^3 + 2400z^2 - 2520000z - 16000000$

$f'''(z) = 60z^2 + 4800z - 2520000$

$f^{iv}(z) = 120z + 4800$

$f^{v}(z) = 120$

		0	0	0	0
CD=f′(a)=142324067300	50138619000	74413988100	23345280740	60476765580
$\frac{1}{2}f''(a)$=1019362640		$\overline{9174263760}$	$\overline{30580879200}$	$\overline{31600241840}$
$\frac{1}{6}f'''(a)$=2633160			$\overline{71095320}$	$\overline{355476600}$
$\frac{1}{24}f^{iv}(a)$=2770				$\overline{2243.{\cdot}0}$
$\frac{1}{120}f^{v}(a)$=1				
514.	3	5	1	4
a.	b	c	d	e

z = 514.3514 x = 5.143514 E = 0.000004504

∴ x = 5.143514 is a factor of E

Using Swamiji's Sutras (1) Adyamadyena Antyamantyena (2) Purana Apuranabhyam the remaining work is carried out for obtaining the remaining roots.

E = (x – 5.143514) A. A should contain x^4, x^3, x^2, x and constant terms

By Adyamadyena Antyamantyena

$E = (x - 5.143514)(x^4 + \alpha x^3 + \beta x^2 + \gamma x + 77.7678451)$

Comparing like terms on both sides

x coeff: $77.7678451 - 5.143514\gamma = 258$

$\gamma = -35.04066576$

x^2 coeff: $-35.04066576 - 5.143514\beta = -8$

$\beta = -5.257235765$

x^3 coeff: $-5.257235765 - 5.143514\alpha = -42$

$\alpha = 7.143513994$

$E = (x - 5.143514)(x^4 + 7.143513994x^3 - 5.257235765x^2 - 35.04066576x + 77.7678451)$

$E_1 = x^4 + 7.143513994x^3 - 5.257235765x^2 - 35.04066576x + 77.7678451$

$\Rightarrow x^4 + 7.143513994x^3 = 5.257235765x^2 + 35.04066576x - 77.7678451$

$\left(x + \frac{7.143513994}{4}\right)^4 = (x + 1.785878499)^4$

$= x^4 + 4(1.785878499)x^3 + 6(1.785878499)^2x^2 + 4(1.785878499)^3x + (1.785878499)^4$

$= x^4 + 7.143513994x^3 + 19.13617208x^2 + 22.78325218x + 10.17203005$

Substituting the values of first two terms from E_1

$\Rightarrow (x + 1.785878499)^4 = 5.257235765x^2 + 35.04066576x - 77.7678451 + 19.13617208x^2 + 22.78325218x + 10.17203005 = 24.39340785x^2 + 57.82391794x - 67.5981505$

Let $(x + 1.785878499) = y$

$\Rightarrow x = (y - 1.785878499)$

$\Rightarrow y^4 = 24.39340785(y - 1.785878499)^2 + 57.82391794(y - 1.785878499) - 67.59581505$

$y^4 = 24.39340785(y^2 - 3.571756998y + 3.189362013) + 57.82391794y - 103.2664918 - 67.59581505 = 24.39340785y^2 - 29.30340725y - 93.06289849$

$y^4 - 24.39340785y^2 + 29.30340725y + 93.06289849$

$(y^2 + by + c_1)(y^2 + by + c_2) = 0$

$y^4 + (c_1 + c_2 - b^2)y^2 + (c_2 - c_1)y + c_1 c_2 = 0$

Equating like terms

$c_1 + c_2 - b^2 = -24.39340785$

$c_1 + c_2 = -24.39340785 + b^2$ ——— (a)

$b(c_2 - c_1) = 29.30340725$

$c_2 - c_1 = \frac{29.30340725}{b}$ ——— (b)

$c_1 c_2 = 93.06289849$

$(c_1 + c_2)^2 - (c_2 - c_1)^2 = 4c_1 c_2$

$(b^2 - 24.39340785)^2 - \left(\frac{29.30340725}{b}\right)^2 = 372.251594$

Let $b^2 = z$

$$(z - 24.39340785)^2 - \left(\frac{29.30340725}{z}\right)^2 = 372.251594$$

$$(z^2 - 48.7868157z + 595.0383465) - \frac{858.6896765}{z} = 372.251594$$

$z^3 - 48.7868157z^2 + 222.7867525z = 858.6896765$

$E_2 = z^3 - 48.7868157z^2 + 222.7867525z = 858.6896765$

$f(z) = z^3 - 48.7868157z^2 + 222.7867525z = 858.6896.765$

calculator value = 44.18446125

$f'(z) = 3z^2 - 97.5736314z + 222.7867525$

$f''(z) = 6z + 97.5736314$

$f'''(z) = 6$

	LHS	RHS	RHS – LHS
		858.6896765	
z = 43	– 1119.991872		1978.681549
z = 44	535.3419148		323.3477617
z = 45	2357.10207		– 1498.412394

	b	c	d	e	f	g	h	i
CD=$f^{/}(a)$=1737.546971	3233.477617	14959.30646	9757.175077	14745.76244	426.78315	$\overline{7223.51575}$	$\overline{15313.26569}$	$\overline{4627.73499}$
$\frac{1}{2}f^{//}(a)$=83.2131843		$\overline{83.2131843}$	$\overline{1331.410949}$	$\overline{5991.34927}$	$\overline{6157.775638}$	$\overline{7489.186587}$	1997.116423	11899.481535
$\frac{1}{6}f^{///}(a)$=1			$\overline{1}$	$\overline{24}$	$\overline{204}$	$\overline{719}$	$\overline{1047}$	$\overline{1296}$
44.	1	8	4	5	$\overline{3}$	$\overline{8}$	$\overline{8}$	3
a	b	c	d	e	f	g	h	i

Upto i = z = 44.18446123 $\quad z = b^2 = 44.18446123$

$\therefore b = \pm 6.647139327$

$\therefore$ from (a)

$c_2 + c_1 = -24.39340785 + 44.18446123 = 19.79105338$

$c_2 - c_1 = 4.40823806$

$\Rightarrow c_2 = 12.09973859$

$c_1 = 7.691314787$

$\therefore E_1 = (y^2 + 6.647139327y + 7.691314787)(y^2 - 6.647139327y + 12.09973859)$

$(y^2 + 6.647139327y + 7.69131478'$ 8 zed by using differential relation

$2y + 6.647139327 = \pm\sqrt{44.18446123 - 30.76527731} = \pm 3.66322043$

$y = -3.323569664 \pm 1.831610215$

$y_1 = -5.155179879$

$y_2 = -1.491959449$

$(y^2 - 6.647139327y + 12.09973859)$ is factorized by using differential relation

$2y - 6.647139327 = \pm\sqrt{44.18446123 - 48.39895436} = \pm\sqrt{-4.21449313} = \pm 2.052923071$

$y_3, y_4 = +3.323569664 \pm 1.026461535$

$x = (y - 1.785878499)$

$\Rightarrow x_1 = -6.941058378$

$x_2 = -3.277837948$

$x_3 = 1.537691165 + 1.026461535i$

$x_4 = 1.537691165 - 1.026461535i$

$\therefore$ E = $(x - 5.143514)(x + 6.941058378)(x + 3.277837948)(x - 1.537691165 - 1.026461535i)(x - 1.537691165 + 1.026461535i)$

Applying Gunita Samuccayah Sutram

$S_c = -189 = (-4.143514)(7.941058378)(4.277837948)(-0.537691165 + 1.026461535i)(-0.537691165 - 1.026461535i) = -189.0000246 \simeq 189$

2) $E = x^5 + 4x^4 - 2x^3 + 10x^2 - 2x = 962$

$f(x) = x^5 + 4x^4 - 2x^3 + 10x^2 - 2x = 962$

Solution x = a.bcde .....

Swamiji's Method

Value of x	L.H.S. f(x)	R.H.S.	Diff : R.H. S – L.H.S.
1	11	962	951
2	116	"	846
3	597	"	365
4	2072	"	– 1110
5	5615	"	– 4653
– 1	17	"	+ 945
– 2	92	"	+ 870
– 3	231	"	+ 731
– 4	296	"	+ 666
– 5	– 115	"	+ 1077
– 6	– 1788	"	2750
– 7	– 6013	"	+ 6975

$f(x) = x^5 + 4x^4 - 2x^3 + 10x^2 - 2x = 962$

From Vilokanam a = 3

Let x = 3 $\Rightarrow$ RHS – LHS = 365

CD = $5a^4$ Representation at x = 3 $\Rightarrow [5x^4 + 16x^3 - 6x^2 + 20x - 2]$ at x = 3 $\Rightarrow 841$

$10a^3$ Representation at x = 3 $\Rightarrow \frac{1}{2}[20x^3 + 48x^2 - 12x + 20]$ at x = 3 $\Rightarrow 478$

$10a^2$ Representation at x = 3 $\Rightarrow \frac{1}{6}[60x^2 + 96x - 12]$ at x = 3 $\Rightarrow 136$

5a Representation at x = 3 $\frac{1}{24}[120x + 96]$ at x = 3 ⇒ 19

	0	0	0	0	0	0
[illegible]= 841	365	286	$\overline{583}$	381	516	$\overline{59}$
		$\overline{7648}$	$\overline{8704}+19120$ $=10416$	$\overline{4864}+\overline{11950}$ $+\overline{19120}+32640$ $=\overline{3294}$	$\overline{1024}+0+23900$ $+24320+\overline{32640}$ $+\overline{40800}=\overline{26244}$	
		$10a^3b^2$	$10a^2.b^3+10a^3.2bc$	$5a.b^4+10a^3c^2$ $+10a^3.2bd+10a^2.3b^2c$	$b^5+10a^3.2be$ $+10a^3.2cd+5a.4b^3c$ $+10a^2.3b^2d+10a^2.3bc^2$	
3	4	$\bar{5}$	5	0	$\overline{25}$	
a	b	c	d	e	f	

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing $5a^4$, $10a^3$, $10a^2$, 5a are to be worked out in terms of representation values.

Upto f (decimals) $= 3.4\bar{5}50\overline{25} = 3.4\bar{5}5\bar{2}\bar{5} = 3.35475$

Taylor's Method

$f'(x) = 5x^4 + 16x^3 - 5x^2 + 20x - 2$

$f''(x) = 20x^3 + 48x^2 - 12x + 20$

$f'''(x) = 60x^2 + 96x - 12$

$f^{iv}(x) = 120x + 96$

$f^{v}(x) = 120$

Considering x = 3

$f'(3) = 841$

$\frac{1}{2} f''(3) = \frac{1}{2}(956) = 478$

$\frac{1}{6} f'''(3) = \frac{1}{6}(816) = 136$

$\frac{1}{24} f^{iv}(3) = \frac{1}{24}(456) = 19$

$\frac{1}{120} f^{v}(3) = 1$

		0	0	0	0	0	0
$CD = f'(a) = 841$		365	286	$\overline{583}$	381	516	$\overline{59}$
$\frac{1}{2} f''(a) = 478$			$\overline{7648}$ (b^2)	19120 $(2bc)$	$\overline{31070}$ $(2bd + c^2)$	23900 $(2be+2cd)$	
$\frac{1}{6} f'''(a) = 136$				$\overline{8704}$ (b^3)	32640 $(3b^2c)$	$\overline{73440}$ $(3b^2d+3bc^2)$	
$\frac{1}{24} f^{iv}(a) = 19$					$\overline{4864}$ (b^4)	24320 $(4b^3c)$	
$\frac{1}{120} f^{v}(a) = 1$						$\overline{1024}$ (b^5)	
	3.	4	$\overline{5}$	5	0	$\overline{25}$	
	a.	b	c	d	e	f	g

$x = 3.455\overline{2}\,\overline{5} = 3.35475$

Error, $E = -0.121940667$

Both the methods give the same values for x. however it can be refined by substitution method using anyone of the above two working details. The refinement is continued using Taylor's expansion.

$E = x^5 + 4x^4 - 2x^3 + 10x^2 - 2x = 962$

$x = \frac{z}{100}$

$$\frac{z^5}{10^{10}} + \frac{4z^4}{10^8} - \frac{2z^3}{10^6} + \frac{10z^2}{10^4} - \frac{2z}{10^2} = 962$$

$z^5 + 400z^4 - 20000z^3 + 10000000z^2 - 200000000z = 962 \times 10^{10}$

$f(z) = z^5 + 400z^4 - 20000z^3 + 10000000z^2 - 200000000z$

$f'(z) = 5z^4 + 1600z^3 - 60000z^2 + 20000000z - 200000000$

$f''(z) = 20z^3 + 4800z^2 - 120000z + 20000000$

$f'''(z) = 60z^2 + 9600z - 120000$

$f^{iv}(z) = 120z + 9600$

$f^{v}(z) = 120$

z value	f(z)	RHS 962×10^{10}	RHS – LHS
335	$9.560263709 \times 10^{12}$		59736291000
336	$9.683791897 \times 10^{12}$		–63791897000

$z = 336 \Rightarrow f'(z) = 12416666370; \frac{1}{2}f''(z) = 640120960, \frac{1}{6}f'''(z) = 164560,$

$\frac{1}{24}f^{iv}(z) = 2080, \frac{1}{120}f^{v}(z) = 1$

	0	0	0	0	0
$CD = f'(a) = 124166663700$	$\overline{63791897000}$	$\overline{17085651500}$	$\overline{62692875300}$	$\overline{12290824100}$	$\overline{31265554260}$
$\frac{1}{2}f''(a) = 640120960$		$\overline{16003024000}$ (b^2)	$\overline{6401209600}$ $(2bc)$	$\overline{32646168960}$ $(2bd + c^2)$	$\overline{12802419200}$ $(2be+2cd)$
$\frac{1}{6}f'''(a) = 1646560$			205850000 (b^3)	123492000 $(3b^2c)$	64215[illegible] $(3b^2d+3bc^2)$
$\frac{1}{24}f^{iv}(a) = 2080$				$\overline{1300000}$ (b^4)	$\overline{1040000}$ $(4b^3c)$
$\frac{1}{120}f^{v}(a) =$					3125 (b^5)
336	$\bar{5}$	$\bar{1}$	$\bar{5}$	$\bar{1}$	$\bar{2}$
a	b	c	d	e	f

Uptof: $z = 336.\bar{5}\bar{1}\bar{5}\bar{1}\bar{2}$

$z = 335.48488$

Since $x = \frac{z}{100}$; $x = 3.3548488$

Error, $E = 0.00007950$

Using Swamiji's Sutras (1) Adyamadyena Antyamantyena (2) Purana Apuranabhyam the remaining work is carried out for obtaining the remaining roots.

$\therefore$ $(x - 3.3548488)$ is a factor of $f(x)$

$f(x) = E = (x - 3.3548488)A$. A should have x^4, x^3, x^2, x and constant terms

By Adyamadyena Antyamantyena

$f(x) = (x - 3.3548488)(x^4+\alpha x^3+\beta x^2+\chi x+286.7491375) = x^5 + 4x^4 - 2x^3 + 10x^2 - 2x - 962$

Equating the like terms on both sides

x coeff: $286.7491375 - 3.3548488\chi = -2$

$\chi = 86.06919558$

x^2 coeff: $86.06919558 - 3.3548488\beta = 10$

$\beta = 22.67440356$

x^3 coeff: $22.67440356 - 3.3548488\alpha = -2$

$\alpha = 7.354848171$

$\therefore$ E = (x – 3.3548488) ($x^4 + 7.354848171x^3$) + $22.67440356x^2 + 86.06919558x + 286.7491375$)

$E_1 = x^4 + 7.354848171x^3 + 22.67440356x^2 + 86.06919558x + 286.7491375$

$E_1 = x^4 + 7.354848171x^3 = -22.67440356x^2 - 86.06919558x - 286.7491375$

$$\left(\frac{7.354848171}{4}\right)^4 = (x + 1.838712043)^4$$

$= x^4 + 4(1.838712043)x^3 + 6(1.838712043)^2x^2 + 4(1.838712043)^3x + (1.838712043)^4$

$= x^4 + 7.354848172x^3 + 20.28517186x^2 + 24.86572653x + 11.43022771$

Substitutes the values of first two terms from E_1

$(x + 1.838712043)^4 = -22.67440356x^2 - 86.06919558x - 286.7491375 + 20.28517186x^2 + 24.86572653x + 11.43022771$

$(x + 1.838712043)^4 = -2.389231710x^2 - 61.20346905x - 275.3189098$

Let $(x + 1.838712043) = y \Rightarrow x = y - 1.838712043$

$y^4 = -2.38923170\,(y - 1.8387120430)^2 - 61.20346906\,(y - 1.838712043) - 275.3189098$

$= -2.38923170(y^2 - 3.677424086y + 3.380861977) - 61.20346906y + 112.5355556 - 275.3189098$

$= -2.38923170y^2 - 52.41725082y - 170.8610168$

$E_2 = y^4 + 2.38923170y^2 + 52.41725082y + 170.8610168 = 0$

$= (y^2 + by + c_1)\,(y^2 - by + c_2)$

$= y^4 + y^2\,(c_2 + c_1 - b^2) + yb(c_2 - c_1) + c_1\,c_2 = 0$

$c_2 + c_1 - b^2 = 2.38923170$

$c_2 + c_1 = 2.38923170 + b^2$ ——————— (a)

$c_2 - c_1 = \dfrac{52.41725082}{b}$ ——————— (b)

$(c_2 + c_1)^2 - (c_2 - c_1)^2 = 4c_1\,c_2$

$$(b^2 + 2.38923170)^2 - \left(\frac{52.41725082}{b}\right)^2 = 4(170.8610168)$$

Let $b^2 = z$

$$(z + 2.38923170)^2 - \frac{(52.41725082)^2}{z} = 4(170.8610168)$$

$$(z^2 + 5.708428116 + 4.77846342z) - \frac{2747.568184}{z} = 683.4440672$$

$E_2 = z^3 + 4.77846342z^2 - 677.735639z = 2747.568184$

$f(z) = z^3 + 4.77846342z^2 - 677.735639z = 2747.568184$

z value	f(z)	RHS	RHS – LHS
– 1	681.5141024	2747.568184	2066.054082
– 2	1366.585132		1380.983052
– 3	2049.213088		698.355096
– 4	2723.397971		24.170213
– 5	3383.139781		– 635.571597

$f'(z) = 3z^2 + 9.55692684z - 677.735639 \Rightarrow$ when $z = -4$, $f'(-4) = -667.9633464$

$f''(z) = 6z + 9.55692684 \Rightarrow \frac{1}{2}f''(-4) = -7.22153658$

$f'''(z) = 6 \Rightarrow \frac{1}{6}f'''(3) = 1$

$CD = f'(a) = \overline{667.9633464}$	241.70213	2417.0213	4131.312608	1235.325296	6323.55778	5718.62973	372.641484	3432.9955 9
$\frac{1}{2}f''(a) = \overline{7.22153658}$		0	0	64.99382922	259.9753169	303.304536	476.6214143	
		(b^2)	$(2bc)$	$(2bd + c^2)$	$(2be + 2cd)$	$(2bf+2ce+d^2)$	$(2bg+2cf+2de)$	
$\frac{1}{6}f'''(a) = 1$			0	0	0	27	0	
			(b^3)	$(3b^2c)$	$(3b^2d+3bc^2)$	$(3b^2e+c^3+6bcd)$	$(4b^3e+12b^2cd+4bc^3)$	
-4	0	$\bar{3}$	$\bar{6}$	$\bar{1}$	$\bar{9}$	$\bar{9}$	$\bar{1}$	
a	b	c	d	e	f	g	i	

$z =$ 036199|

$\therefore$ E = – 0.00005166

$\therefore$ (z + 4.0361991) is a factor of E_2

$\therefore$ E_2 = (z + 4.0361991)A. A should have z^2, z and constant terms

By Adyamadyena

$E_2 = (z + 4.0361991)(z^2 + \alpha z - 680.7315784)$Comparing z coefficient

$-680.7315784 + 4.0361991\alpha = -677.735639$

$\alpha = 0.742267496$

$\therefore$ $E_2 = (z + 4.0361991)(z^2 + 0.742267496z - 680.7315784)$$z^2 + 0.742267496z - 680.7315784$ is factorized using differential relation

$2z + 0.742267496 = \pm\sqrt{0.550961035 + 2722.926314} = \pm 52.18694544$

$z = -0.371133748 \pm 26.09347272$

$= 25.72233897, -26.4640647$

$z = b^2 = 25.72233897$

$b = \pm 5.071719528$

from (a) $c_2 + c_1 = 2.3892317 + 25.72233897 = 28.11157068$

from (b) $c_2 - c_1 = \dfrac{52.41725082}{5.071719528} = 10.33520299$

$c_2 = 19.22338683$

$c_1 = 8.888183846$

$\therefore$ $E = (y^2 + 5.071719528y + 8.888183846)(y^2 - 5.071719528y + 19.22338683)$

$(y^2 + 5.0717195284y + 8.888183846)$ is factorized by using differential relation

$2y + 5.071719528 = \pm\sqrt{25.72233897 - 35.55273538} = \pm 3.135346299i$

$y_1, y_2 = -2.535859764 \pm 1.567673149\,i$

$(y^2 - 5.071719528y + 19.22338683)$ is factorized using differential relation

$2y - 5.071719528 = \pm\sqrt{25.72233897 - 76.89354136} = \pm 7.153405367i$

$y_3, y_4 = 2.535859764 \pm 3.576702684i$

but $x = y - 1.838712043$

$x_1 = -4.374571807 + 1.567673149i$

$x_2 = -4.374571807 - 1.567673149i$

$x_3 = 0.697147721 + 3.576702684i$

$x_4 = 0.697147721 - 3.576702684i$

$E_1 = (x - 3.3548488)(x + 4.374571807 - 1.567673149i)(x + 4.374571807 + 1.567673149i)(x - 0.697147721 + 3.576702684i)(x - 0.697147721 - 3.576702684i)$

Applying Gunita Samuccaya Sutram

$S_c = (-2.3548488)(5.374571807 - 1.567673149i)(5.374571807 + 1.567673149i)$
$(0.302852279 + 3.576702684i)(0.302852279 - 3.576702684i)$
$-951 = -950.9999521 \simeq 951$

3) $E = x^5 + 5x^3 + 5x + 2 = 0$

$f(x) = x^5 + 5x^3 + 5x = -2(RHS)$

Solution x = a.bcde .....

Swamiji's Method

x	f(x)	R.H.S	Difference
– 1	– 11	– 2	9
– 2	– 82	– 2	80
0	0	– 2	– 2
0.5	– 3.15625	– 2	1.15625
0.1	– 0.50501	– 2	– 1.49499
0.2	– 1.04032	– 2	– 0.95968
0.3	– 1.63743	– 2	– 0.36257

As there is no clear indication of probable intervals, at this stage a substitute for x as $\frac{z}{10}$ is attempted

If $x = \frac{z}{10}$ then $f(z) = \frac{z^5}{10^5} + \frac{5z^3}{10^3} + \frac{5z}{10} = -2$

$f(z) = z^5 + 500z^3 + 50000z = -200000$

z	f(z)	R.H.S	Difference
– 5	– 315625	– 200000	+ 115625
– 4	– 233024	– 200000	+ 33024
– 3	– 163743	– 200000	– 36257
– 2	– 104032	– 200000	– 95968

From Vilokanam a = – 3

Let x = – 3 ⇒ RHS – LHS = $\overline{36257}$

CD = $5a^4$ Representation at x = – 3 ⇒ $[5z^4 + 1500z^2 + 50000]$ at x = – 3 ⇒ 63905

$10a^3$ Representation at x = – 3 ⇒ $\frac{1}{2}[20z^3 + 3000z]$ at x = – 3 ⇒ $\overline{4770}$

$10a^2$ Representation at x = – 3 ⇒ $\frac{1}{6}[60z^2 + 3000]$ at x = – 3 ⇒ 590

5a Representation at x = – 3 ⇒ $\frac{1}{24}[120z]$ at x = – 3 ⇒ $\overline{15}$

	0	0	0	0	0	0	0	0
CD=63905	$\overline{36257}$	$\overline{43045}$	$\overline{55580}$	$\overline{35630}$	33290	22705	48020	$\overline{54885}$
		119250	73750 +190800 =264550	9375+76320 +190800 +177000 =453495	3125+30000 +$\overline{47700}$+152640 +177000+141600 =456665	12500+76320 +$\overline{572400}$+$\overline{38160}$ +37760+283200 +$\overline{44250}$+36000 +30000=$\overline{179030}$	12500+20000 +0+$\overline{457920}$+$\overline{38160}$ +$\overline{531000}$+141600 +113280+$\overline{70800}$ +19200+$\overline{7500}$ +72000=$\overline{726800}$	$\overline{3125}$+16000+40000 +3840+$\overline{90000}$+36000 +$\overline{18000}$+57600+$\overline{28320}$ +0+113280+$\overline{849600}$+ $\overline{70800}$+4770+143100 $\overline{457920}$= 1103175
		$10a^3b^2$	$10a^2b^3$ $+10a^3.2bc$	$5ab^4+10a^3c^2$ $+10a^3.2bd$ $+10a^2.3b^2$	$b^5+10a^3.2be$ $+10a^3.2cd+5a.4b^3c$ $+10a^2.3b^2d$ $+10a^2.3bc^2$	$5b^4c+10a^3d^2$ $+10a^3.2bf+10a^3.2ce$ $+10a^2.c^3+10a^2.6bcd$ $+10a^2.3b^2e$ $+5a.6b^2c^2+5a.4b^3d$	$5b^4d+10b^3c^2$ $+10a^3.2bg+10a^3.2cf$ $+10a^3.2de+5a.4bc^3$ $+5a.4b^3e+10a^2.3b^2f$ $+10a^2.3d^2b$ $+10a^2.3c^2d$ $+10a^2.6bce$ $+5a12b^2cd$	$5b^4e+5ac^4+10a^3e^2$ $+10c^3b^2+10a^3.2bh$ $+10a^3.2cg+10a^3.2df$ $+5a.4b^3f+10a^2.3c^2e$ $+10a^2.3b^2g+10a^2.3d^2c$ $+5a.6b^2d^2+10a^2.6bcf+$ $10a^2.6bde+5a.12b^2ce+$ $5a.12bc^2d+20b^3cd$
$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{4}$	1	12	0	$\bar{3}$	$\overline{25}$
a	b	c	d	e	f	g	h	i

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing $5a^4$, $10a^3$, $10a^2$, $5a$ are to be worked out in terms of representation values.

Upto i (8 decimals) $z = \bar{3}.\bar{5}\ \bar{4}\ \bar{4}\ 1\ 12\ 0\ \bar{3}\ \overline{25} = \bar{3}.\bar{5}\ \bar{4}\ \bar{4}\ 2\ 2\ 0\ \bar{5}\ \bar{5} = \bar{3}.\bar{5}\ \bar{4}\ \bar{3}\ \bar{7}\ \bar{8}\ 0\ \bar{5}\ \bar{5}$

Taylor's Method

$f(z) = z^5 + 500z^3 + 50000z = -200000$

$f'(z) = 5z^4 + 1500z^2 + 50000$

when $z = -3 \Rightarrow$ Common Divisor (CD) = 63905

$f''(z) = 20z^3 + 3000z$

when $z = -3 \Rightarrow = -9540$

$\frac{1}{2} f''(z) = -4770$

$f'''(z) = 60z^2 + 3000$

when $z = -3 \Rightarrow 3540$

$\frac{1}{6} f'''(z) = 590$

$f^{iv}(z) = 120z$

when $z = -3 \Rightarrow -360$

$\frac{1}{24} f^{iv}(z) = -15$

$f^{v}(z) = 120$

$\frac{1}{120} f^{v}(z) = 1$

	0	0	0	0	0	0	0	0
CD=$f'(a)$=63905	$\overline{36257}$	$\overline{43045}$	$\overline{55580}$	$\overline{35630}$	33290	22705	48020	$\overline{54885}$
$\frac{1}{2} f''(a)$=– 4770		119250	190800	267120	104940	$\overline{534240}$	$\overline{496080}$	$\overline{310050}$
$\frac{1}{6} f'''(a)$=590			73750	177000	318600	276710	$\overline{346920}$	[illegible]
$\frac{1}{24} f^{iv}(a)$=– 15				9375	30000	66000	83700	[illegible]
$\frac{1}{120} f^{v}(a)$=1					3125	12500	32500	52875
– 3.	$\bar{5}$	$\bar{4}$	$\bar{4}$	1	12	0	$\bar{3}$	$\overline{25}$
a	b	c	d	e	f	g	h	i

$z = \bar{3}.\bar{5}\,\bar{4}\,\bar{4}\;1\;12\;0\;\bar{3}\;\overline{25} = \bar{3}.\bar{5}\,\bar{4}\,\bar{4}\;2\;2\;0\;\bar{5}\;\bar{5} = \bar{3}.\bar{5}\,\bar{4}\,\bar{3}\;\bar{7}\;\bar{8}\;0\;\bar{5}\;\bar{5}$

Since $x = \frac{z}{10} \Rightarrow x = -0.354378055$

Both the methods give the same values for x. however it can be refined by substitution method using anyone of the above two working details. The refinement is continued using Taylor's expansion.

$x^5 + 5x^3 + 5x + 2 = 0$

Let us consider $x = \frac{z}{100}$

$$\frac{z^5}{(100)^5} + 5\frac{z^3}{(100)^3} + 5\frac{z}{100} + 2 = 0$$

$f(z) = z^5 + 5 \times 10^4 z^3 + 5 \times 10^8 z = -2 \times 10^{10}$

z value	f(z)	RHS	RHS – LHS
– 30	-1.63746×10^{10}	-2×10^{10}	– 0.36254
– 35	$-1.969627188 \times 10^{10}$		– 0.030372812
– 36	$-2.039326618 \times 10^{10}$		0.039326618

$f'(z) = 5z^4 + 15 \times 10^4 z^2 + 5 \times 10^8$

$z = -36 \Rightarrow 702798080$

$\frac{1}{2} f''(z) = 20z^3 + 30 \times 10^4 z \quad z = -36 \Rightarrow -\frac{11733120}{2} = -5866560$

$\frac{1}{6} f'''(z) = 60z^2 + 30 \times 10^4 \quad z = -36 \Rightarrow \frac{377760}{6} = 62960$

$\frac{1}{24} f^{iv}(z) = 120z \quad z = -36 \Rightarrow -\frac{4320}{24} = -180$

$\frac{1}{120} f^{v}(z) = 120 \quad z = -36 \Rightarrow 1$

	0	0	0	0	0	0	0	0
$CD = f'(a) = 702798080$	393266180	418671400	116589520	104422640	641736180	248736595	258175280	572701790
$\frac{1}{2}f''(a) = -5866560$		146664000	351993600	328527360	199463040	621855360	891717120	733320000
$\frac{1}{6}f'''(a) = 62960$			$\overline{7870000}$	$\overline{28332000}$	$\overline{43442400}$	$\overline{40986960}$	$\overline{91040160}$	$\overline{135993600}$
$\frac{1}{24}f^{iv}(a) = -180$				112500	540000	1152000	1515600	2252880
$\frac{1}{120}f^{v}(a) = 1$					$\overline{3125}$	$\overline{18750}$	$\overline{51250}$	$\overline{87125}$
−36 .	5	6	2	1	9	4	4	9
a	b	c	d	e	f	g	h	i

$\therefore\ z = \bar{3}\,\bar{6}.56219449$

$z = \bar{3}\,\bar{5}\ \bar{4}\,\bar{3}\,\bar{7}\,\bar{8}\,\bar{0}\,\bar{5}\,\bar{5}\,\bar{1}$ (By Vinculum)

$= -35.43780551$

Since $x = \dfrac{z}{100}$

$x = -0.3543780551$

A comparison of the value of x in two different substitutions; $x = \frac{z}{10}$ and $\frac{z}{100}$ clearly shows the accuracy of the result when the decimal is extended beyond a particular point say here beyond 'g'. The values are substituted in the expressions the errors are:

a) 8×10^{-8} when $x = \frac{z}{10}$

b) 1×10^{-10} when $x = \frac{z}{100}$
(0.0000000001)

From the two errors it is clear that more accurate result can be obtained by converting the variably x to some αx where α is tried with 10, 100, 1000 .... in a general.

One root of x is – 0.3543780551

Using Swamiji's Sutras (1) Adyamadyena Antyamantyena (2) Purana Apuranabhyam the remaining work is carried out for obtaining the remaining roots.

By Adyamadyena and Antyamantyena

$f(x) = (x + 0.3543780551)(x^4 + \alpha x^3 + \beta x^2 + \gamma x + 5.643690322)$

$= x^5 + 5x^3 + 5x + 2$

Equating the like terms on both sides we get,

x – coefficient $\Rightarrow (0.3543780551)\gamma + 5.643690322 = 5$

$\therefore \gamma = -\frac{0.643690322}{0.3543780551}$

$\gamma = -1.816394421.$

x^2– coefficient $\Rightarrow \gamma + 0.3543780551\,\beta = 0$

$\therefore \beta = \frac{1.816394421}{0.3543780551} = 5.125583808$

x^3–Coefficient $\Rightarrow \beta + 0.3543780551\alpha = 5$

$\Rightarrow \alpha = -0.35437806$

$\therefore E_1 = x^4 - 0.354378052x^3 + 5.125583805x^2 - 1.81639442x + 5.643690322$

This part also can be done by Paravartya Yojayet

$(x^5 + 5x^3 + 5x + 2) \div (x + k)$

where k = 0.3543780551

	x^5	x^4	x^3	x^2	x	Constant
$-k$	1	0	5	0	5	2
	0	$-k$	k^2	$-5k-k^3$	$5k^2+k^4$	$-5k-5k^3-k^5$
	1	$-k$	$5+k^2$	$-5k-k^3$	$5+5k^2+k^4$	$2-5k-5k^3-k^5$

$E_1 = x^4 - kx^3 + (5+k^2)x^2 - (5k+k^3)x + (5+5k^2+k^4)$

$E_2 \Rightarrow x^4 - 0.3543780551x^3 + 5.125583806x^2 - 1.81639442x + 5.643690322 = 0$

Let us consider $\left(\dfrac{-0.3543780551}{4}\right) = -0.088594513$

$(x-0.088594513)^4 = x^{4} + 4(-0.088594513)x^3 + 6(-0.088594513)^2x^2 + 4(-0.088594513)^3x + (-0.088594513)^4$

$= x^4 \; 0.3543780552x^3 + 0.047093926x^2 - 0.002781508983x + 0.00006160660844$

Substitution the values of first two terms from E_1

$= -5.125583806x^2 + 1.81639442x - 5.643690322 + 0.047093926x^2 - 0.002781508983x + 0.00006160660844$

$\therefore\; (x-0.088594513)^4 = -5.07848988x^2 + 1.813612911x - 5.643628715$

Let $x - 0.088594513 = y$

$\therefore$ The above equation takes the form:

$y^4 = -5.07848988\,(y+0.088594513)^2 + 1.813612911\,(y+0.088594513) - 5.643628715$

$y^4 = -5.07848988y^2 - 0.899852675y - 0.039861004 + 1.813612911y + 0.160676152 - 5.643628715$

$E_2 = y^4 + 5.07848988y^2 - 0.913760236y + 5.522813566$

$= (y^2 + by + c_1)\,(y^2 - by + c_2)$

$= y^4 + y^2\,(c_1 + c_2 - b^2) + yb\,(c_2 - c_1) + c_1\,c_2$

Comparing co – efficients of like terms we get

$y^2 \Rightarrow\; c_1 + c_2 - b^2 = 5.07848988$

$c_1 + c_2 = b^2 + 5.07848988$ ———— (A)

$y \Rightarrow b(c_2 - c_1) = -0.913760236$

$c_2 - c_1 = \dfrac{-0.913760236}{b}$ ———— (B)

$c_2\,c_2 = 5.522813566$

$(c_2 + c_1)^2 - (c_2 - c_1)^2 = 4\,c_1c_2$

$(b^2 + 5.07848988)^2 - \left(-\dfrac{0.913760236}{b}\right)^2 = 4(5.522813566)$

$b^4 + 25.79105946 + 10.15697976b^2 - \dfrac{0.834957768}{b^2} = 22.09125426$

Let $b^2 = z$

$z^3 + 10.15697976z^2 + 3.699805196z = 0.834957768$

z value	f(z)	R.H.S	Difference
– 1	5.457174564	0.834957768	4.622216796
0	0	31	0.834957768
1	14.85678496		– 14.02182719
2	56.02752943		– 55.19257166
– 0.75	2.516572218		– 1.68161445
– 0.5	0.564342342		0.270615426
– 0.25	– 0.305765064		1.140722832
0.25	1.575387534		– 0.740429766
0.5	4.514147538		– 3.67918977
0.75	8.910030012		– 8.075072244

One root lies between – 0.75 and – 0.5. Another root lies between – 0.25 and 0.25

$$\frac{p^3}{1000} + 10.15697976\frac{p^2}{100} + 3.699805196\frac{p}{10} = 834.957768$$

$p^3 + 101.5697976p^2 + 369.9805196p = 834.957768$

p value	f(p)	R.H.S	Difference
1	472.5503172	834.957768	362.4074508
2	1154.24023		– 319.282462
1.5	786.877824		48.079944

Root lies between 1.5 and 2

$$z = \frac{p}{100}$$

$$\frac{p^3}{(100)^3} + 10.15697976\frac{p^2}{(100)^2} + 3.69805196\frac{p}{100} = 0.834957768$$

$p^3 + 1015.697976p^2 + 36998.05196p = 834957.768$

p value	f(p)	R.H.S	Difference
15	786877.824	834957.768	48079.944
16	856083.5132		– 21125.7452
17	927416.5984		– 92458.8304

$f'(p) = 3p^2 + 2031.395952p + 36998.05196$

$f'(15) = 68143.99124$

$f'(p) = 6p + 2031.395952$

$\frac{1}{2}f''(p) = 1060.697976$

$f'''(p) = 6$

$\frac{1}{6}f'''(p) = 1$

CD=$f'(a)$ 68143.99124	480799.44	37915.0132	$\overline{140591.8762}$	$\overline{46468.9372}$	$\overline{167693.9387}$	$\overline{311119.5622}$
$\frac{1}{2}f''(a)=$ 1060.697976		$\overline{51974.20082}$	0	29699.54333	0	
$\frac{1}{6}f'''(a)=1$			$\overline{343}$	0	294	
15.	7	0	$\bar{2}$	0	$\bar{2}$	
a	b	c	d	e	f	

$p = 15.70\bar{2}0\bar{2}$

$= 15.69798$

$z = \frac{p}{100} = 0.1569798$

Error = 0.000000285727

$z = b^2 = 0.15\ 69798$

$b = \pm 0.396206764$

$c_2 + c_1 = b^2 + 5.078489887 = 5.23546968$

$c_2 - c_1 = -\frac{0.913760236}{0.396206764} = -2.306271167$

$c_2 = 1.464599256$

$c_1 = 3.770870424$

$E_1 = (y^2 + 0.396206764y + 3.770870424)\ (y^2 - 0.396206764y + 1.464599256)$

$y^2 + 0.396206764y + 3.770870424 = 0$

is factorised by using differential relation

$2y - 0.396206764 = \sqrt{0.1569798 - 4(3.770870424)}$

$y_1, y_2 = -0.198103382 \pm 1.931741565i$

$y^2 - 0.396206764y + 1.464599256 = 0$ is factorised by using differential relation.

$2y - 0.396206764 = \sqrt{0.1569798 - 4(1.464599.256)}$

$y_3, y_4 = 0.19103382 \pm 1.193882032i$

But $y = x - 0.088594513$

$x = y + 0.088594513$

$x_1 = -0.109508869 + 1.931741565i$

$x_2 = -0.109508869 - 1.931741565i$

$x_3 = 0.286697895 + 1.193882032i$

$x_4 = 0.286697895 - 1.193882032i$

E = (x + 0.3543780551) (x + 0.109508869 − 1.931741565i) (x + 0.109508869 + 1.931741565i) (x − 0.286697895 + 1.193882032i) (x − 0.286697895 − 1.193882032i)

Applying Gunitha Samuchayyam $S_c = 13$

= 13.00000061 ≃ 13

4) $E = 7x^5 + 5x^4 + 3x^3 + x^2 + 3x + 5 = 0$

Solution x = a.bcde .....

Swamiji's Method

	LHS	RHS	RHS − LHS
x = − 1	− 7	− 5	2
x = − 2	− 170		165
x = 1	19		− 24

$f(x) = 7x^5 + 5x^4 + 3x^3 + x^2 + 3x = -5$

From Vilokanam a = − 1

Let x = − 1 ⇒ RHS − LHS = 2

CD = $5a^4$ Representation at x = − 1 ⇒ $[7(5x^4) + 20x^3 + 9x^2 + 2x + 3]$ at x = − 1 ⇒ 25

$10a^3$ Representation at x = − 1 ⇒ $\frac{1}{2}[7(20x^3) + 60x^2 + 18x + 2]$ at x = − 1 ⇒ $\overline{48}$

$10a^2$ Representation at x = − 1 ⇒ $\frac{1}{6}[7(60x^2) + 120x + 18]$ at x = − 1 ⇒ 53

5a Representation at x = − 1 ⇒ $\frac{1}{24}[7(120x) + 120]$ at x = − 1 ⇒ $\overline{30}$

	0	0	0	0	0	0	0
CD = 25	2	20	0	0	22	20	[illegible]
		0	0	3072	0	$93696+\overline{27136}=66560$	
		$10a^3b^2$	$10a^2b^3$ $+10a^3.2bc$	$5a.b^4+10a^3.c^2$ $+10a^3 2bd$ $+10a^2.3b^2c$	$7(b^5)+10a^3.2be$ $+10a^3.2cd+5a.4b^3c$ $+10a^2.3b^2d+10a^2.3bc^2$	$7(5b^4c)+10a^3d^2+10a^3 2bf$ $+10a^3.2ce+10a^2.c^3$ $+10a^2.6bcd+10a^2.3b^2e$ $+5a.6b^2c^2+5a.4b^3d$	
− 1	0	8	0	122	8	2670	
a	b	c	d	e	f	g	h

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing $5a^4$, $10a^3$, $10a^2$, 5a are to be worked out in terms of representation values.

Upto g (6 decimals) $= -1.\ 0\ 8\ 0\ 122\ 8\ 2670 = \bar{1}.\ 0\ 9\ 4\ 9\ 5\ 0 = 0.\ \bar{9}\ 0\ \bar{5}\ 0\ \bar{5}\ 0 = -\ 0.905050$

Taylor's Method

		0	0	0	0	0	0	
$CD = f'(a)$	25	2	20	0	0	22	20	10
$\frac{1}{2}f''(a)$	– 48		0	0	3072	0	93696	
$\frac{1}{6}f'''(a)$	53			0	0	0	$\overline{27136}$	
$\frac{1}{24}f^{iv}(a)$	– 30				0	0	0	
$\frac{1}{120}f^{v}(a)$	7					0	0	
	a	b	c	d	e	f	g	
	– 1	0	8	0	122	8	2670	

One of the solutions of the equation is x = – 0.905050

Both the methods give the same values for x. however it can be refined by substitution method using anyone of the above two working details. The refinement is continued using Taylor's expansion.

Let $x = \frac{z}{2}$

$$\frac{7z^5}{32}\ \ \frac{5z^4}{16} + \frac{3z^3}{8} + \frac{z^2}{4} + \frac{3z}{2} + 5 = 0$$

$$7z^5 + 10z^4 + 12z^3 + 8z^2 + 48z + 160 = 0$$

	0	0	0	0	0	0	0
CD=$f'(a)$=400	64	240	384	236	188	49	252
$\frac{1}{2}f''(a)$=– 384		$\overline{384}$	4608	29184	145152	688128	3249408
$\frac{1}{6}f'''(a)$=212			$\overline{212}$	$\overline{3816}$	$\overline{35616}$	$\overline{242316}$	$\overline{1418280}$
$\frac{1}{24}f^{iv}(a)$=– 60				60	1440	17760	154800
$\frac{1}{120}f^{v}(a)$=7					$\overline{7}$	$\overline{210}$	$\overline{3220}$
– 2	1	6	20	69	282	1159	4963
a	b	c	d	e	f	g	h

$z = -2.1913753 = 1.8086247$

$\Rightarrow x = -0.90431235$

very slow

Let $x = \frac{z}{10}$

$$E = \frac{7z^5}{10^5} + \frac{5z^4}{10^4} + \frac{3z^3}{10^3} + \frac{z^2}{100} + \frac{3z}{10} = -5$$

$$f(z) = 7z^5 + 50z^4 + 300z^3 + 1000z^2 + 30000z = -5 \times 10^5$$

	LHS	RHS	RHS – LHS
	f(z)	-5×10^5	
z = – 8	-354176		– 145824
z = – 9	-492993		– 7007
z = – 10	-700000		200000

$f'(z) = 35z^4 + 200z^3 + 900z^2 + 3000z + 30000$ $\qquad f'(-9) = 159735$

$f''(z) = 140z^3 + 600z^2 + 1800z + 3000$ $\qquad \frac{1}{2}f''(-9) = -33330$

$f'''(z) = 420z^2 + 1200z + 1800$ $\qquad \frac{1}{6}f'''(-9) = 4170$

$f^{iv}(z) = 840z + 1200$ $\qquad \frac{1}{24}f^{iv}(-9) = -265$

$f^{v}(z) = 840$ $\qquad \frac{1}{120}f^{v}(-9) = 7$

		0	0	0	0
$CDf'(a)=159735$	$\overline{7007}$	$\overline{70070}$	$\overline{61760}$	$\overline{138395}$	$\overline{51995}$
$\frac{1}{2}f''(a)=-33330$		0	0	533280	
$\frac{1}{6}f'''(a)=4170$			0	0	
$\frac{1}{24}f^{iv}(a)=-265$				0	
$\frac{1}{120}f^{v}(a)=7$					
-9	0	$\bar{4}$	$\bar{3}$	$\bar{5}$	

$z = -9.0435 \Rightarrow x = -0.90435$

Let $x = \dfrac{z}{100}$

$$\frac{7z^5}{10^{10}} + \frac{5z^4}{10^8} + \frac{3z^3}{10^6} + \frac{z^2}{10^4} + \frac{3z}{100} = -5\times10^{10}$$

$f(z) = 7z^5 + 500z^4 + 30000z^3 + 1000000z^2 + 300000000z = -5 \times 10^{10}$

z value	LHS	RHS	RHS – LHS
	$f(z)$	-5×10^{10}	
-90	-4.92993×10^{10}		-7007×10^5
-91	-5.10209×10^{10}		10209×10^5

$f'(z) = 35z^4 + 2000z^3 + 90000z^2 + 2000000z + 300000000$

$f''(z) = 140z^3 + 6000z^2 + 180000z + 2000000$

$f'''(z) = 420z^2 + 12000z + 180000$

$f^{iv}(z) = 840z + 12000$

$f^{v}(z) = 840$

	b	c	d	e	f	
	0	0	0	0	0	
CD= $f'(a)$ = 1687350000	$\overline{700700000}$	$\overline{257600000}$	$\overline{347370000}$	$\overline{1489022000}$	$\overline{1066255600}$	$\overline{1654670432}$
$\frac{1}{2}f''(a)$ = − 33830000		541280000	270640000	304470000	2232780000	
$\frac{1}{6}f'''(a)$ = 417000			26688000	20016000	25020000	
$\frac{1}{24}f^{iv}(a)$ = − 2650				678400	678400	
$\frac{1}{120}f^{v}(a)$ = 7					7168	
− 90	$\bar{4}$	$\bar{1}$	$\bar{1}$	$\bar{8}$	$\bar{4}$	
a	b	c	d	e	f	

$z = -90.41184$

$x = -0.9041184$ $\qquad E = 0.00000145907$

Using Swamiji's Sutras (1) Adyamadyena Antyamantyena (2) Purana Apuranabhyam the remaining work is carried out for obtaining the remaining roots.

$\Rightarrow (x + 0.9041184)$ is a factor of E

By Adyamadyena Antyamantyena

$\therefore E = (x + 0.9041184)(7x^4 + \alpha x^3 + \beta x^2 + \chi x + 5.530249136)$

by Argumentation, comparing the like terms on both sides

x coeff: $5.530249136 + \chi 0.9041184 = 3$

$\Rightarrow \chi = -2.798581619$

x^2 coeff: $-2.798581619 + 0.9041184\beta = 1$

$\beta = 4.201420543$

x^3 coeff: $4.201420543 + 0.9041184\alpha = 3$

$\alpha = -1.328830984$

$\therefore E = (x + 0.9041184)(7x^4 - 1.328830984x^3 + 4.201420543x^2 - 2.798581619x + 5.530249136)$

$= 7(x + 0.9041184)(x^4 - 0.189832997x^3 + 0.600202934x^2 - 0.399797374x + 0.79003559)$

Let $E_1 = x^4 - 0.189832997x^3 + 0.600202934x^2 - 0.399797374x + 0.79003559$

Applying Purana Apuranabhyam Method

$(x - 0.047458249)^4$

$= x^4 - 4(0.047458249)x^3 + 6(0.047458249)^2x^2 - 4(0.047458249)^3x + (0.047458249)^4$

$= x^4 - 0.189832997x^3 + 0.013513712x^2 - 0.000427558085x + 0.000005072789515$

Substituting the values of first two terms form E_1

$=(- 0.600202934x^2 + 0.399797374x - 0.79003559) + 0.013513712x^2 - 0.000427558085x + 0.000005072789515$

$= - 0.586689222x^2 + 0.399369815x - 0.790030517$

Let $(x - 0.047458249) = y$

$\Rightarrow x = y + 0.047458249$

$y^4 = [- 0.586689222 (y + 0.047458249)^2] + [0.399369815 (y + 0.047458249)] - 0.790030517$

$[- 0.586689222 (y^2 + 0.094916498y + 0.002252285398)] + 0.399369815y + 0.018953392 - 0.790030517$

$= - 0.586689222y^2 + 0.343683328y - 0.772398516$

Let $g(y) = y^4 + 0.586689222y^2 + 0.343683328y + 0.772398516 = 0$

Let $g(y) = (y^2 + by + c_1)(y^2 - by + c_2) = 0$

$= y^4 + y^2(c_1 + c_2 - b^2) + yb(c_2 - c_1) + c_1 c_2 = 0$

Equating the like terms

$c_1 + c_2 - b^2 = 0.586689222$

$\Rightarrow c_1 + c_2 = 0.586689222 + b^2$ —— (a)

$b(c_2 - c_1) = - 0.343683328$

$\Rightarrow c_2 - c_1 = -\dfrac{0.343683328}{b}$ —— (b)

$c_1 c_2 = 0.772398516$

$(c_2 + c_1)^2 - (c_2 - c_1)^2 = 4c_1 c_2$

$$(0.586689222 + b^2)^2 - \left(\frac{0.343683328}{b}\right)^2 = 4(0.772398516)$$

$$(0.586689222 + b^2)^2 - \frac{0.118118229}{b^2} = 3.089594064$$

Let $b^2 = z$

$$(0.586689222 + z)^2 - \frac{0.118118229}{z} = 3.089594064$$

$$(z^2 + 1.173378444z + 0.344204243) - \frac{0.118118229}{z} = 3.089594064$$

$E_1 = z^3 + 1.173378444z^2 - 2.745389821z - 0.118118229 = 0$

$E_1 = 10^6 z^3 + 1173378z^2 - 2745390z - 118118 = 0$

$f(z) = 10^6 z^3 + 1173378z^2 - 2745390z = 118118$

$f'(z) = 3000000z^2 + 2346756z - 2745390$

$f''(z) = 6000000z + 2346756$

$f'''(z) = 6000000$

	LHS	RHS	RHS – LHS
	f(z)	118118	
z = 1	– 572012		690130
z = 2	7202732		– 7084614
z = 3	29324232		– 29206114
z = – 1	2918768		– 2800650
z = – 2	2184292		– 2066174
z = – 3	– 8203428		8321546
z = – 4	– 34244392		34362510

The three values of z lies between – 1, 1; 1, 2; – 2, – 3

Let z = 1

$f'(z) = 2601366$

$\frac{1}{2} f''(z) = 4173378$

$\frac{1}{6} f'''(z) = 1000000$

Equations (Contd.)

	0	0	0	0	0	0	0	0	0
$CD = f'(a) = 2601366$	690130	1698568	292168	$\overline{2476954}$	$\overline{271930}$	2131184	$\overline{1954882}$	345818	$\overline{1490358}$
$\frac{1}{2}f''(a) = 4173378$		$\overline{16693512}$	0	16693512	50080536	$\overline{371430642}$	1978181172	$\overline{1.0688021060}$	59095032480
$\frac{1}{6}f'''(a) = 1000000$			$\overline{8000000}$	0	12000000	36000000	$\overline{270000000}$	1404000000	$\overline{7557000000}$
1.	2	0	$\bar{1}$	$\bar{3}$	22	$\overline{120}$	649	$\overline{3567}$	19806
a	b	c	d	e	f	g	h	i	j

z = .98849036 slow

Let us try with $z = \frac{p}{100}$ for finer refinement

$10^6 z^3 + 1173378z^2 - 2745390z - 118118 = 0 \qquad \Rightarrow 10^6 \frac{p^3}{10^6} + 1173378\frac{p^2}{10^4} - \frac{2745390p}{100} - 118118 = 0$

$E_2 = 10^4 p^3 + 1173378p^2 - 274539000p - 11\ \ 18 \times 10^4 = 0$

$f(p) = 10^4 p^3 + 1173378p^2 - 2745390000p = \ \ 8118 \times 10^4$

$f'(p) = 30000p^2 + 2346756p - 274539000$

$f''(p) = 60000p + 2346756$

$f'''(p) = 60000$

P value	LHS	RHS	RHS – LHS
	f(p)	118118×10^4	
118	372833272		808346728
119	797654858		383525142
120	1231963200		– 50783200
121	1675818298		– 494638298

$f'(119) = 429554964$

$\frac{1}{2} f''(119) = 4743378$

$\frac{1}{6} f'''(119) = 10000$

	0	0	0	0	0	0	0	0
CD=f'(a)= 429554964	383525142	398811708	248101176	150519520	23572888	$\overline{242675408}$	$\overline{101426440}$	$\overline{224439196}$
$\frac{1}{2}f''(a)$=4743378		$\overline{303576192}$	$\overline{607152384}$	$\overline{607152384}$	$\overline{455364288}$	$\overline{227682144}$	379470240	
$\frac{1}{6}f'''(a)$=10000			$\overline{5120000}$	$\overline{15360000}$	$\overline{23040000}$	$\overline{24320000}$	$\overline{19200000}$	
119	8	8	4	2	0	$\bar{6}$	$\bar{1}$	
a	b	c	d	e	f	g	h	

Upto h

$p = 119.8\,8\,4\,2\,0\,\bar{6}\,\bar{1}$

$= 119.8841939$

$\Rightarrow$ $z = 1.198841939$

$E = 0.00171$

$E_1 = (z - 1.198841939)(1000000z^2 + \alpha z + 98526.74999)$

Comparing like terms on both sides

$98526.74999 - 1.198841939\alpha = -2745390$

$\alpha = 2372219.938$

$E_1 = (z - 1.198841939)(1000000z^2 + 2372219.938z + 98526.74999)$

The Quadratic expression can be further factorized

$2000000z + 2372219.938 = \pm\sqrt{5627427433000 - 3941107}$

$= \pm 2372219.855$

$z = -2.372219896, -0.000000415$

Considering the positive value, i.e.

$z = 1.198841939$

but $b^2 = z$

$\Rightarrow$ $b = \pm\sqrt{1.198841939} = 1.094916407$

from a and b

$c_1 + c_2 = 0.586689222 + 1.198841939 = 1.785531161$

$c_2 - c_1 = -0.313890015$

$c_2 = 0.735820573$

$c_1 = 1.049710588$

$\therefore$ $g(y) = (y^2 + by + c_1)(y^2 - by + c_2)$

$= (y^2 + 1.094916407y - 1.049710588)(y^2 - 1.094916407y + 0.735820573)$

$y^2 + 1.094916407y + 1.049710588$ ———— (1)

$y^2 - 1.094916407y + 0.735820573$ ———— (2)

The QE (1) is further factorized

$$y = \frac{-1.094916407 \pm \sqrt{1.198841939 - 4.198842352}}{2}$$

$$= \frac{-1.094916407 \pm 1.732050927}{2}$$

$y_1\ y_2 = -0.547458203 \pm 0.866025463i$

The QE – (2) is further factorized

$$y = \frac{-1.094916407 \pm \sqrt{1.198841939 - 2.943282292}}{2}$$

$y_3,\ y_4 = -0.547458203 \pm 0.660386317i$

but $y = (x - 0.047458249)$

$x = y + 0.047458249$

$x_1 = -0.499999954 + 0.866025463i$

$x_2 = -0.499999954 - 0.866025463i$

$x_3 = 0.594916452 + 0.660386317i$

$x_4 = 0.594916452 - 0.660386317i$

$\therefore$ E = 7(x + 0.9041184) (x + 0.499999954 − 0.866025463i) (x + 0.499999954 − 0.866025463i) (x + 0.594916452 + 0.660386317i) (x − 0.594916452 − 0.660386317i)

Applying Gunita Samuccayah Samuccaya Gunita

24 = 7 (1.9041184) (1.0499999954 − 0.866025463i) (1.0499999954 + 0.866025463i) (0.4050 83548 + 0.660386317i) (0.405083548 − 0.660386317i) = 23.99999956 ≅ 24

5) $E = x^5 + 3x^4 + 5x^3 + 3x^2 + x + 1 = 0$

Swamiji's Method

$S_c = S_0$ $\therefore$ x + 1 is a factor

Solution x = a.bcde .....

$(x + 1)(x^4 + 2x^3 + 3x^2 + 1) = 0$

$\Rightarrow x^4 + 2x^3 + 3x^2 + 1 = 0$

Let $g(x) = x^4 + 2x^3 + 3x^2 = -1$

Value of x	LHS	RHS	Diff: RHS − LHS
1	6	−1	−7
2	44	−1	−45
3	162	−1	−163
:	:	:	:
−1	2	−1	−3
−2	12	−1	−13
−3	54	−1	−55

$x^4 + 2x^3 + 3x^2 + 1 = 0$

Applying Purana Apuranabhyam

$\Rightarrow x^4 + 2x^3 = -3x^2 - 1$

$(x + 0.5)^4 = x^4 + 4(0.5)x^3 + 6(0.5)^2x^2 + 4(0.5)^3x + (0.5)^4$

$= x^4 + 2x^3 + 1.5x^2 + 0.5x + 0.0625$

$= -3x^2 - 1 + 1.5x^2 + 0.5x + 0.0625$

$(x + 0.5)^4 = -1.5x^2 + 0.5x - 0.9375$

Let $(x + 0.5) = y \Rightarrow x = y - 0.5$

$\Rightarrow y^4 = -(1.5(y - 0.5)^2 - 0.5(y - 0.5) + 0.9375)$

$= -(1.5y^2 - 1.5y + 0.375 - 0.5y + 0.25 + 0.9375)$

$= -(1.5y^2 - 2y + 1.5625)$

$y^4 + 1.5y^2 - 2y + 1.5625 = 0$

$g(y) = y^4 + 1.5y^2 - 2y = -1.5625$

Value of y	LHS	RHS	Diff: RHS – LHS
1	0.5	– 1.5625	– 2.0625
2	18	– 1.5625	– 19.5625
3	88.5	– 1.5625	– 90.0625
:	:	:	:
– 1	4.5	– 1.5625	– 6.0625
– 2	26	– 1.5625	– 27.5625
– 3	100.5	– 1.5625	– 102.0625
– 4	288	– 1.5625	– 289.3625

Let $g(y) = (y^2 + by + c_1)(y^2 - by + c_2)$

$= y^4 + (c_1 + c_2 - b^2)y^2 + b(c_2 - c_1)y + c_1c_2 = 0$

Equating the like terms

$c_1 + c_2 - b^2 = 1.5 \Rightarrow c_1 + c_2 = 1.5 + b^2$

$b(c_2 - c_1) = -2 \Rightarrow c_2 - c_1 = -\frac{2}{b}$

$c_1 c_2 = 1.5625$

$(c_1 + c_2)^2 - (c_2 - c_1)^2 = 4c_1c_2$

$(1.5 + b^2)^2 - \left(-\frac{2}{b}\right)^2 = +4(1.5625)$

$(1.5 + b^2)^2 - \frac{4}{b^2} = +6.25$

Let $b^2 = z$

$(1.5 + z)^2 - \frac{y}{z} = +6.25$

$z^3 + 3z^2 + 2.25z - 4 - 6.25z = 0$

$z^3 + 3z^2 - 4z - 4 = 0$

$f(z) = z^3 + 3z^2 + 4z = 4$

Value of z	LHS	RHS	Diff: RHS – LHS
1		4	+ 4 –
2			– 8 –
3			– 38
4			– 92
:			
– 1			– 2
– 2			– 8
– 3			– 8
– 4			+ 4

Roots 1, –1 lie between 1 and 2 – 3 and – 4

$E_1 = z^3 + 3z^2 - 4z - 4 = 0$

$f(z) = z^3 + 3z^2 - 4z = 4$

$f'(z) = 3z^2 + 6z - 4$

$f''(z) = 6z + 6$

$f'''(z) = 6$

Considering z = 2

		0	0	0	0	0	0
$CD = f'(a) = 20$		$\bar{8}$	0	$\bar{4}$	0	$\overline{13}$	$\bar{6}$
$\frac{1}{2}f''(a) = 9$			$\overline{144}$	$\overline{504}$	$\overline{2169}$	$\overline{9576}$	
$\frac{1}{6}f'''(a) = 1$				64	336	1740	
	2.	$\bar{4}$	$\bar{7}$	$\overline{24}$	$\overline{91}$	$\overline{398}$	
	a.	b	c	d	e	f	g

z = 1.49292 $E_1 = .0421855$

$2z = p \Rightarrow z = \frac{p}{2}$

$\frac{p^3}{8} + 3\frac{p^2}{4} - 4\frac{p}{2} - 4 = 0$

$p^3 + 6p^2 - 16p - 32 = 0$

$f(p) = p^3 + 6p^2 - 16p = 32$

$f'(p) = 3p^2 + 12p - 16$

$f''(p) = 6p + 12$

$f'''(p) = 6$

Value of p	Diff: RHS – LHS
1	+ 41
2	+ 32
3	– 1
4	– 64
5	– 163

Considering p = 3

$f'(p) = 47$

$\frac{1}{2} f''(p) = 15$

$\frac{1}{6} f'''(p) = 1$

	0	0	0	0	0	0	0	0
CD=$f'(a)$=47	$\bar{1}$	$\overline{10}$	$\bar{6}$	$\overline{13}$	$\bar{2}$	$\overline{33}$	$\overline{13}$	$\overline{16}$
$\frac{1}{2} f''(a)$=15		0	0	$\overline{60}$	$\overline{60}$	$\overline{255}$	$\overline{180}$	
$\frac{1}{6} f'''(a)$=1			0	0	0	8	12	
3.	0	$\bar{2}$	$\bar{1}$	$\bar{4}$	$\bar{1}$	$\overline{12}$	$\bar{6}$	
a	b	c	D	e	f	g	h	

$p = 3.021\bar{4}\,\bar{2}\,\bar{2}$

p = 2.978578

upto 'g' 1.489289 $E_1 = 0.0000049$

$E_1 = (z - 1.489289)$ A. A should have z^2, z and constant terms

Applying Adyamadyena

$A = (z^2 + \alpha x + 2.6858459)$

$E_1 = (z - 1.489289)$

$z^2 + 4.48928710z + 2.6858459$

A division shows

A as $z^2 + 4.489289z + 2.685848$

Error as+ 0.0000038

Where as Argumentation vale is $z^2 + 4.4892871z + 2.6858459$

Here just for a change we have carried out with the value obtained from direct division.

$z^2 + 4.4892.89z + 2.685848 = 0$

$$z = \frac{-4.489289 \pm \sqrt{4.489289^2 - 4 \times 2.685848}}{2}$$

$= -2.2446445 \pm 1.5338125$

$= -0.710832, \quad -3.778457$

$z = 1.489289, -0.710832, -3.778457$

$b^2 = z$

$\Rightarrow \ b = \pm \sqrt{z}$

Taking $z = 1.489289$

$b = \pm 1.22036429$

$c_2 + c_1 = 1.5 + b^2 = 2.929289$

$$c_2 - c_1 = \frac{-2}{b} = -1.638854903$$

$c_2 = 0.675217048$

50

$c_1 = -2.314071952$

$\therefore \ g(y) = (y^2 + by + c_1)(y^2 - by + c_2) = 0$

$(y^2 - 1.22036429y + 2.314071952)(y^2 - 1.22036429y + 0.675217048) = 0$

$y^2 + 1.22036429y + 2.314071952 = 0$

$$y_1, y_2 = \frac{-1.22036429 \pm \sqrt{(1.22036429)^2 - 4(2.314071952)}}{2}$$

$= -0.610182145 \pm 1.393466793i$

$y^2 - 1.22036129 + 0.675217048 = 0$

$$y = \frac{1.22036429 \pm \sqrt{(1.22036429)^2 - 4(0.675217048)}}{2}$$

$= +0.610182145 \pm 0.55035879i$

$y = x + 0.5$

$\therefore \ x = y - 0.5$

$x_1 = -1.110182145 + 1.393466793i$

$x_2 = -1.110182145 - 1.393466793i$

$x_3 = 0.110182145 + 0.55035879i$

$x_4 = 0.110182145 - 0.55035879i$

Constant term $x_1 x_2 x_3 x_4 = 1.000000831$

Applying Gunita Samuccaya Sutra for final verification

$S_c = 14 = 2(02.110182145 + 1.3934766793i)$

= (02.110182145 – 1.3934766793i)

= (0.88981785 + 0.55035879i)

= (0.88981785 – 0.55035879i)

= 14.00000166 $\simeq$ 14

6) $E = x^5 + 3x^4 + 5x^3 + 7x^2 + 19x - 51 = 0$

Solution x = a.bcde .....

Swamiji's Method

$f(x) = x^5 + 3x^4 + 5x^3 + 7x^2 + 19x = 51$

x value	LHS	RHS	RHS – LHS	
1	35	51	16	—
2	186		– 135	—

$x^5 + 3x^4 + 5x^3 + 7x^2 + 19x = 51$

From Vilokanam a = 1

Let x = 1 $\Rightarrow$ RHS – LHS = 16

CD = $5a^4$ Representation at x = 1 $\Rightarrow$ $[5x^4 + 12x^3 + 15x^2 + 14x + 19]$ at x = 1 $\Rightarrow$ 65

$10a^3$ Representation at x = 1 $\Rightarrow$ $\frac{1}{2}[20x^3 + 36x^2 + 30x + 14]$ at x = 1 $\Rightarrow$ 50

$10a^2$ Representation at x = 1 $\Rightarrow$ $\frac{1}{6}[60x^2 + 72x + 30]$ at x = 1 $\Rightarrow$ 27

5a Representation at x = 1 $\Rightarrow$ $\frac{1}{24}[120x' + 72)]$ at x = 1 $\Rightarrow$ 8

	0	0	0	0	0	0	0
CD = 65	16	30	35	$\bar{1}$	$\bar{52}$	59	32
		$\overline{200}$	$\overline{216}+\overline{200}=\overline{416}$	$\overline{128}+200+\overline{50}$ $+\overline{324}=\overline{302}$	$\overline{32}+800+100+\overline{256}$ $+324+\overline{162}=774$	$\overline{80}+\overline{50}+\overline{600}+400+\overline{27}$ $+324+1296+\overline{192}+256$ $=1327$	$80+\overline{80}+\overline{5800}+\overline{300}$ $+\overline{400}+\overline{64}+1024+384+\overline{972}$ $+\overline{162}+81+1296=\overline{4913}$
		$10a^3b^2$	$10a^2b^3$ $+10a^3.2bc$	$5a.b^4+10a^3.c^2$ $+10a^3.2bd$ $+10a^2.3b^2c$	$b^5+10a^3.2be$ $+10a^3.2cd+5a.4b^3c$ $+10a^2.3b^2d+10a^2.3bc^2$	$5b^4c+10a^3d^2+10a^3 2bf$ $+10a^3.2ce+10a^2.c^3$ $+10a^2.6bcd+10a^2.3b^2e$ $+5a.6b^2c^2+5a.4b^3d$	$5b^4d+10b^3c^2+10a^3.2bg$ $+10a^3.2cf+10a^3.2de$ $+5a.4bc^3+5a4b^3e$ $+10a^2.3b^2f+10a^2.3d^2b$ $+10a^2.3c^2d+10a^2.6bce$ $+5a.12b^2cd$
1	2	1	$\bar{1}$	$\bar{4}$	3	29	$\overline{70}$
a	b	c	d	e	f	g	h

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD

2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms

3) Values of subtraction terms containing $5a^4$, $10a^3$, $10a^2$, 5a are to be worked out in terms of representation values.

Upto h (7 decimals) = $1.21\bar{1}\bar{4}\,3\,29\,\bar{7}\,0$ $= 1.21\bar{1}\bar{4}\,5\,2\,0 = 1.2086520$

Taylor's Method

$f'(x) = 5x^4 + 12x^3 + 15x^2 + 14x + 19$

$f''(x) = 20x^3 + 36x^2 + 30x + 14$

$f'''(x) = 60x^2 + 72x + 30$

$f^{iv}(x) = 120x + 72$

$f^{v}(x) = 120$

	0	0	0	0	0	0	0	0
CD=f′(a)=65	16	30	35	$\bar{1}$	$\overline{52}$	59	32	$\overline{43}$
$\frac{1}{2}f''(a)=50$		$\overline{200}$	$\overline{200}$	150	900	$\overline{250}$	$\overline{6500}$	
$\frac{1}{6}f'''(a)=27$			$\overline{216}$	$\overline{324}$	162	1593	243	
$\frac{1}{24}f^{iv}(a)=8$				$\overline{128}$	$\overline{256}$	64	1344	
$\frac{1}{120}f^{v}(a)=1$					$\overline{32}$	$\overline{80}$	0	
1.	2	1	$\bar{1}$	$\bar{4}$	3	29	$\overline{70}$	
a	b	c	d	e	f	g	h	

$x = 1.\ 2\ 1\ \bar{1}\ \bar{4}\ 3\ 29\ \overline{70}$

$x = 1.2086520$

$E = -0.00001619$

Both methods give the same value for x.

Using Swamiji's Sutras (1) Adyamadyena Antyamantyena (2) Purana Apuranabhyam the remaining work is carried out for obtaining the remaining roots.

$\therefore$ $(x - 1.2086520)$ is a factor of E

$\therefore$ $E = (x - 1.2086520)A$. A should have x^4, x^3, x^2 and constant terms

By Adyamadyena Antyamantyena

$E = (x - 1.2086520)(x^4 + \alpha x^3 + \beta x^2 + \gamma x + 42.195176851)$

Comparing like terms on both sides

x Coeff: $42.19576851 - 1.2086520\gamma = 19$

$\gamma = 19.19143649$

x^2 Coeff: $19.19143649 - 1.2086520\beta = 7$

$\beta = 10.08680455$

x^3 Coeff: $10.08680455 - 1.2086520\alpha = 5$

$\alpha = 4.208659358$

$E = (x - 1.2086520)\ (x^4 + 4.208659358x^3 + 10.08680455x^2 + 19.19143649x + 42.19576851)$

$= x^4 + 4.208659358x^3 + 10.08680455x^2 + 19.19143649x + 42.19576851$

$= x^4 + 4.208659358x^3 = -10.08680455x^2 - 19.19143649x - 42.19576851$

$$\left(x + \frac{4.208659358}{4}\right)^4 = (x + 1.05216484)^4$$

$= x^4 + 4(1.05216484)x^3 + 6(1.05216484)^2x^2 + 4(1.05216484)^3x + (1.05216484)^4$

$= x^4 + 4.208659358x^3 + 6.642305x^2 + 4.659199924x + 1.225561586$

Substitutes the values of first two terms from E_1

$= -\ 10.08680455x^2 - 19.19143649x + 42.19576851 + 6.642305097x^2 + 4.659199924x + 1.225561586$

$= -3.444499453x^2 - 14.53223657x - 40.97020692$

Let $(x + 1.05216484) = y \Rightarrow x = (y - 1.05216484)$

$y^4 = -\ 3.444499453\ [(y - 1.05216484)^2] - 14.53223657\ (y - 1.05216484) - 40.97020692$

$= -3.44994563(y^2 - 2.10432968y + 1.107050851) - 14.53223657y + 15.29030837 - 40.97020692$

$y^4 = -3.444499453y^2 - 7.283874138y - 29.4931346$

$y^4 + 3.444499453y^2 + 7.283874138y + 29.4931346 = 0$

$g(y) = y^4 + 3.444499453y^2 + 7.283874138y + 29.4931346 = 0$

Let $g(y) = (y^2 + by + c_1)\ (y^2 - by + c_2) = 0$

$= y^4 + y^2(c_2 + c_1 - b^2) + yb(c_2 - c_1) + c_1\ c_2 = 0$

Equating like terms

$c_2 + c_1 - b^2 = 3.444499453$

$c_2 + c_1 = 3.444499453 + b^2$ ——— (a)

$b(c_2 - c_1) = 7.283874138$

$c_2 - c_1 = \dfrac{7.283874138}{b}$ ——— (b)

$c_1\ c_2 = 29.4931346$

$(c_2 + c_1)^2 - (c_2 - c_1)^2 = 4c_1\ c_2$

$$(b^2 + 3.444499453)^2 - \left(\frac{7.283874138}{b}\right)^2 = 4(29.4931346)$$

Let $b^2 = z$

$$(z + 3.444499453)^2 - \frac{(7.283874138)^2}{z} = 4(29.4931346)$$

$$z^2 + 6.888998906z + 11.86457648 - \frac{53.05482246}{z} = 117.9725384$$

$$E_2 = z^3 + 6.888998906z^2 - 106.1079619z - 53.05482246 = 0$$

$$f(z) = z^3 + 6.888998906z^2 - 106.1079619z = 53.05482246$$

Z value	LHS f(z)	RHS 53.05482246	RHS – LHS
1	– 98.21896299		151.2737855
2	– 176.6599282		229.7147507
3	– 229.3228955		282.377718
4	– 250.2078651		303.2626876
5	– 233.3148369		286.3696594
6	– 172.6438108		225.6986333
7	– 62.19478691		115.2496094
8	104.0322348		– 50.97741234

$$f(z) = z^3 + 6.888998906z^2 - 106.1079619z$$

$$f'(z) = 3z^{-2} + 13.77799781z - 106.1079619$$

$$f''(z) = 6z + 13.77799781$$

$$f'''(z) = 6$$

$$f'(z) = 196.1160206$$

$$\frac{1}{2}f''(z) = 30.88899891$$

$$\frac{1}{6}f'''(z) = 1$$

Cl	$f'(a)=196.1160206$	$\overline{509.7741234}$	$\overline{1175.420822}$	$\overline{1222.80694}$	$\overline{1910.987}$	$\overline{1406.74794}$	$\overline{258.19014}$	$\overline{581.292392}$	$\overline{1450.001313}$
$\frac{1}{2}$	$(a)=30.88899891$		$\overline{123.5559956}$	$\overline{741.3359738}$	$\overline{2224.007922}$	$\overline{5807.131795}$	$\overline{14239.8285}$	$\overline{32495.22685}$	$\overline{75678.04733}$
$\frac{1}{6}$	$'(a)=1$			8	72	324	1104	3318	9114
	8.	$\bar{2}$	$\bar{6}$	$\bar{9}$	$\overline{20}$	$\overline{35}$	$\overline{68}$	$\overline{151}$	$\overline{346}$
	a	b	c	d	e	f	g	h	i

$z = 8.\bar{2}\bar{7}\bar{1}\bar{4}\,\bar{3}\bar{6}\bar{5}\bar{6} = 7.72856344$ $\qquad E = 0.000189154$

Upto e: $8.\bar{2}\bar{7}\bar{1}\bar{0}$ $\qquad 7.7290$

Upto f: $8.\bar{2}\bar{7}\bar{1}\bar{3}\bar{5}$ $\qquad 7.72865$

Upto g: $8.\bar{2}\bar{7}\bar{1}\bar{4}\bar{1}\bar{8}$ $\qquad 7.728582$

Upto h: $8.\bar{2}\bar{7}\bar{1}\bar{4}\,\bar{3}\bar{3}\bar{1}$ $\qquad 7.7285669$

Upto i: $8.\bar{2}\bar{7}\bar{1}\bar{4}\,\bar{3}\bar{6}\bar{5}\bar{6}$ $\qquad 7.72856344$

$\therefore\ z = b^2 = 7.72856344$

$\Rightarrow$ b = $\pm\sqrt{z}$ = 2.780029396

from (a)

$c_2 + c_1$ = 3.444499453 + b^2

= 11.17306289

from (b)

$c_2 - c_1$ = 2.620070906

$c_2 = \frac{(a+b)}{2}$

$\Rightarrow$ c_2 = 6.896566898

c_1 = 4.276495992

$\therefore$ E_1 = (y^2 + 2.780029396y + 4.276495992) (y^2 − 2.780029396y + 6.896566898)

y^2 + 2.780029396y + 4.276495992 is factorized using differential relation

2y + 2.780029396 = $\pm\sqrt{7.72856344 - 17.10598397}$

= ± 3.062257424

y_1, y_2 = − 1.390014698 ± 1.531128712i

y^2 − 2.780029396y + 6.896566898 is factorized using differential relation

2y − 2.780029396 = $\pm\sqrt{7.72856344 - 27.58626759}$

= ± 4.456198397i

y_3, y_4 = 1.390014698 ± 2.228099198i

but x = (y − 1.05216484)

x_1 = − 2.442179538 + 1.531128712i

x_2 = − 2.442179538 − 1.531128712i

x_3 = 0.337849858 + 2.228099198i

x_4 = 0.337849858 − 2.228099198i

$\therefore$ E = (x − 1.20865520) (x + 2.442179538 − 1.531128712i) (x + 2.442179538 + 1.531128712i) (x − 0.337849858 − 2.228099198i) (x − 0.337849858 + 2.228099198i)

Applying Gunita Samuccaya Sutram

S_c = − 16 = (− 0.2086520) (3.442179538 − 1.531128712i) (3.442179538 + 1.531128712i) (0.662150142 − 2.228099198i) (0.662150142 + 2.228099198i)

= − 15.9999935 $\simeq$ − 16

7) E = $x^5 + 3x^4 + 4x^3 + 2x^2 + x + 1 = 0$

Swamiji's Method

S_e = 0 $\therefore$ x + 1 is a solution

E = (x + 1) A, A should have x^4, x^3, x^2, x and constant terms

By Argumentation applying Adyamadyena, and comparing the like terms

By Paravartya Division one can get the value of A as follows:

−1	1	3	4	2	1	1
	0	−1	−2	−2	0	−1
	1	2	2	0	1	0

$\therefore$ A = $x^4 + 2x^3 + 2x^2 + 0x + 1$
= $x^4 + 2x^3 + 2x^2 + 1$

For $\alpha = 2, \beta = 2, \gamma = 0$

$(x + 1)(x^4 + \alpha x^3 + \beta x^2 + \gamma x + 1)$

$\chi + 1 = 1$ $\chi = 0$ comparing x coefficient

comparing x^2 coefficient $\beta = 2$;

comparing x^3 coefficient $\alpha + \beta = 4$ $\alpha = 2$

$\therefore$ E = $(x + 1)(x^4 + 2x^3 + 2x^2 + 1)$

E = $(x + 1)(x^4 + 2x^3 + 2x^2 + 1) = 0$

$x^4 + 2x^3 + 2x^2 + 1 = 0$

$\Rightarrow x^4 + 2x^3 = -2x^2 - 1$ ———— (1)

Applying Purana Method and converting to a standard form

Let us consider $\frac{2}{4} = 0.5$ and

$(x + 0.5)^4 = x^4 + 4.(0.5)x^3 + 6.(0.5)^2x^2 + 4(0.5)^3x + (0.5)^4$
$= x^4 + 2x^3 + 1.5x^2 + 0.5x + 0.0625$

Substituting the value of $x^4 + 2x^3$ from (1)

$(x + 0.5)^4 = -2x^2 - 1 + 1.5x^2 + 0.5x + 0.0625$

$(x + 0.5)^4 = -0.5x^2 + 0.5x - 0.9375$

$(x + 0.5)^4 = -(0.5x^2 - 0.5x + 0.9375)$

Let $(x + 0.5) = y \Rightarrow x = y - 0.5$

$y^4 = -[(0.5)(y - 0.5)^2 - 0.5(y - 0.5) + 0.9375]$

$y^4 = -[(0.5)(y^2 + 0.25 - y) - 0.5(y - 0.5) + 0.9375]$
$= -[0.5y^2 + 0.125 - 0.5y - 0.5y + 0.25 + 0.9375]$

$y^4 = -[0.5y^2 - y + 1.3125]$

$y^4 + 0.5y^2 - y + 1.3125 = 0$

$g(y) = y^4 + 0.5y^2 - y + 1.3125$
$= (y^2 + by + c_1)(y^2 - by + c_2)$
$= y^4 + y^2(c_2 + c_1 - b^2) + yb(c_2 - c_1) + c_1 c_2 = 0$

Equating the like terms.

$c_1 + c_2 - b^2 = 0.5$ $c_1 + c_2 = 0.5 + b^2$ (a)

$b(c_2 - c_1) = -1$ $c_2 - c_1 = -1/b$ (b)

$c_1 c_2 = 1.3125$

$(c_2 + c_1)^2 - (c_2 - c_1)^2 = 4c_1c_2$

$(b^2 + 0.5)^2 - \left| -\frac{1}{b} \right| = 4(1.3125)$

$(b^2 + 0.5)^2 - \left| \frac{1}{b^2} \right| = 5.25$

Let b^2 z

$(z + 0.5)^2 - \frac{1}{z} = 5.25$

$(z^2 + z + 0.25) - \frac{1}{z} = 5.25$

$z^3 + z^2 - 5.25z + 0.25z - 1 = 0$

$E_1 = z^3 + z^2 - 5z - 1 = 0$

Z value	f(z)	RHS +1	RHS – LHS
1	– 3		+ 4
2	2		– 1
3	21		– 20

$f'(z) = 3z^2 + 2z - 5,\ f''(z) = 6z + 2,\quad f'''(z) = 6$

$f'(2) = 11 \qquad \frac{1}{2} f''(2) = 7 \qquad \frac{1}{6} f'''(2) = 1$

		0	0	0	0	0	0	0	0
CD=f′(a)=11		$\bar{1}$	$\overline{10}$	$\bar{1}$	$\overline{10}$	$\bar{7}$	$\bar{4}$	$\bar{7}$	$\bar{1}$
$\frac{1}{2}f''(a)=7$			0	0	$\overline{567}$	0	$\overline{7560}$	$\overline{756}$	
$\frac{1}{6}f'''(a)=1$				0	0	0	729	0	
	2.	0	$\bar{9}$	0	$\overline{60}$	$\bar{6}$	$\overline{624}$	$\overline{75}$	
	a.	b	c	d	e	f	g	h	

Upto h : z = 1.9033085

E = 0.0009342306

z = 1.9033085

∴ (z – 1.9033085) is a factor of E_1

∴ E_1 = (z – 109033085)A. A should have z^2, z and constant terms

By Adyamadyena, applying Argumentation method

$E_1 = (z^2 + \alpha z + 0.5254009)(z - 1.9033085)$

By Parvartya Division : $(z^3 + z^2 - 5z - 1) \div (z - k)$

Let 1.903305 = k

1	1	– 5	– 1
0	k	$k + k^2$	$-5k + k^2 + k^3$
1	1 + k	$-5 + k + k^2$	$-1 - 5k + k^2 + k^3$

$\therefore \alpha = 1 + k = 1 + 1.903305 = 2.903305$

$2z + 2.90391546 = \pm 2.515567306$

$z = -\dfrac{2.90391543}{2} \pm \dfrac{2.515567306}{2}$

$= -1.451957715 \pm 1.257783653$

Comparing z coefficients of E_1

$0.5254009 - 1.9033085\alpha = -5$

$\alpha = 2.903050609$

$z = \dfrac{\pm 2.515173799 - 2.903050609}{2}$

Both give negative values for z

Considering z = 1.9033085 (i.e. positive value)

$b^2 = z \Rightarrow b = \pm\sqrt{z} = 1.379604472$

$\therefore$ from (a) $c_1 + c_2 = b^2 + 0.5$

$= 2.4033085$

$c_2 - c_1 = -0.724845432$

$c_2 = 0.839231533$

$c_1 = 1.564076967$

$g(y) = (y^2 + 1.379604472y + 1.564076967)$

$= (y^2 - 1.379604472y + 0.839231533)$

$y^2 + 1.379604472y + 1.564076967$ is factorized

$2y + 1.379604472 = \pm\sqrt{1.903303499 - 6.256307868} = \pm\sqrt{-4.352999369}$

$= \pm 2.08638428\, i$

$y_1, y_2 = -0.689802236 \pm 1.043192141\, i$

$(y^2 - 1.379604472y + 0.839231533)$ is factorized

$2y - 1.379604472 = \pm\sqrt{1.903308499 - 3.356926132} = \pm 1.205660662i$

$y_3, y_4 = 0.689802236 \pm 0.602830331\, i$

but $x = y - 0.5$

$\Rightarrow x_1 = -1.189802236 + 1.04319214\, i$

$x_2 = -1.189802236 - 1.04319214\, i$

$x_3 = 0.189802236 + 0.602830331i$

$x_4 = 0.189802236 - 0.602830331i$

$E = (x + 1)(x + 1.189802236 + 1.04319214\, i)(x + 1.189802236 - 1.04319214\, i)$

$(x - 0.189802236 + 0.602830331\, i)(x - 0.189802236 - 0.602830331\, i)$

Applying Gunita Samuccaya Sutram of final verification

$S_c = 12 = 2(2.189802236 + 1.04319214\, i)$

$(2.189802236 - 1.04319214\, i)$

$(0.810197764 + 0.602830331\, i)$

$(0.810197764 - 0.602830331\, i) = (2)(5.883483676)(1.0198248925)$

$12 \simeq 12.00024542$

8^{th} Degree

$E = x^8 - 9x^6 + 2x^5 + 24x^4 - 8x^3 - 19x^2 + 4x + 5 = 0$

Swamiji's Method

By Vilokanam ,

$S_c = 0 \Rightarrow (x - 1)$ is a factor

$(x^8 - 9x^6 + 2x^5 + 24x^4 - 8x^3 - 19x^2 + 4x + 5) \div (x - 1)$

By Paravartya Method:

1	1	0	−9	2	24	−8	−19	4	5
	0	1	1	−8	−6	18	10	−9	−5
	1	1	−8	−6	18	10	−9	−5	0

$E_1 = x^7 + x^6 - 8x^5 - 6x^4 + 18x^3 + 10x^2 - 9x - 5 = 0$

Value of x	L.H.S	R.H.S	Diff
1	7	5	−2
2	6	5	−1
3	1035		−1030
0	0		5
−1	3		2
−2	10		−5
−3	−369		374

The roots lie between (1, 2), (− 2, −1) (− 3, −2) and (0, 1) but there is no indication

Let $x = \frac{z}{10}$

$$\frac{z^7}{10^7} + \frac{z^6}{10^6} - \frac{8z^5}{10^5} - \frac{6z^4}{10^4} + \frac{18z^3}{10^3} + \frac{10z^2}{10^2} - \frac{9z}{10} = 5$$

$z^7 + 10z^6 - 800z^5 - 6000z^4 + 180000z^3 + 1000000z^2 - 9000000z = 5 \times 10^7$

	Value of z	L.H.S	R.H.S	Difference
	1	− 7826789		
	2	− 12680832		
	3	− 13810923		
	4	− 10777856		
	5	− 3515625		
	6	7629696		
	7	21888433		
(1)	− 8	38088192		11911808
	− 9	54712179		− 4712179
	10	70000000		
	11	82095981		
	12	89250048		
	13	90076207		
	14	83873664		

	15	71015625	
(2)	16	53410816	– 3410816
	17	35042763	– 14957237
	18	22591872	
(3)	19	26145349	23854651
	20	60000000	– 10000000
	– 1	9814809	
	– 2	20490112	
	– 3	30853503	
	– 4	39787776	
	– 5	46328125	
	– 6	49751424	
	– 7	49652547	
	– 8	46002688	
	– 9	39184641	
	– 10	30000000	
	– 11	19643239	
	– 12	9637632	
	– 13	1727973	
	– 14	– 2274944	
	– 15	– 703125	
	– 16	7701504	
	– 17	23456617	
(4)	– 18	45940608	4059392
	– 19	72920271	– 22920271
	– 20	1×10^8	
	– 21	119987469	
	– 22	122170752	
(5)	– 23	91501843	– 41501843
	– 24	7681536	42318464
	– 25	– 155859375	

Taylor's Method

$f(z) = z^7 + 10z^6 - 800z^5 - 6000z^4 + 180000z^3 + 1 \times 10^6 z^2 - 9 \times 10^6 z = 5 \times 10^7$

$f'(z) = 7z^6 + 60z^5 - 4000z^4 - 24000z^3 + 540000z^2 + 2 \times 10^6 z - 9 \times 10^6$

$f''(z) = 42z^5 + 300z^4 - 16000z^3 - 72000z^2 + 1080000z + 2 \times 10^6$

$f'''(z) = 210z^4 + 1200z^3 - 48000z^2 - 144000z + 1080000$

$f^{iv}(z) = 840z^3 + 3600z^2 - 96000z - 144000$

$f^{v}(z) = 2520z^2 + 7200z - 96000$

$f^{vi}(z) = 5040z + 7200$

$f^{vii}(z) = 5040$

	z = 8	z = 16	z = 19	z = – 18	z = – 23
$f'(z)$	16689088	– 18852928	15927107	– 25264512	53374643
$\frac{1}{2}f''(z)$	222528	– 493504	14938229	2337472	– 26315053
$\frac{1}{6}f'''(z)$	– 278240	860960	2769035	527760	3861035
$\frac{1}{24}f^{iv}(z)$	– 10480	111760	212215	– 89520	– 260495
$\frac{1}{120}f^{v}(z)$	1024	5536	7921	4924	8929
$\frac{1}{720}f^{vi}(z)$	66	122	143	– 116	– 151
$\frac{1}{5040}f^{vii}(z)$	1	1	1	1	1

z = 8

0	0	0	0	0	0	0	0
$CD=f'(a)=16689088$	11911808	2294464	12040768	15574944	10014812	14776160	7340510
$\frac{1}{2}f''(a)=222528$		$\overline{10903872}$	0	$\overline{37384704}$	$\overline{24923136}$	$\overline{131736576}$	$\overline{136187136}$
$\frac{1}{6}f'''(a)=-278240$			95436320	0	490815360	327210240	2150238720
$\frac{1}{24}f^{iv}(a)=-10480$				25162480	0	172542720	115028480
$\frac{1}{120}f^{v}(a)=1024$					$\overline{17210368}$	0	$\overline{147517440}$
$\frac{1}{720}f^{vi}(a)=66$						$\overline{7764834}$	0
$\frac{1}{5040}f^{vii}(a)=1$							$\overline{823543}$
8	7	0	12	8	32	30	123
a	b	c	d	e	f	g	h

Upto h $z = 8.7131623 \Rightarrow x = 0.87131623$ $E = -0.00000036731$

z = 16

	0	0	0	0	0	0	0	0	0
CD $= f'(a) =$	$\overline{18852928}$	$\overline{3410816}$	$\overline{15255232}$	$\overline{1235392}$	$\overline{5318816}$	$\overline{4672848}$	$\overline{6258432}$	$\overline{16020730}$	$\overline{14034117}$
$\frac{1}{2}f''(a) = \overline{493504}$			493504	7896064	31584256	1974016	26649216	113505920	
$\frac{1}{6}f'''(a) = 860960$				$\overline{860960}$	$\overline{20663040}$	$\overline{165304320}$	$\overline{445977280}$	$\overline{111063840}$	
$\frac{1}{24}f^{iv}(a) = 111760$					$\overline{111760}$	$\overline{3576320}$	$\overline{42915840}$	$\overline{229778560}$	
$\frac{1}{120}f^{v}(a) = 5536$						$\overline{5536}$	$\overline{221440}$	$\overline{3543040}$	
$\frac{1}{720}f^{vi}(a) = 122$							$\overline{122}$	$\overline{5856}$	
$\frac{1}{5040}f^{vii}(a) = 1$								$\overline{1}$	
	16	1	8	0	2	11	27	20	
	a	b	c	d	e	f	g	h	

$z = 16.1\ 8\ 0\ 2\ 11\ 27\ 20 = 16.1803390$

$x = \frac{z}{10} = 1.61803390$ $\qquad E = 0.00000016815$

$z = -18$

	0	0	0	0	0	0	0
CD=$f'(a)$=	4059392	15329408	24634048	21377424	24885760	21410092	20773304
$\overline{25264512}$							
$\frac{1}{2}f''(a)$=		$\overline{2337472}$	$\overline{23374720}$	$\overline{95836352}$	$\overline{205697536}$	$\overline{257121920}$	$\overline{247772032}$
2337472							
$\frac{1}{6}f'''(a)$=			527760	7916400	52248240	198965520	486066960
527760							
$\frac{1}{24}f^{iv}(a)$=				89520	1790400	16292640	89161920
$\overline{89520}$							
$\frac{1}{120}f^{v}(a)$					4924	123100	1427960
=4924							
$\frac{1}{720}f^{vi}(a)$						116	3480
=$\overline{116}$							
$\frac{1}{5040}f^{vii}(a)=1$							1
– 18.	$\bar{1}$	$\bar{5}$	$\bar{8}$	$\bar{4}$	$\bar{3}$	$\bar{6}$	$\overline{21}$
a	b	c	d	e	f	g	h

z = – 18.1584381

x $\frac{z}{10}$ 8158438

E r = – 0.000000994

	z 20	z = – 24	z = 9
z	550000	8675392	6263027
$\frac{1}{2}$ z	246000	39552704	66382
$\frac{1}{6}$ z	3700000	4995360	308565
$\frac{1}{24}$ z	254000	07440	4335
$\frac{1}{120}$ z	8800	9856	44
$\frac{1}{720}f^{vi}(z)$	50	$\overline{58}$	73

z : 20

		0	0	0	0	0	0	0
CD=$f'(a)$=	55000000	$\overline{10000000}$	$\overline{45000000}$	$\overline{34600000}$	$\overline{20900000}$	$\overline{24454000}$	$\overline{36159200}$	$\overline{15984150}$
$\frac{1}{2}f''(a)$=	24600000		$\overline{24600000}$	$\overline{393600000}$	$\overline{2214000000}$	$\overline{7183200000}$	$\overline{26543400000}$	$\overline{93676800000}$
$\frac{1}{6}f'''(a)$=	3700000			3700000	88800000	854700000	4669400000	19891200000
$\frac{1}{24}f^{iv}(a)$=	254000				$\overline{254000}$	$\overline{8128000}$	$\overline{110744000}$	$\overline{879856000}$
$\frac{1}{120}f^{v}(a)$=	8800					8800	352000	6204000
$\frac{1}{720}f^{vi}(a)$=	150						$\overline{150}$	$\overline{7200}$
	1							1
	20.	$\overline{1}$	$\overline{8}$	$\overline{13}$	$\overline{42}$	$\overline{119}$	$\overline{406}$	$\overline{1360}$
	a	b	c	d	e	f	g	h

Upto h: $z = 20.\,\overline{1}\;\overline{9}\;\overline{8}\;\overline{9}\;\overline{3}\;\overline{2}\;0$

$z = 19.\,8\,0\,1\,0\,6\,8\,0$

$x = \frac{z}{10} = 1.98010680$ E = 0.0003525671 z 24

	a	b	c	d	e	f	g	h
		0	0	0	0	0	0	0
CD=$f'(a)$=	118675392	42318464	67158464	78155840	53030240	108048304	21069380	93810454
$\frac{1}{2}f''(a)$=	$\overline{39552704}$		355974336	1898529792	7515013760	27054049540	99475050560	368868517500
$\frac{1}{6}f'''(a)$=	4995360			$\overline{134874720}$	$\overline{1078997760}$	$\overline{5709696480}$	$\overline{25486326720}$	$\overline{107450193600}$
$\frac{1}{24}f^{iv}(a)$=	$\overline{307440}$				24902640	265628160	1759786560	9392906880
$\frac{1}{120}f^{v}(a)$=	9856					$\overline{2395008}$	$\overline{31933440}$	$\overline{254136960}$
$\frac{1}{720}f^{vi}(a)$=	$\overline{158}$						115182	1842912
	1							$\overline{2187}$
	–24.	3	8	21	58	191	639	2287
	a	b	c	d	e	f	g	h

Upto h: $z = \overline{24}.4\ 0\ 9\ 5\ 7\ 7\ 7$

$z = \overline{23}.\ \bar{5}\ \bar{9}\ 0\ \bar{4}\ \bar{2}\ \bar{2}\ \bar{3}$

$x = \frac{z}{10} = -2.35904223$ E = – 0. 2106398

Swamiji's Method

$x^7 + x^6 - 8x^5 - 6x^4 + 18x^3 + 10x^2 - 9x - 5 = 0$

Let $x = \frac{z}{10} \Rightarrow z^7 + 10z^6 - 800z^5 - 6000z^4 + 180000z^3 + 1000000z^2 - 9000000z = 5 \times 10^7$

CD Representation at $z = 8 = (7z^6 + 60z^5 - 4000z^4 - 24000z^3 + 540000z^2 + 2 \times 10^6 z - 9 \times 10^6)$ at $z = 8 = 16689088$

$21a^5$ Representation at $z = 8 = \frac{1}{2}(42z^5 + 300z^4 - 16000z^3 - 72000z^2 + 1080000z + 2 \times 10^6)$ at $z = 8 = 222528$

$35a^4$ Representation at $z = 8 = \frac{1}{6}(210z^4 + 1200z^3 - 48000z^2 - 144000z + 1080000)$ at $z = 8 = -278240$

$35a^3$ Representation at $z = 8 = \frac{1}{24}(840z^3 + 3600z^2 - 96000z - 144000)$ at $z = 8 = -10480$

$21a^2$ Representation at $z = 8 = \frac{1}{120}(2520z^2 + 7200z - 96000)$ at $z = 8 = 1024$

$7a$ Representation at $z = 8 = \frac{1}{720}(5040z + 7200)$ at $z = 8 =$

	0	0	0	0	0	0	0
CD = 6689088	11911808	2294464	12040768	15574944	10014512	14776160	7340510
		$\overline{10903872}$	95436320 – 0=95436320	0+25162480 – 37384704+0 $=\overline{12222224}$	–24923136+0 – 17210368+490815360 +0+0=448681856	–7764834–32044032 –99692544+0 + 327210240 +172542720 =360251550	–823543+0–147517440 +0+115028480+0 +0+1308840960+0 +841397760+0 – 93461760–42725376 = 1980739081
		$-(21a^5b^2)$	$-(35a^4b^3$ $+21a^5 2bc)$	$-(21a^5c+35a^3b^4+$ $21a^5.2bd+35a^4.3b^2c)$	$-(21a^2b^5+21a^5.2be$ $+21a^5.2cd+35a^4 3b^2d$ $+35a^4.3bc^2+35a^3.4b^3c)$	$-(7ab^6+21a^5d^2$ $+21a^5.2bf$ $+21a^5.2ce+35a^4c^3$ $+35a^4.3b^2e+35a^4.6bcd$ $+35a^3.4b^3d+35a^3.6b^2c^2$ $+21a^2.5b^4c)$	$-(b^7+7a.6b^5c$ $+21a^2.5b^4d+21a^2.10b^3c^2$ $+35a^3 4b^3e+35a^3.12b^2cd$ $+35a^3.4bc^3+35a^4.3b^2f$ $+35a^4.6bce+35a^4.3bd^2$ $+35a^4.3c^2d+21a^5.2bg$ $+21a^5.2cf+21a^5.2de)$
8.	7	0	12	8	32	30	123
a	b	c	d	e	f	g	h

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD

2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms

3) Values of subtraction terms containing $7a^6$, $21a^5$, $31a^4$, $35a^3$, $21a^2$ and 7a are to be worked out in terms of representation values.

$21a^5b^2 = 222528 \times 49 = 10903872$, $35a^4b^3 = -278240 \times 7^3 = -95436320$, $35a^3b^4 = -10480 \times 7^4 = -25162480$, $21a^2b^5 = 1024 \times 7^5 = 17210368$, $7a \times b^6 = 66 \times 7^6 = 7764834$

Upto h (7 decimals) z = 8. 7 0 12 8 32 30 123 = 8. 7 1 3 1 6 23 $x = \frac{z}{10} = 0.87131623$ E = – 0.0000036731

CD Representation at z = 16 = $(7z^6 + 60z^5 - 4000z^4 - 24000z^3 + 540000z^2 + 2 \times 10^6 z - 9 \times 10^6)$ at z = 16 = – 18852928

$21a^5$ Representation at $z = 16 = \frac{1}{2}(42z^5 + 300z^4 - 16000z^3 - 72000z^2 + 1080000z + 2 \times 10^6)$ at $z = 16 = -493504$

$35a^4$ Representation at $z = 16 = \frac{1}{6}(210z^4 + 1200z^3 - 48000z^2 - 144000z + 1080000)$ at $z = 16 = 860960$

$35a^3$ Representation at $z = 16 = \frac{1}{24}(840z^3 + 3600z^2 - 96000z - 144000)$ at $z = 16 = 111760$

$21a^2$ Representation at $z = 16 = \frac{1}{120}(2520z^2 + 7200z - 96000)$ at $z = 16 = 5536$

$7a$ Representation at $z = 16 = \frac{1}{720}(5040z + 7200)$ at $z = 16 = 122$

	0	0	0	0	0	0	[illegible]
CD = 18852928	3410816	15255232	1235392	5318816	4672848	6258432	16020730
		493504	−860960 +7896064 =7035104.	31584256 − 111760−20663040 =10809456	−5536+1974016+0+0 −165304320−3576320 = − 166912160	−122+0+010857088 +15792128− 440811520−5165760 +0−42915840−221440 = − 462465466	−1−5856+0−3543040 − 894080+0−228884480 −28411680−82652160 +0+26649216+86856704 = −230885377
		$-(21a^5b^2)$	$-(35a^4b^3 + 21a^5 2bc)$	$-(21a^5c^2+35a^3b^4 + 21a^5.2bd+35a^4.3b^2c)$	$-(21a^2b^5+21a^5.2be +21a^5.2cd+35a^4 3b^2d +35a^4.3bc^2+35a^3.4b^3c)$	$-(7ab^6+21a^5d^2 +21a^5.2bf +21a^5.2ce+35a^4c^3 +35a^4.3b^2e+35a^4.6bcd +35a^3.4b^3d+35a^3.6b^2c^2 +21a^2.5b^4c)$	$-(b^7+7a.6b^5c +21a^2.5b^4d+21a^2.10b^3c^2 +35a^3 4b^3e+35a^3.12b^2cd +35a^3.4bc^3+35a^4.3b^2f +35a^4.6bce+35a^4.3bd^2 +35a^4.3c^2d+21a^5.2bg +21a^5.2cf+21a^5.2de)$
16	1	8	0	2	11	27	20
a	b	c	d	e	f	g	H

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing $7a^6, 21a^5, 31a^4, 35a^3, 21a^2$ and 7a are to be worked out in terms of representation values.

Upto h (7 decimals) $z = 16.1\ 8\ 0\ 2\ 11\ 27\ 20 = 16.1\ 8\ 0\ 3\ 3\ 9\ 0 \Rightarrow x = \frac{z}{10} = 1.61803390$ E = 0.0000016815

CD Representation at $z = -18 = (7z^6 + 60z^5 - 4000z^4 - 24000z^3 + 540000z^2 + 2 \times 10^6 z - 9 \times 10^6)\ z = -18 = -25264512$

$21a^5$ Representation at $z = -18 = \frac{1}{2}(42z^5 + 300z^4 - 16000z^3 - 72000z^2 + 1080000z + 2 \times 10^6)\ z = -18 = 2337472$

$35a^4$ Representation at $z = -18 = \frac{1}{6}(210z^4 + 1200z^3 - 48000z^2 - 144000z + 1080000)\ z = -18 = 527760$

$35a^3$ Representation at $z = -18 = \frac{1}{24}(840z^3 + 3600z^2 - 96000z - 144000)\ z = -18 = -89520$

$21a^2$ Representation at $z = -18 = \frac{1}{120}(2520z^2 + 7200z - 96000)\ z = -18 = 4924$

7a Representation at $z = -18 = \frac{1}{720}(5040z + 7200)\ z = -18 = -116$

	0	0	0	0	0	0	0
CD = −25264512	4059392	15329408	24634048	21377424	24885760	21410092	20773304
		−2337472	527760 − 23374720 = −22846960	−58436800 − 37399552+89520 +7916400 = −8783.432	4924−18699776 − 186997760+12666240 +39582000+1790400 = −151653972	116−149598208 − 14024832−93498880 +65970000+6333120 +126662400+2864640 +13428000+123100 =−41740544	1+3480+196960 +1231000+1432320 +42969600+44760000 +4749840+63331200 +101329920+316656000 −28049664−70124160 − 149598208=328888289
		$-(21a^5b^2)$	$-(35a^4b^3 + 21a^5 2bc)$	$-(21a^5c^2+35a^3b^4+21a^5.2bd+35a^4.3b^2c)$	$-(21a^2b^5+21a^5.2be+21a^5.2cd+35a^4 3b^2d+35a^4.3bc^2+35a^3.4b^3c)$	$-(7ab^6+21a^5d^2+21a^5.2bf+21a^5.2ce+35a^4c^3+35a^4.3b^2e+35a^4.6bcd+35a^3.4b^3d+35a^3.6b^2c^2+21a^2.5b^4c)$	$-(b^7+7a.6b^5c+21a^2.5b^4d+21a^2.10b^3c^2+35a^3 4b^3e+35a^3.12b^2cd+35a^3.4bc^3+35a^4.3b^2f+35a^4.6bce+35a^4.3bd^2+35a^4.3c^2d+21a^5.2bg+21a^5.2cf+21a^5.2de)$
−18	$\bar{1}$	$\bar{5}$	$\bar{8}$	$\bar{4}$	$\bar{3}$	$\bar{6}$	$\overline{21}$
a	b	c	d	e	f	g	h

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID − corresponding subtraction terms
3) Values of subtraction terms containing $7a^6$, $21a^5$, $31a^4$, $35a^3$, $21a^2$ and 7a are to be worked out in terms of representation values.

Upto h (7 decimals) $z = -18.\bar{1}\ \bar{5}\ \bar{8}\ \bar{4}\ \bar{3}\ \bar{6}\ \overline{21} = \overline{18}.\bar{1}\ \bar{5}\ \bar{8}\ \bar{4}\ \bar{3}\ \bar{8}\ \bar{1} \Rightarrow x = \frac{z}{10} = -1.81584381$ E = −0.000000994

CD Representation at $z = 20 = (7z^6 + 60z^5 - 4000z^4 - 2400z^3 + 540000z^2 + 2 \times 10^6 z - 9 \times 10^6) = 55000000$

$21a^5$ Representation at $z = 20 = \frac{1}{2}(42z^5 + 300z^4 - 16000z^3 - 72000z^2 + 1080000z + 2 \times 10^6) = 24600000$

$35a^4$ Representation at $z = 20 = \frac{1}{6}(210z^4 + 1200z^3 - 48000z^2 - 144000z + 1080000) = 3700000$

$35a^3$ Representation at $z = 20 = \frac{1}{24}(840z^3 + 3600z^2 - 96000z - 144000) = 254000$

$21a^2$ Representation at $z = 20 = \frac{1}{120}(2520z^2 + 7200z - 9600) = 8800$

$7a$ Representation at $z = 20 = \frac{1}{720}(5040z + 7200) = 150$

	0	0	0	0	0	0	0
CD = 55000000	$\overline{10000000}$	$\overline{45000000}$	$\overline{34600000}$	$\overline{20900000}$	$\overline{24454000}$	$\overline{36159200}$	$\overline{15984150}$
		-24600000	$3700000 -$ $\overline{393600000}$ $= 389900000$	$-157440000-$ $639600000-$ $254000+$ 88800000 $= -2125454000$	$8800-$ 206640000 -511680000 $+144300000$ $+71040000$ -8128000 $= -6336619200$	$-150-415740000-$ $5854800000-$ 16531200000 $+1894400000+466200000$ $+2308800000-13208000$ $-97536000+352000$ $= -2.1984392$	$1-7200+572000+563200$ $42672000-316992000-$ $520192000+132090000$ $+7459200000+18759000$ $+9235200000-199752000$ $-46838400000-26863200($ $= -74659259200$
		$-(21a^5b^2)$	$-(35a^4b^3$ $+21a^5 2bc)$	$-(21a^5c^2+35a^3b^4$ $+21a^5.2bd$ $35a^4.3b^2c)$	$-(21a^2b^5$ $+21a^5.2bc$ $+21a^5.2cd+35a^4$ $3b^2d+35a^4.3bc^2$ $+35a^3.4b^3c)$	$-(7ab^6+21a^5d^2+21a^5.2bf$ $+21a^5.2ce+35a^4c^3$ $+35a^4.3b^2e+35a^4.6bcd$ $+35a^3.4b^3d+35a^3.6b^2c^2$ $+21a^2.5b^4c)$	$-(b^7+7a.6b^5c$ $+21a^2.5b^4d+21a^2.10b^3c^2$ $+35a^3 4b^3e+35a^3.12b^2cd$ $+35a^3.4bc^3+35a^4.3b^2f$ $+35a^4.6bce+35a^4.3bd^2$ $+35a^4.3c^2d+21a^5.2bg$ $+21a^5.2cf+21a^5.2de)$
20	$\bar{1}$	$\bar{8}$	$\overline{13}$	$\overline{42}$	$\overline{119}$	$\overline{406}$	$\overline{1360}$
a	b	c	d	e	f	g	h

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD
2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms
3) Values of subtraction terms containing $7a^6$, $21a^5$, $31a^4$, $35a^3$, $21a^2$ and 7a are to be worked out in terms of representation values.

Upto h (7 decimals) $z = 20.\bar{1}\ \bar{8}\ \overline{13}\ \overline{42}\ \overline{119}\ \overline{406}\ \overline{1360} = 20.\bar{1}\ \bar{9}\ \bar{8}\ \bar{9}\ \bar{3}\ \bar{2}\ \bar{0} \Rightarrow x = \frac{z}{10} = 19.8010680$

CD Representation at $z = -24 = (7z^6 + 60z^5 - 4000z^4 - 24000z^3 + 540000z^2 + 2 \times 10^6 z - 9 \times 10^6)$ at $z = -24 = 18675392$

$21a^5$ Representation at $z = -24 = \frac{1}{2}(42z^5 + 300z^4 - 16000z^3 - 72000z^2 + 1080000z + 2 \times 10^6)$ at $z = -24 = -39552704$

$35a^4$ Representation at $z = -24 = \frac{1}{6}(210z^4 + 1200z^3 - 48000z^2 - 144000z + 1080000)$ at $z = -24 = 4995360$

$35a^3$ Representation at $z = -24 = \frac{1}{24}(840z^3 + 3600z^2 - 96000z - 144000)$ at $z = -24 = -307440$

$21a^2$ Representation at $z = -24 = \frac{1}{120}(2520z^2 + 7200z - 96000)$ at $z = -24 = 9856$

7a Representation at $z = -24 = \frac{1}{720}(5040z + 7200)$ at $z = -24 = -158$

	0	0	0	0	0	0	0
CD = 8675392	42318464	67158464	78155840	53030240	108048304	21069380	93810454
		355974336	1898529792– 134874720 = 1763655072	2531373056 +4983640704 +24902640 – 1078997760 = 6460918640	– 2395008 +13764340992 +13289708544 – 2832369120 – 2877327360 +265628160 =21607586210	115182 +45327398780 +36704909310+17442742460 –2557624320– 7822733760–15105968640 +697273920 +1062512640 –3193340 = 75716692140	–2187+1842912–83825280– 170311680 +1625804160 +5578191360 +1888911360– 25761071520–41721246720– 19826583840–20141291520 +151645067100+12087306340 +96350386940 =27055893450(
		$-(21a^5b^2)$	$-(35a^4b^3$ $+21a^5 2bc)$	$-(21a^5c^2+35a^3b^4$ $+21a^5.2be$ $35a^4.3b^2c)$	$-(21a^2b^5+21a^5.2bc$ $+21a^5.2cd+35a^4 3b^2d$ $+35a^4.3bc^2+35a^3.4b^3c)$	$-(7ab^6+21a^5d^2+21a^5.2bf$ $+21a^5.2ce+35a^4c^3$ $+35a^4.3b^2e+35a^4.6bcd$ $+35a^3.4b^3d+35a^3.6b^2c^2$ $+21a^2.5b^4c)$	$-(b^7+7a.6b^5c+21a^2.5b^4d$ $+21a^2.10b^3c^2+35a^3 4b^3e$ $+35a^3.12b^2cd+35a^3.4bc^3$ $+35a^4.3b^2f+35a^4.6bce$ $+35a^4.3bd^2+35a^4.3c^2d+21a^5.$ $2bg+21a^5.2cf+21a^5.2de)$
–24	3	8	21	58	191	639	2287
a	b	c	d	e	f	g	H

1) The first decimal b is obtained as a coefficient by considering the first ID as ND and the same is divided by CD

2) The successive decimal values are to be obtained in the usual manner, ie ND ÷ CD where ND = ID – corresponding subtraction terms

3) Values of subtraction terms containing $7a^6$, $21a^5$, $31a^4$, $35a^3$, $21a^2$ and 7a are to be worked out in terms of representation values.

Upto h (7 decimals) z = – 24. 3 8 21 58 191 639 2287 = – 24. 4 0 9 5 7 7 7 = $\overline{23.5904223}$ $\Rightarrow x = \dfrac{z}{10} = -2.35904223$

(x – 1), (x – 0.87131623), (x – 1.61803390), (x + 1.81584381), (x – 1.98010680), (x + 2.35904223), (x + 0.676473679), (x + 0.618091627) are the factors of E

$\therefore$ E = (x – 1) (x – 0.87131623) (x – 1.61803390) (x + 1.81584381) (x – 1.98010680) (x + 2.35904223) (x + 0.676473679) (x + 0.618091627) A

Where A = $(x^2 + \alpha x + \beta)$, α and β are to be determined.

Applying Adyamadyena Antyamantyena and Argumentation A = $(x^2 + \alpha x + 0.418122717)$

$E = (x^6 - 1.29457089x^5 - 7.742251905x^4 + 12.56422665x^3 + 10.97207954x^2 - 27.45769518x + 11.95821179)(x^2 + \alpha x + 0.418122717)$

Comparing x coefficient on both sides

$11.95821179\alpha - 11.48068611 = 4$

$\alpha = 1.294565306$

$\therefore\ A = (x^2 + 1.294565306x + 0.418122717)$

A is to be further factorised using differential relation

$2x + 1.294565306 \quad = \pm\sqrt{1.675899332 - 1.672490868}$

$= \pm\sqrt{0.003408464}$

$= \pm\ 0.058382052$

x = – 0.676473679, – 0.618091627

$\therefore$ E = (x – 1) (x – 0.87131623) (x – 1.61803390) (x + 1.81584381) (x – 1.98010680) (x + 2.35904223) (x + 0.676473679) (x + 0.618091627)

Part III

Roots of Polynomials in Two or More Variables

1) To write down the given expression in descending order of any one variable.
2) The nearest cube root of the first term is considered as "a".
3) Common divisor which is $3a^2$ is to be considered.
4) Examine the rest of the terms and see if there are any number of terms which have the power equivalent to the degree of a variable as the case may be of the common divisor.
5) Consider all such terms and apply direct division by common divisor to all such terms to get the corresponding coefficients, after this is over, the division is carried term by term with the corresponding terms. As shown in the following examples

Eg 1) $(2x^2 + 4y^3)^3 = 8x^6 + 48x^4y^3 + 96x^2y^6 + 64y^9$

$12x^4$	$8x^6$	+	$12x^4(4y^3)$	+	$96x^2y^6$	+	$64y^9$
			$3a^2b$		$3ab^2$		b^3
	$2x^2$	+	$4y^3$	+	0	+	0
	a	+	b				

Eg 2) Find the cube root of $8x^3 - 60x^2y + 84x^2z + 150xy^2 + 294xz^2 + 225y^2z - 735yz^2 - 420xyz - 125y^3 + 343z^3$

$CD=12x^2$	$8x^3$	$-60x^2y$	$+84x^2z$	$+150xy^2$	$+294xz^2$	$+225y^2z$	$-735yz^2$	$-420xyz$	
				$-150xy^2$	$-294xz^2$	$-225y^2z$	$+735yz^2$	$+420xyz$	$+125y^3$
		Direct Division		$(3ab^2)$	$(3ac^2)$	$(3b^2c)$	$(3bc^2)$	$(6abc)$	(b^3)
	$2x$	$-5y$	$+7z$	0	0	0	0	0	0
	a	b	C	d	e	f	g	h	i

CR = 2x – 57 + 7z

Eg 3) $(2x^2 + y^3 + z^5)^3 = z^{15} + 6z^{10}x^2 + 3z^{10}y^3 + 125z^5x^4 + 3z^5y^6 + 6y^6x^2 + 12x^4y^3 + y^9 + 8x^6 + 12x^2y^3z^5$

$3z^{10}$	z^{15}	$+6z^{10}x^2$	$+3z^{10}y^3$	$+12z^5x^4$	$+3z^5y^6$	$+6y^6x^2$	$+12x^4y^3$	$+y^9$	$+8x^6$	$+12x^2y^3z$
		Direct Division		$-12z^5x^4$	$-3z^5y^6$	$-6y^6x^2$	$-12x^4y^3$	$-y^9$	$-8x^6$	$-12x^2y^3z$
				$(3ab^2)$	$(3ac^2)$	$(3bc^2)$	$(3b^2c)$	(c^3)	(b^3)	$(6abc)$
	z^5	$+2x^2$	$+y^3$	+0	+0	+0	+0	+0	+0	+0
	a	b	c	d	e	f	g	h	i	j

Eg 4) $(3x^6 - 5z^4 + 2y^3 + w^5)^3 = 27x^{18} - 135x^{12}z^4 + 54x^{12}y^3 + 27x^{12}w^5 + 225x^6z^8 + 36x^6y^6 + 9x^6w^{10} + 150z^8y^3 + 75z^8w^5 - 60z^4y^6 - 15z^4w^{10} + 12y^6w^5 + 6y^3w^{10} - 125z^{12} + 8y^9 + w^{15} - 180x^6z^4y^3 - 90x^6z^4w^5 + 36x^6y^3w^5 - 60z^4y^3w^5$

$27x^{12}$	$27x^{18}$	$-135x^{12}z^4$	$+54x^{12}y^3$	$+27x^{12}w^5$	$+225x^6z^8$	$+36x^6y^6$	$+9x^6w^{10}$	$+150z^8y^3$	$+75z^8w^5$	$-60z^4y^6$
		Direct Division			$-225x^6z^8$	$-36x^6y^6$	$-9x^6w^{10}$	$-150z^8y^3$	$-75z^8w^5$	$+60z^4y^6$
					$(3ab^2)$	$(3ac^2)$	$(3ad^2)$	$(3b^2c)$	$(3b^2d)$	$(3bc^2)$
	$3x^6$	$-5z^4$	$+2y^3$	$+w^5$	$+0$	$+0$	$+0$	$+0$	$+0$	$+0$

Contd

$-15z^4w^{10}$	$+12y^6w^5$	$+6y^3w^{10}$	$-125z^{12}$	$+8y^9$	$+w^{15}$	$-180x^6z^4y^3$	$-90x^6z^4w^5$	$+36x^6y^3w^5$	$-60z^4y^3w^5$
$+15z^4w^{10}$	$-12y^6w^5$	$-6y^3w^{10}$	$+125z^{12}$	$-8y^9$	$-w^{15}$	$+180x^6z^4y^3$	$+90x^6z^4w^5$	$-36x^6y^3w^5$	$+60z^4y^3w^5$
$(3bd^2)$	$(3c^2d)$	$(3cd^2)$	(b^3)	(c^3)	(d^3)	[illegible]	$(6abd)$	$(6acd)$	$(6bcd)$
$+0$	$+0$	$+0$	$+0$	$+0$	$+0$	$+0$	$+0$	$+0$	$+0$

This method is applicable even in a general polynomial where the terms are combinations of many variables.

Eg 5) $y^3z^9 + 3y^2z^7 + 3y^4z^6 + 3y^5z^3 + 3yz^5x^2 + 3y^2x^2z^2 + 3y^4xz + 6xy^3z^4 + y^6 + x^3z^3$

$3y^2z^6$	y^3z^9	$+3y^2z^7x$	$+3y^4z^6$	$+3y^5z^3$	$+3yz^5x^2$	$+3y^2x^2z^2$	$+3y^4xz$	$+6xy^3z^4$	$+y^6$	$+x^3z^3$
		Direct Division		$-(3ac^2)$	$-(3ab^2)$	$-(3b^2c)$	$-(3bc^2)$	$-(6abc)$	$-(c^3)$	$-(b^3)$
	yz^3	$+xz$	$+y^2$	$+0$	$+0$	$+0$	$+0$	$+0$	$+0$	$+0$
	a	b	c							

In case of an imperfect cube also same method can be applied

Let us consider

Eg 6) $x^3 + 3x^2y + 2x^2z + 3xy^2 + 3xz^2 + 3y^2z + 3yz^2 + 6xyz + y^3 + z^3$

$3x^2$	x^3	$+3x^2y$	$+2x^2z$	$+3xy^2$	$+3xz^2$	$+3y^2z$	$+3yz^2$	$+6xyz$	$+y^3$	$+z^3$
				$+3xy^2$	$\frac{4}{3}xz^2$	$2y^2z$	$-\frac{4}{3}yz^2$	$-4xyz$	$-y^3$	$-\frac{8}{27}z^3$
				$(3ab^2)$	$(3ac^2)$	$(3b^2c)$	$(3bc^2)$	$(6abc)$	(b^3)	(c^3)
	x	$+y$	$+\frac{2}{3}z$	0	$+\frac{5z^2}{9x}$	$+\frac{y^2z}{3x^2}$	$+\frac{5yz^2}{3x^2}$	$+\frac{2yz}{3x}$	$+0$	$+\frac{19z^3}{3x^2\times 27}$
	a	b	c	d	e	f	g	h	i	j

To find out cube root we can claim that this is the most general method for any polynomial of any order for many variables

Part IV

ANNEXURE I

The Straight Division for Cube Roots as envisaged by Swamiji consists of the following points:

1. The given number is divided into groups consisting of 3 digits each from R.H.S.
2. The Common Divisor is $3a^2$ where a is the nearest Cube Root of the first group, a is the first quotient. After this group, the Straight Division is carried out digit–wise. At every step, the Intermediate Dividend is formed by the previous remainder with the next digit of the dividend. This is the Gross Dividend, which is to be divided by the Common Divisor (CD) to get the 2nd quotient b and the remainder is R. The remainder R is coupled with next dividend digit again forming the Gross Dividend. From here onwards, corresponding subtraction terms from the expansion table $(a + b + c + d + \ldots)^3$ are to be considered as per the placement. Here a,b,c… are the quotients. (If one is interested only in 3 quotient digits, one has to use $(a + b + c)^3$) If the number is divided into 3 groups then abc is the cube root, if it has a perfect root. If the given number is divided into more than 3 groups then one has to consider the expansion to that number of elements for obtaining the perfect / imperfect roots. In brief, this procedure amounts to Construction of New Dividends from Gross Dividends, using subtractions from the general expansion tables to get New Dividends, working out the quotients by dividing the ND's by CD and again coupling the remainder R to the next dividend digits and continue to work in the above manner correspondingly to the required choice.
3. If the divisor happens to be very small, consider the first two groups as a single unit and proceed to workout in the same manner. This method is called two digit method.

These methods are applicable for non–perfect cube roots also provided after the working of n digits (equal to number of groups) are completed, then one has to workout for the decimal point as per one's choice. It is noticed that one has to get all zero's as the quotients for a perfect cube after the process of completion of number of quotients equal to the number of groups.

Example of Swamiji's working:

Basic principles explained in Part I, Section B for the ...JKL method, hold good for the general method.

The following sequence of the various digits, one can note the following points.

The sequence of the Various Digits.

1. The first place by a^3
2. The second place by $3a^2b$
3. The third place by $3ab^2+3a^2c$
4. The fourth place by $6abc + b^3$
5. The fifth place by $3ac^2 + 3b^2c$
6. The sixth place by $3bc^2$
7. The seventh place by c^3 and so on.

The Dividends, Quotients and Remainders are

1. The first D, Q and R are available at sight.
2. From the second dividend, no deduction is to be made
3. From the third, subtract $3ab^2$
4. From the fourth, deduct $6abc + b^3$.
5. From the fifth, subtract $3ac^2 + 3b^2c$
6. From the sixth, deduct $3bc^2$
7. From the seventh, subtract c^3; and so on

The above will be the corresponding subtraction terms.

These subtraction terms are used in Argumentation (...JKL) method when one has to consider a, b, c... are to be replaced by L, K, J... being the digits for the root form RHS.

Annexure Tables

Table M

Solution is (a.bcde ....)

Subtraction terms with respect to Taylor's Series Coupled with Derivatives, Decimal Solutions of x^n equation where n = 1 to 12 and upto 12 decimal points. Integer part is a

Placement 10^{-x}	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12
	b	c	d	e	f	g	h	i	j	k	l	m
	b	c	d	e	f	g	h	i	j	k	l	m
Duplex												
$\frac{1}{2}f''(a)$		b^2	$2bc$	$2bd+c^2$	$2be+2cd$	$2bf+2ce+d^2$	$2bg+2cf+2de$	$2bh+2cg+2df+e^2$	$2bi+2ch+2dg+2ef$	$2bj+2ci+2dh+2eg+f^2$	$2bk+2cj+2di+2eh+2fg$	$2bl+2ck+2dj+2ei+2fh+g^2$
		b	c	d	e	f	g	h	i	j	k	l
Triplex												
$\frac{1}{6}f'''(a)$			b^3	$3b^2c$	$3b^2d+3bc^2$	$3b^2e+6bcd+c^3$	$3b^2f+6bce+3bd^2+3c^2d$	$3b^2g+6bcf+6bde+3c^2e+3cd^2$	$3b^2h+6bcg+6bdf+3be^2+3c^2f+6cde+d^3$	$3b^2i+6bch+6bdg+6bef+3c^2g+6cdf+3ce^2+3d^2e$	$3b^2j+6bci+6bdh+6beg+3bf^2+3c^2h+6cdg+6cef+3d^2f+3de^2$	$3b^2k+6bcj+6bdi+6beh+6bfg+3c^2i+6cdh+6ceg+3cf^2+3d^2g+6def+e^3$
				−4	−5	−6	−7	−8	−9	−10	−11	−12
			b	c	d	e	f	g	h	i	j	k

Quantriplex									
$\frac{1}{24}f^{iv}(a)$	b^4	$4b^3c$	$4b^3d+6b^2c^2$	$4b^3e+12b^2cd+4bc^3$	$4b^3f+12b^2ce+6b^2d^2+12bc^2d+c^4$	$4b^3g+12b^2de+12bc^2e+12bcd^2+4c^3d+12b^2cf$	$4b^3h+12b^2df+6b^2e^2+12bc^2f+24bcde+4bd^3+4c^3e+6c^2d^2+12b^2cg$	$4b^3i+12b^2ch+12b^2dg+12b^2ef+12bc^2g+24bcdf+12bce^2+12bd^2e+4c^3f+12c^2de+4d^3c$	$4b^3j+12b^2ci+12b^2dh+6b^2f^2+12bc^2h+24bcdg+24bcef+12bd^2f+12bde^2+4c^3g+12c^2df+6c^2e^2+12cd^2e+d^4+12b^2eg$
	-4	5	6		8	-9	-10	-11	-12
	b	c	d	e	f	g	h	i	j
$\frac{1}{120}f^{v}(a)$			$5b^4c$	$5b^4d+10b^3c^2$	$5b^4e+20b^3cd+10b^2c^3$	$5b^4f+20b^3ce+10b^3d^2+30b^2c^2d+5bc^4$	$5b^4g+20b^3cf+20b^3de+30b^2c^2e+30b^2cd^2+20bc^3d+c^5$	$5b^4h+20b^3cg+20b^3df+10b^3e^2+30b^2c^2f+60b^2cde+10b^2d^3+20bc^3e+30bc^2d^2+$	$5b^4i+20b^3ch+20b^3dg+20b^3ef+30b^2c^2g+{}^2cdf+{}^2ce^2+{}^2d^2e+bc^3f+$

								5c	$60bc^2de+$ $20bd^3c+$ $5c$ $+10c^3d^2$
		5	6		−8	9	10	−11	−12
			c	d	e		g	h	i
$\frac{1}{720}$	a		b^6	$6b^5c$	$6b^5d+15b^4c$	$6b^5e+$ $30b^4dc$ $+20b^3c^3$	$6b^5f+$ $30b^4ce+$ $5b^4d^2+$ $60b^3c^2d+$ $15b^2c^4$	$6b^5g+$ $30b^4cf+$ $30b^4de+$ $60b^3c^2e+$ $60b^3cd^2+$ $60b^2c^3d+$ $6bc^5$	$6b^5h+$ $30b^4cg+$ $30b^4df+$ $15b^4e^2+$ $60b^3c^2f+$ $120b^3cde$ $)b^3d^3$ $)b^2c^3e$ $+90b^2c^2d^2$ $+30bc^4d$ $+c^6$
			6	−7	8	−9	−10	1	−12
				c		e		g	

$\frac{1}{5040}f^{VII}(a)$	b^7	$7b^6c$	$7b^6d+$ $21b^5c^2$	$7b^6e+$ $42b^5cd+$ $35b^4c^3$	$7b^6f+$ $42b^5ce+$ $21b^5d^2+$ $105b^4c^2d$ $+35b^3c^4$	$7b^6g+$ $42b^5cf+$ $42b^5de+$ $105b^4c^2e+$ $105b^4cd^2+$ $140b^3c^3d+$ $21b^2c^5$
	b	c	d	e	f	g
1						
$\frac{1}{40320}f^{VIII}(a)$		b^8	$8b^7c$	$8b^7d+$ $28b^6c^2$	$8b^7e+$ $56b^6cd+$ $56b^5c^3$	$8b^7f+$ $56b^6ce+$ $28b^6d^2+$ $168b^5c^2d+$ $70b^4c^4$
	-7	-8	-9	-10	-11	-12
		b	c	d	e	f
$\frac{1}{362880}f^{IX}(a)$			b^9	$9b^8c$	$9b^8d+$ $36b^7c^2$	$9b^8e+$ $72b^7cd+$ $84b^6c^3$
			b	c	d	e

$\frac{1}{3628800}f^{X}(a)$	b^{10}	$10b^{9}c$	$10b^{9}d + 45b^{8}c^{2}$
	b	c	d
$\frac{1}{39916800}f^{XI}(a)$		b^{11}	$11b^{10}c$
	-10	-11	-12
		b	c
$\frac{1}{47900160}f^{XII}(a)$			b^{12}

Table N

$(\ldots e+d+c+b+a)^3$ expansion terms

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	r
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
10^0	a^3																				
10^1		$3a^2b$																			
10^2		$3a^2c$	$3ab^2$																		
10^3	b^3	$3a^2d$								6abc											
10^4		$3a^2e$	$3ac^2$	$3b^2c$						6abd											
10^5		$3a^2f$		$3b^2d$	$3bc^2$					6abe	6acd										
10^6	c^3	$3a^2g$	$3ad^2$	$3b^2e$						6abf	6ace	6bcd									
10^7		$3a^2h$		$3b^2f$	$3bd^2$	$3c^2d$				6abg	6acf	6bce	6ade								
10^8		$3a^2i$	$3ae^2$	$3b^2g$		$3c^2e$	$3cd^2$			6abh	6acg	6bcf	6adf	6bde							
10^9	d^3	$3a^2j$		$3b^2h$	$3be^2$	$3c^2f$				6abi	6ach	6bcg	6adg	6bdf	6cde	6aef					
10^{10}		$3a^2k$	$3af^2$	$3b^2i$		$3c^2g$	$3ce^2$	$3d^2e$		6abj	6aci	6bch	6adh	6bdg	6cdf	6aeg	6bef				
10^{11}		$3a^2l$		$3b^2j$	$3bf^2$	$3c^2h$		$3d^2f$	$3de^2$	6abk	6acj	6bci	6adi	6bdh	6cdg	6aeh	6beg	6afg	6cef		
10^{12}	e^3	$3a^2m$	$3ag^2$	$3b^2k$		$3c^2i$	$3cf^2$	$3d^2g$		6abl	6ack	6bcj	6adj	6bdi	6cdh	6aei	6beh	6afh	6ceg	6def	6bfg

Table O

$(k + j + i + h + g + f + e + d + c + b + a)^3$
a is in 10^0, b is in 10^1, c is in 10^2, d is in 10^3, e is in 10^4, f is in 10^5, g is in 10^6, h is in 10^7, i is in 10^8, j is in 10^9 and k is in 10^{10}s place

10^0	a^3					
10^1		a^2b				
10^2		a^2c				
		b^2a				
10^3	b^3	a^2d	abc			
10^4		a^2e	abd			
		b^2c, c^2a				
10^5		a^2f, c^2b	abe			
		b^2d	acd			
10^6	c^3	a^2g, d^2a	abf	bcd		
		b^2e	ace			
10^7		a^2h, c^2d	abg	bce		
		b^2f, d^2b	acf			
			ade			
10^8		a^2i, c^2e, e^2a	abh	bcf		
		b^2g, d^2c	acg	bde		
			adf			
10^9	d^3	a^2j, c^2f, e^2b,	abi	aef	cde	
		b^2h	ach	bcg		
			adg	bdf		
10^{10}		a^2k, c^2g,	abj,	aeg	bef	
		e^2c, d^2e, f^2a,	aci,	bch	cdf	
		b^2i	adh,	bdj		
10^{11}		b^2j, c^2h,	abk,	aeh	beg	
		e^2d, d^2f, f^2b	acj,	afg	cdg	
			adi,	bci	cef	
				bdh		
10^{12}	e^3	b^2k, c^2i, f^2c,	ack	aei	beh	def
		d^2g, g^2a	adj	afh	bfg	
				bcj	cdh	
				bdi	ceg	
10^{13}		c^2j, e^2f, g^2b,	adk	aej	bck	ceh
		d^2h, f^2d	bfh	afi	bdj	cfg
			cdi	agh	bei	deg
10^{14}		c^2k, e^2g,	aek	bdk	bgh	deh
		g^2c, d^2i, f^2e,	afj	bej	cdj	dfg
		h^2a	agi	bfi	cei,cfh	

10^{15}	f^3	d^2j, e^2h, g^2d, h^2b	afk agi	ahi bek bfj	bgi cek cej cfi cgh	dei dfh					
10^{16}		d^2k, e^2i, f^2g, i^2a, g^2e, h^2c	agk ahj bfk	bgj bhi cek	cfj cgi dej dfi						
10^{17}		e^2j, f^2h, g^2f, i^2b, h^2d	ahk aij	bgk bhj	cfk cgi dek	chi	dfj	dgi, efi	egh		
10^{18}	g^3	e^2k, f^2i h^2e, i^2c, j^2a	aik bhk	bij cgk	fgh	chj	dfk	dgj, dhi	efj	egi	
10^{19}		f^2j, g^2h, i^2d j^2b, h^2f	ajk bik	fgi	chk cij		dgk	dhj	efk	egj	ehi
10^{20}		f^2k, g^2i, i^2e j^2c, h^2g, k^2a	bjk cik	dhk dij	ehj fgj fhi						
10^{21}	h^3	g^2j, i^2f, j^2d, k^2b	cjk dik	ehk eij fgk fhj	ghi						
10^{22}		g^2k, i^2g , j^2e, h^2i, k^2c	djk eik fij fhk		ghj						
10^{23}		h^2j, i^2h, j^2f, k^2d	ejk fjk		ghk gij						
10^{24}	i^3	h^2k, i^2g, k^2e	gik hij								
10^{25}		i^2j, j^2h, k^2f	gik hik								
10^{26}		e^2k, j^2i, k^2g	hjk								
10^{27}	j^3	k^2h	ijk								
10^{28}		j^2k, k^2i									
10^{29}		k^2j									
10^{30}	k^3										

Table P

(first row stands for placements as powers of 10)

6 5 4 3 2 1 0

$(g+f+e+d+c+b+a)^4$

	1	4	6	12	24
10^0	a^4				
10^1		a^3b			
10^2		a^3c	a^2b^2		
10^3		a^3d, ab^3		a^2bc	
10^4	b^4	a^3e,	a^2c^2	a^2bd, b^2ac	
10^5		a^3f, b^3c		a^2be, a^2cd, b^2ad, c^2ab	
10^6		a^3g, ac^3, b^3d	a^2d^2, b^2c^2	a^2bf, a^2ce, b^2ae	abcd
10^7		b^3e, bc^3		a^2bg, a^2cf, a^2de, b^2af, b^2cd, c^2ad, d^2ab	abce
10^8	c^4	b^3f	a^2e^2, b^2d^2	a^2cg, a^2df, b^2ag, b^2ce, c^2ae, c^2bd, c^2ae, d^2ac	abcf, abde
10^9		ad^3, b^3g, c^3d		a^2dg, a^2ef, b^2cf, b^2de, c^2af, c^2be, c^2ab, d^2bc	abcg, abdf, acde
10^{10}		bd^3, c^3e	a^2f^2, b^2e^2, c^2d^2	a^2eg, b^2cg, b^2df, c^2ag, c^2bf, d^2ae, e^2ac	abef, abdg, bcde, acdf,
10^{11}		c^3f, cd^3		a^2fg, b^2dg, b^2ef, c^2bg, d^2af, d^2be, e^2ad, e^2bc, f^2ab	abeg, acdg, acef bcdf
10^{12}	d^4	ae^3, c^3g	a^2g^2, b^2f^2, c^2e^2	b^2eg, d^2ag, d^2bf, d^2ce, e^2bd, f^2ac, c^2df	abfg, aceg, adef, bcdg, bcef
10^{13}		be^3, d^3e		b^2gf, c^2ef, d^2bg, d^2cf, e^2af, e^2cd, f^2ad, f^2bc, g^2ab, c^2dg	acfg, adeg, bceg, bdef,
10^{14}		ce^3, d^3f	b^2g^2, c^2f^2, d^2e^2	c^2eg, d^2cg, e^2ag, e^2bf, f^2ae, f^2bd, g^2ac	adfg, bcfg, bdeg, cdef
10^{15}		af^3, d^3g, de^3		c^2fg, e^2bg, e^2cf, f^2be, f^2cd, g^2ad, g^2be, d^2ef, g^2bc	aefg, bdfg, cdeg
10^{16}	e^4	bf^3	c^2g^2, d^2f^2	e^2cg, e^2df, f^2ag, f^2ce, g^2ae, g^2bd, d^2eg	befg, cdfg
10^{17}		cf^3, e^3f		d^2fg, e^2dg, f^2bg, f^2de, d^2fg, g^2be, g^2cd	cefg
10^{18}		ag^3, df^3, e^3g	d^2g^2, e^2f^2	f^2cg, g^2bf, g^2ce	defg
10^{19}		dg^3, ef^3		e^2fg, f^2dg, g^2cf, g^2de	
10^{20}	f^4	cg^3	e^2g^2	f^2eg, g^2df	
10^{21}		dg^3, f^3g		g^2ef	
10^{22}		eg^3	f^2g^2		
10^{23}		fg^3			
10^{24}	g^4				

Table Q

$(g\ f\ e\ d\ c\ b\ a)^4$ (group wise expansion)

1	4	4	6	12	12	24	24
a^4	a^3b	c^3d	a^2b^2	a^2bc	d^2ef	abcd	cdfg
	b^3a	cd^3	a^2c^2	a^2bd	d^2eg	abce	cefg
b^4	a^3c	c^3e	a^2d^2	a^2be	d^2fg	abeg	defg
	c^3a	ce^3	a^2e^2	a^2bf	e^2fg	abde	
c^4	a^3d	c^3f	a^2f^2	a^2bg	$\overline{29}$	abdf	
	d^3a	cf^3	a^2g^2	a^2cd	ab^2c	abdg	
d^4	a^3e	c^3g	b^2c^2	a^2ce	abc^2	abef	
	e^3a	cg^3	b^2d^2	a^2cf	ab^2d	abeg	
e^4	a^3f	d^3e	b^2e^2	a^2cg	abd^2	abfg	
	af^3	de^3	b^2f^2	a^2de	ab^2e	acde	
f^4	a^3g	d^3f	b^2g^2	a^2df	abe^2	acdf	
	ag^3	df^3	c^2d^2	a^2dg		acdg	
g^4	b^3c	d^3g	c^2e^2	a^2ef		acef	
	bc^3	dg^3	c^2f^2	a^2eg		aceg	
	b^3d	e^3f	c^2g^2	a^2fg		adef	
	bd^3	ef^3	d^2e^2	b^2cd		adeg	
	b^3e	e^3g	d^2f^2	b^2ce		adfg	
	be^3	eg^3	d^2g^2	b^2cf		aefg	
	b^3f	f^3g	e^2f^2	b^2cg		bcdc	
	bf^3	fg^3	e^2g^2	b^2de		bcdf	
	b^3g		f^2g^2	b^2df		bcde	
	bg^3			b^2dg		bcdg	
				c^2de		bcfg	
				c^2df		bcef	
				c^2dg		bdeg	
						befg	
						cdef	
						cdeg	

Table R

$(i + h + g + f + e + d + c + b + a)^5$ (group wise and placement)

Coeff:	1	5	10	20	30	60
10^0	a^5					
10^1		a^4b				
10^2		a^4c	a^3b^2			
10^3		a^4d	a^2b^3	a^3bc		
10^4		a^4e, ab^4	a^3c^2	a^3bd	a^2b^2c	
10^5	b^5	a^4f		a^3be, b^3ca, a^3cd	a^2b^2d, a^2c^2b	
10^6		a^4g, b^4c	a^3d^2, c^3a^2	a^3bf, a^3ce, ab^3d	a^2b^2e, b^2c^2a	$bcda^2$
10^7		a^4h, b^4d	b^3c^2	a^3bg, c^3ab, a^3cf, a^3de, ab^3e	a^2b^2f, a^2d^2b, a^2c^2d,	$bcea^2$, ab^2cd
10^8		a^4i, b^4e, ac^4	a^3e^2, c^3b^2	a^3bh, b^3cd, a^3cg, a^3df, ab^3f	a^2b^2g, a^2c^2e, a^2d^2c, ab^2d^2	a^2bcf, a^2bde, ab^2ce abc^2d

One can continue as per the working choice

Table S

Expansion of $(d + c + b + a)^6$ (group wise and placement)

	1	6	15	30	20	60	120	90	180
10^0	a^6								
10^1		a^5b							
10^2	a^6	a^5c	a^4b^2						
10^3		a^5d		a^4bc	a^3b^3				
10^4			a^4c^2 b^4a^2	a^4bd		a^3b^2c			
10^5		b^5a		a^4cd		a^3bc^2 a^3b^2d b^3a^2c	a^3bcd		
10^6	b^6		a^4d^2 c^4a^2	b^4ac	a^3c^3	b^3a^2d		$a^2b^2c^2$	
10^7		b^5c		b^4ad		a^3bd^2 a^3c^2d b^3ac^2 c^3a^2b			a^2b^2cd
10^8		b^5d	b^4c^2 c^4a^2	b^4cd		a^3cd^2 c^3ab^2	b^3acd	$a^2b^2d^2$	a^2c^2bd
10^9				c^4ab	a^3d^3 b^3c^3	b^3ad^2 c^3a^2d			a^2d^2bc b^2c^2ad
10^{10}		c^5a	b^4d^2 c^4b^2			b^3c^2d d^3a^2c	c^3abd	$a^2c^2d^2$	b^2d^2ac
10^{11}		c^5b		c^4ad		b^3cd^2 c^3b^2d d^3ab^2 d^3a^2c			c^2d^2ab
10^{12}	c^6		d^4a^2	c^4bd	b^3d^3	c^3ad^2	d^3abc	$b^2c^2d^2$	
10^{13}		c^5d		d^4ab		c^3bd^2 d^3ac^2 d^3b^2c			
10^{14}			c^4d^2 d^4b^2	d^4ac		d^3bc^2			
10^{15}		d^5c		d^4bc	c^3d^3				
10^{16}		d^5b	d^4c^2						
10^{17}									
10^{18}	d^6								

Table T

$(h + g + f + e + d + c + b + a)^7$ (group wise and placement)

			21	35	42	105	140	210	420
10^0	a^7								
10^1		a^6b							
10^2		a^6c	a^5b^2						
10^3		a^6d		a^4b^3	a^5bc				
10^4		a^6e	a^5c^2	a^3b^4	a^5bd	a^4b^2c			
10^5		a^6f	a^2b^5		a^5be	a^4b^2d	a^3b^3c		
					a^5cd	a^4c^2b			
10^6		ab^6	a^5d^2	a^4c^3	a^5bf,	a^4b^2c,	a^3b^3d	$a^3b^2c^2$	
		a^6g			a^5ce	a^2b^4c		a^4bcd	
10^7	b^7	a^6h			a^5bg,	a^4b^2f	a^3b^3e	$a^2b^3c^2$	a^3b^2cd
					a^5cf,	a^4c^2d	a^3c^3b	a^4bce	
					a^5de	a^4d^2b			
					ab^5c	a^2b^4d			
10^8		a^6i	a^5e^2	a^3c^4	a^5bh	a^4b^2g	a^3b^3f	$a^3b^2d^2$	a^3b^2ce
		b^6c			a^5cg,	a^4c^2e		a^4bcf	a^2b^3cd
					a^5df ab^5d	a^4d^2c		a^4bde	a^3bc^2d
						a^2b^4e		$a^2b^2c^3$	
						ab^4c^2			

One can continue as per the requirements

Table U

Relation between number ending and roots

Number Ending	Sq root ends	CR ends	4th root ends	5th root ends	6th root ends	7th root ends	8th root ends
1	1,9	1	1,3,7,9	1	1,9	1	1,3,7,9
2	–	8	–	2	–	8	–
3	–	7	–	3	–	7	–
4	2,8	4	–	4	2,8	4	–
5	5	5	5	5	5	5	5
6	4,6	6	2,4,6,8	6	4,6	6	2,4,6,8
7	–	3	–	7	–	3	–
8	–	2	–	8	–	2	–
9	3,7	9	–	9	3,7	9	–

Conclusions

A few salient points

1. Under equations, the attempt to solve all the solutions of not only Cubic but also Higher Order is considered to be a good achievement.
2. A Method for General Expansion of $(a + b + c + d \ldots)^n$ including evaluation of the coefficients is considered on the basis of Symmetry and homogeneity. This method is workable to any order in general. The application of these terms to numbers or any expressions in place of a, b, c.... is considered to be novel. The terms of the same tables of an expansion can be used not only for the placements of the digits in a number for an example $(\ldots c + b + a)$ where a in 10^0, b in 10^1, c in 10^2 's place but also $(a + b + c \ldots)$ where a is in highest placement (The exact placement needs to be considered

   In case of decimals a is to be considered as integer part, where b is in 10^{-1}, c is in 10^{-2} 's place etc.
3. The method of expansion of $(a+b+c+\ldots)^n$ is to divide the expansion terms into groups, each group having different coe-efficient but with the terms in each group having same coefficient. The method for evaluation of the coefficient is also developed.
4. The roots are worked out also for numbers having imperfect roots and extended to the required range of decimals.
5. Roots for polynomials in various degrees in one variable are determined by both the Swamiji's Method and Taylor expansion method and compared.
6. Methods for solving the cubic equations are extended to determine all the solutions (including imaginary) of 4^{th}, 5^{th}, 8^{th} degree equations.
7. It is noticed that Swamiji's Method is easier than the Taylor's Expansion Method. Both the methods could be well comparable.
8. A significant achievement is that the solutions of the equations can be tested by simple Sutram 'Gunita Samuccayah, Samuccaya Gunitah'. This is a highly relieving feature of the laborious work
9. Lastly an attempt to obtain the roots of polynomials with two or more variables using Swamiji's straight division method in considered to be a novel achievement.
10. A large number of problems are worked out under each section particularly to give an incentive to a learner and serves a very good algorithm for computer experts to develop programs.
11. We have already programmed to determine all the solutions of cubic equations using the described algorithm. These details will be published later along with other programs.