LECTURE NOTES – V

RECURRING DECIMALS, AUXILIARY FRACTIONS, DIVISIBILITY AND THEORY OF OSCULATION, POWER SERIES, PARTIAL FRACTIONS, DIFFERENTIAL EQUATIONS INCLUDING PARTIAL, LOGARITHMS & EXPONENTIALS, TRIGONOMETRIC FUNCTIONS INCLUDING INVERSE AND HYPERBOLIC, HCF

ON

JAGADGURU SANKARACHARYA

SRI BHARATI KRISHNA TIRTHAJI MAHARAJA'S WORK

VEDIC MATHEMATICS

By

Sri Ch.V. Saradhi , Smt. N. Bhanusri, Kum. Chaitanya, Sri D. Sarveswara Rao, Prof. P. Gopalakrishna Murthy, Prof. T.K.V. Iyengar, Prof. L.S.R.K. Prasad, Prof. S.V. Krishna, Prof. M.G.K. Murthy, Prof A. Sithapathi, Sri. K. V. Prasad and Prof. C. Santhamma

Sponsored By

DR. GOKULAKUMARI GHANASHYAMDAS SHAH

VEDIC MATHEMATICS PROJECT

*Contact Person: **Prof. C.Santhamma**, Project Director,

8-60-1, Vidyanagar, Visakhapatnam – 530 003,

☎: (0891)- 2755884, 6457515

e_mail : santhamma_c@yahoo.co.in

Publishers

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Lord Dattatreya

THIS LECTURE NOTES IS DEDICATED TO

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PUBLISHER'S NOTE

Bharateeya Vidya Kendram, Visakhapatnam is a well-known educational organization in the North Eastern part of Andhra Pradesh. It is affiliated to Vidya Bharathi, Akhila Bharateeya Sikshana Sansthan, Lucknow.

Vidya Bharathi is running approximately 20,000 educational institutions throughout the country. Vidya Bharathi tempers the educational system with Bharateeyata. A national orientation to the educational system is the need of the day. This orientation is based on five aspects-Sanskrit, Sangeet, Sharirik, Yoga and Naithik Sikshana.

Vidya Bharathi has several activities-publication of materials concerned with Sishuvatika, Samskara kendrams, Yoga etc., Sanskrit education is one such.

Presently, Bharateeya Vidya Kendram undertakes propagation of the principles explained by Sri Jagadguru Sankaracharya in his work on Vedic Mathematics. Consequently Bharateeya Vidya Kendram has taken up the publication of series of Lecture notes on Jagaduru Sankaracharya, Sri Bharathi Krishna Tirthaji Maharaja's work on Vedic Mathematics.

Dr. Smt. Gokul Kumari Ghana Shyam Das Shah Vedic Mathematics project has undertaken the onerous job of preparing the lecture notes. The project graciously handed over the first, 2nd, 3rd(a), 3rd(b) (treated as 4th in the series) volumes in the series of the lecture notes to the Bharateeya Vidya Kendram for publication. It augurs well for Bharateeya Vidya Kendram accepted its publications with the sacred work of Lecture Notes-I Multiplication, Lecture Notes II on Division, Lecture Notes III(a) on Equations, Lecture Notes III(b) on Squares, Cubes, expansions, Roots (Numbers and Polynomials) and Equations (Contd.) and the last in the series as Volume V on Recurring Decimals, Auxiliary Fractions, Divisibility And Theory Of Osculation, Partial Fractions, Powers Series, Differential Equations, Logarithms & Exponentials, Trigonometric Functions Including Inverse and Hyperbolic, HCF on Jagadguru Sankaracharya's work, Vedic Mathematics.

We are grateful to the Project for choosing us as the publishers.

We hope the publication will be well received and our efforts will be fruitful in reaching the wider circle of public.

Bharateeya Vidya Kendram
Visakhapatnam

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DIRECTOR'S NOTE

I had the good fortune of meeting Mananiya Rambhow Satheji, of Bharatiya Sanskriti Prachara Samithi to work on a project on Vedic Mathematics, initiated by Sri Jagadguru Sankaracharya of Puri Mutt, with a view to propagate and popularize the methods, sutras, etc in a systematic manner by bringing them in the form of a series of lecture notes which include basic and advanced mathematical operations. Mananiya Satheji was convinced of the merits of the proposal and assured me of arranging a major part of the finances required. I was very deeply touched by his instantaneous action by contacting Sri Ghanshyam Das Shah, a philanthropist. I was further inspired by the promptness of Sri Ghanshyam Das Shahji to sponsor the project.

I would also like to go back to the genesis of the thought of working on Vedic Mathematics. In the very late seventies, when Sri. M. R. Appa Rao, the then Vice Chancellor of the Andhra University and Dr. M. Gopalakrishna Reddy, the then Registrar who became later the Vice Chancellor of Andhra University sent to me a letter written to the University by Dr. A.P. Nicholas (U.K.) to find out if Swamiji has graced the Andhra University with his presence and also gave lectures on Vedic Mathematics as a part of his lecture tour program both in India and abroad. Since then I was working and contemplating on Vedic Mathematical operations. I participated in several workshops conducted on these topics. I had the opportunity of lecturing on these topics both in India and USA. I had very useful discussions with several professors in mathematics who also are convinced of the idea of a systematic preparation of lecture notes useful not only to the students but also to the mathematicians and persons interested in basic as well as advanced mathematics.

The project is a collective endeavor, includes, in addition to the efforts of project personnel, the scrutiny of the lecture notes, by several experts, of Current (Conventional) mathematical methods both in basic and advanced topics.

The Charitable Trust by name Sri Vangala Narsimha Dikshitulu Srimathi Allevelamma Smaraka Dharma Samstha came forward to provide the project, the office and infrastructure.

The project is named as Dr. (Smt.) Gokulakumari Ghanshyam Das Shah Vedic Mathematics Project, after the name of Late Smt. Gokulakumari as per the desire of her husband Sri Ghanshyam Das Shah, who funded the major part of the project.

It is highly gratifying to receive an amount of Rupees one lakh, through Honorable Former Central Minister Sri Ch. Vidyasagar Rao from

Smt. Ch . Chandramma Memorial Trust founded in the name of his late mother; which is meant exclusively for publishing of the series of lecture notes.

We are extremely indebted to Viswanatha Foundation, Prasanthi Nilayam, Puttaparthi, Anantapur District – 515134 for their liberal donation towards the publishing of Final Volume and other miscellaneous expenditure, an amount of Rs. 45,000/-

The methods that are brought out by Sri Jagadguru Sankaracharya are based on 16 Sutras and 13 Upasutras and corollaries (Sub-Sutras) and their combinations. With the help of these aphorisms, many aspects of basic and advanced topics also could be easily worked out. These are mostly for mental working with ease and elegance. The philosophy behind all this is to consider:

1. That the solution lies in the problem.
2. The symmetry aspect, total, or partial of varied degrees of symmetries in the sense deviation (either deficiency / excess) from a total symmetry.
3. The working out of solution is generally done with one or more bits under one or more different sutras as well.
4. Each method stands the test of proof, which is also aimed.
5. A comparison of the existing methods with the Vedic methods is elaborately worked out. The ease with which one can solve the problem is self-explanatory.
6. Simplification, elegance and ease are strikingly noticed in the application of the Vedic Sutras.
7. Operational techniques to work out Multiplication, Division, Squaring, Square-roots, Cubing, Simple equations, Simultaneous equations of two or more variables, Factorization, Quadratic equations, Fractions, Trigonometry, Recurring Decimals, Divisibility, Partial Fractions, Solid geometry, Co-ordinate geometry, Differential Equations, Integral and Integro Differential Equations, Partial Differential Equations, Logarithms, Exponentials, Combined operations, polynomials and power series etc are detailed in series of Lecture Notes.
8. This is intended for the teachers and also for students and other learners who are to work out the problems and train themselves before these are to be implemented.
9. This is definitely innovative and I opine that the methods undoubtedly promote research into much deeper and more difficult and unexplored areas as well.
10. It is really commendable to see that the problems can be solved by observing certain quanta such as similarities, dissimilarities, deficiencies, excess, conversion

of numbers into polynomials and vice versa, simplifying techniques such as conversion of a larger digit into a smaller one, existence of G.P., A.P., or relations of ratios, writing down the remainders, which are simply followed by the quotients, solving simple equations from certain identities existing among the various parts of the equations, transposing and adjusting, splitting the terms, elimination and retention, addition and subtraction, to make functions complete, Vedic code-decoding, groups and individuals, ultimate and penultimate relations, by mere inspection, certain striking identities in rearrangement, etc.

11. The aim is more towards the explanation and working methods, which our great Seers have passed on to us, with a view that these can be easily understood and worked out as a routine by mathematicians and also by those who have a good taste for mathematics. It is also aimed to see if these can be followed by introducing these methods in our curricula. As we are all trained in the western system, it may be difficult to switch on to this unless persons are first trained in these methods. Keeping this also in view the work on lecture notes is taken up.

12. We are also working on bringing out worked out examples applying Vedic mathematical principles to the problems in various texts. For a comparison, both the Current Method and the Vedic Methods are presented throughout.

13. In case of Multiplication & Division we could successfully perform these methods on the computer using Computer Language, which was attempted by the Project Fellows. For the Lecture Notes III (a), III(b) and V, it will be taken up a later date.

14. We are planning to conduct workshops to train the teachers in these methods. We propose to hold a conference at the end.

It is with this good intention I approached many professors in mathematics who are willing to take part in the work through their suggestions, expertise and scrutiny of the lecture notes.

The project fellows of this Lecture Notes Mrs. N. Bhanusri, Ch.V. Saradhi, D. Sarveswara Rao and Miss Chaitanya and Miss Kalyani and Sai Eswari are very hard working, diligent, efficient, and are found to be extremely deserving candidates in working conscientiously. They really deserve applause. Their persistent work with dedication and sincerity are highly exemplary.

Technical assistance on Computer of Mr. Bhaskar for the general Maintenance is exemplary. We extend our deep thanks to his patience and commitment to the work. We thankful of Computer assistance rendered by Mr. Naresh and Mr. Prasad

My deep indebtedness is to 1) Mananiya Satheji. But for his readiness, this work would not have been even dreamt of, and 2) the Bharatiya Sanskriti Prachara Samithi for their inspiring, acceptance of my proposal and arranging for the financial aid from a philanthropic sponsor, Sri. Ghanshyam Das Shahji.

My grateful thanks to Prof. L.S.R.K. Prasad, Prof. T.K.V. Iyengar, Prof. S.V. Krishna, Prof. M.G.K. Murthy, Prof. D.R.K. Sangameshwara Rao, Prof. A.Sitapati, Prof Gopal Krishna Murthy for their help in going through the manuscript and their valued suggestions.

I thank Dr. M. Gopalakrishna Reddy, Prof. P.V. Krishnaiah, Prof. V. Lakshminarayana, Prof. J. Gopalakrishna and Prof. N. Rajagopala Rao, for their keen interest in the topic, intense appreciation of the work and for their valuable suggestions.

My Special Thanks to the Former Prime Minister Late Sri. P.V. Narasimha Rao and Sri Kiriti Joshi for providing an opportunity for me to attend National Seminars on Vedic Mathematics at Jaipur, Ahmedabad and Meerut.

I am grateful to the honorable (ex) Central Minister of state Sri. Ch. Vidyasagar Rao for his liberal contribution made from the trust named after his other, which facilitated the publication of this work.

Thanks to Ma. Sudarshanji, and Late Ma. Seshadriji and Dr. Murali Manohar Joshiji for taking special interest and giving constant encouragement during this work.

I thank all those elders who have been encouraging and enquiring about the progress of this work.

Quite a number of well-wishers of the project participated in the lecture notes directly or indirectly, but for whose well wishes the project work, which is stupendous, could not be completed.

We are grateful to the Bharateeya Vidya Kendram, Visakhapatnam for accepting to publish all the series of the lecture notes.

I thank my husband, Prof. C. Subrhamanya Sastry, for his constant encouragement. I thank my old mother of 99 yrs who had been taking care of me throughout.

Finally I bow down at the feet of **Lord Dattatreya** for his Divine Directions and Blessings.

C.Santhamma

A WORD TO READERS

1) This attempt of writing lecture notes on Swahili's Vedic Mathematics is mainly to bring out the possibility of working out the mathematical problems using the Sutras and Upasutras whose modus operandi is explained in his book.
2) It is also aimed at working out a number of problems extending to a more general case.
3) General Proofs are given to various methods explained on the basis of sutras.
4) An attempt is also made to compare the current method with the methods explained by using the sutras.
5) The book is written mostly with a view to provide an incentive to a teacher or a researcher and to all those who are interested in mathematics, to understand the operation and to work out the details by themselves.

The reader is requested to go through all the working details given in the Lecture Notes. We are confident that the elegance, simplicity, ease with which the problem can be worked out applying the Vedic Sutras is undoubtedly transparent. This leads definitely to further probing into the unexplored areas in mathematics. This Lecture Notes is the final in the series and deals Recurring Decimals, Auxilliary Fractions, Divisibility and theory of Osculation, Partial Fractions, Powers series, Differential Equations, Vedic Numerical code, Logarithms & Exponentials, Trigonometric Functions including inverse and hyperbolic, HCF.

The various Sutras used, the modus operandi and the working details are to be concentrated unbiased to verify their applicability.

It may be likely that a few errors (Typographical nature by oversight) might have crept in, inspite of our conscientious effort in getting up this Lecture Notes. The Readers are requested kindly to communicate the same, if any, to the publishers

Elaborate procedures or working details, explanations of Sutras are given mostly in the text. For the matter given in the Synopsis, the reader is requested to refer concerned sections for full explanation or working details.

Any suggestions for improvement are earnestly solicited.

INTRODUCTION

His Holiness Jagad Guru Shankaracharya Sri Bharathi Krishna Tirthaji Maharaj of Govardhan Mutt, Puri, who had his Post Graduation in Sanskrit, Philosophy, English, Mathematics, History and Science, had initiated several mathematical formulations through the application of sixteen main sutras and thirteen Upasutras, stated to have been found in the appendix of Adharvana Veda. The modus operandi of these sutras was explained and the application of them in different topics, in working of mathematics, which include not only the general algebraical working but also higher and advanced working in differentials, integrals, polynomials, trigonometry, partial differentiation, solution of equations (simple, quadratic, cubic, biquadratic, etc), squares, cubes, square roots and higher order solutions, problems concerned with coordinate geometry, logarithms, exponential, summation of series, solid geometry, etc. A list of all the sixteen main sutras and thirteen upasutras is given. However in each part of the lecture notes the concerned and relevant sutras and upasutras are explained elaborately together with applications.

The methods are surprisingly simple, at the same time elegant and even able to solve in much less time than the current methods. The philosophy behind these methods is that a total faith in the dictum that

1) The solution of any problem is in the problem itself.
2) A symmetry of some order exists always in the problem which needs to be tapped. (Working is based also on such existing symmetries).
3) A totality and a comprehensive thought of working is clearly depicted.
4) Aim is always towards a mental working, preservation and a lesser complication in working details.

These features are clearly established in operational techniques as explained by the great seer Sri Bharathi Krishna Tirthaji.

We have taken up a project to work out the details and also to explore the possibilities of advancing these methods to arrive at important deductions which are found difficult or which are being worked out elaborately using current methods. The entire work is divided into a number of topics such as multiplication, division, equations, calculus, co-ordinate geometry, logarithms, trigonometry, squares, square-roots, cube-roots, and higher roots and the like. We intend to bring out lecture notes to enable not only mathematicians but also persons who are interested in mathematics to go through these methods and spread this message to the general public.

As the first of the series of Lecture Notes, we have already published Lecture Notes (I) Multiplication. We are pleased to say we have published also second in the series, Lecture Notes-II-Division. As the part of the third Lecture Notes - III(a) equations consisting of Simple Equations, Simple Simultaneous Equations Multiple simultaneous equations, Quadratic Equations, Simultaneous Quadratic Equations, Cubics, Higher Order Equations by applying about 10 Sutras namely Paravartya , Paravartya Yojayet, Sunyam Samya Samuccaye, Sopantyadwayamantyam, Antyayoreva (Upasutram), Anurupyena, Sunyam Anyat, Sankalana Vyavakalanabhyam, Adyamadyenantyamanthyena. (Upasutram followed by Anurupyena), Lopanasthapanabhyam (Upasutram), Purana Apuranabhyam, Gunita Samuccayah Samuccaya Gunitah, Gunaka Samuccayah in working out the solutions of Higher Degree Equations. In the Lecture Notes III (b),

(1) Continuation of solution of cubic equations is extended using two different methods extendable upto any decimal range. A number of examples are worked out together with a comparison. The sutras used are Vilokanam, Adyamadyena Antyamantyena, Argumentation, Purana Apuranabhyam, Gunita Samuccaya Samuccaya Gunitah. These methods are extended to solving higher order equations involving even 8^{th} degree equations, wherein all the solutions are derived.

(2) Squares, Cubes, Higher Order Expansions of numbers and polynomials with a generalisation of expansion to any positive integer are worked out and tabulated.

These tables are found to be useful in working out in Roots of any order with the decimal working and also determining the roots of polynomials with any number of variables as well. The sutras used for this work are Vilokanam, Argumentation, Lopana Sthapanabhyam, Yavadunam, Anurupyena, Dwandhvayoga. A general formulae for the expansions including the co-efficients of the various groups of the terms is worked out. The method for working out the square roots, cube roots of numbers and polynomials is clearly explained. The same is extended to higher order roots as well similarly for polynomials containing more than one variable.

In this lecture notes topics on Recurring Decimals, Auxiliary Fractions, Divisibility and Theory of Osculation, Power Series, Partial Fractions, Differential Equations including Partial, Logarithms & Exponentials, Trigonometric Functions including Inverse and Hyperbolic, and finally HCF have been dealt with the western method and also by applying the sutras pertaining to the Vedic Method.

The introduction of Auxiliary fractions, theory of osculators are considered to be for the first time defined by Swamiji. The logarithms and exponentials, Trigonometric functions including inverse functions have been detailed. The evaluation of highest common factor using (HCF) Vedic method, appears to be much simpler. A few different types of Differential equations are attempted to solve. It's noticed that the evaluation of Particular Integral (P.I) is extremely simpler than in the western method. Solutions of a few Partial differential equations have also been demonstrated.

It is felt that a more exhaustive and extensive work on differential equations is very much-needed using Vedic concepts. It is proposed to work out in detail and completely covering or all types of differential equations, latter.

The work is. carried out with care and precession. We request all the readers kindly to go through the work presented in all the volumes. In case an error is crept in this may be kindly brought to my notice for further correction. So that they can be corrected.

SYNOPSIS ON VEDIC MATHEMATICS – V

Section –1 deals with Recurring Decimals using several Vedic formulae. Full working details are clearly shown for each formula applied. The Vedic code language as explained by Swamiji is also enumerated with examples.

In Section –2 a new concept of Auxiliary fractions (called as Sahayaks) as introduced by Swamiji is explained. The method is simpler and the introduction of Auxiliary fractions, vulgar fractions and the modus operanda of the method are clearly explained. The working when carried out in comparison with the current method is strikingly significant which makes use of the sutras such as Ekadhikas etc. Vedic code method is clearly demonstrated.

Section –3 is another innovative, introduced by Swamiji, of the concept of osculation (theory of osculators, simple, complex, multiplex osculators), which are termed as Vestanas. Swamiji has explained is very interesting question as to how one can determine whether a certain given number, (it may be any long number) is divisible by the given number using Vedic method. The concept of theory of osculation, positive osculator, negative osculator, complex osculator, multiplex osculator has been well defined and applied to determine the divisibility. Quite a good number of problems are worked out.

Section-4 deals with a few problems worked out using both the current method and Vedic method. (Integrations by partial fractions).

Section-5 deals with powers of polynomials. Quite a good number of problems are worked out. The method making use of logarithms and differentiation appears to be elegant.

Section-6 deals with evaluation of logarithms and exponentials. The method is a continuation of the one described in power series.

Section-7 deals with the evaluation of Trigonometric, Hyperbolic and Inverse functions, which could be considered for every small angles as an approximation. The details of a Simpler method are worked out in appendix for the inverse sin, cos and tan functions. This method is also very approximate, particularly for small angles.

Section-8 deals with Differential equations. An attempt is made to solve the particular integral in a few types of differential equations. The method is considered to be novel in the sense that a comparison with current method gives a very surprisingly simple at work. We have extended to study the partial differential equations; the working details of which elegant suggested by the British author. It is felt that a more comprehensive and extensive work on the Differential equations is worthy of attempt. The Director proposes to take it up at a later stages.

Section –9 deals with HCF. It is interesting in the sense that the working by using Vedic methods is worthwhile practicing.

Section-1
RECURRING DECIMALS

One of the marvels of Vedic Mathematics is work on recurring decimals.

Preliminary note

Definition:
i) If a denominator has only 2 or 5 or 10 as factors it gives a non- recurring decimal.
For example:
$\frac{1}{2} = 0.5$; $\frac{1}{5} = 0.2$; $\frac{1}{10} = 0.1$; $\frac{1}{8} = 0.125$

ii)If denominators have only 3, 7, 11 or higher prime numbers as factors they give rise to Recurring decimals and come under a separate group which will be dealt with in this volume.

Similarly a denominator with factors partly of 3, 7 and 9 and partly of 2 and 5 leads to a mixed result which is partly recurring and partly non recurring decimal.

For example:
$\frac{1}{6} = \frac{1}{2\times3} = 0.1\dot{6}$; $\frac{1}{15} = \frac{1}{3\times5} = 0.0\dot{6}$
$\frac{1}{18} = \frac{1}{2\times9} = 0.0\dot{5}$; $\frac{1}{22} = \frac{1}{2\times11} = 0.0\dot{4}\dot{5}$
$\frac{1}{24} = \frac{1}{2\times12} = 0.41\dot{6}$ and so on.

In every nonrecurring decimal fraction with a standard numerator 1, it is observed that the last digit of the denominator and the last digit of the equivalent decimal when multiplied together ending in zero.

Example:
$\frac{1}{25} = 0.04$
∴ Last digit of denominator (5) when multiplied with the last digit of value i.e. 4 gives 20.
∴ The last digit of the product ends in zero.

In every recurring decimal with a standard numerator 1, the last digit of the denominator when multiplied with the last digit of its recurring decimal equivalent gives a product ending in 9.
(The product is actually a continuous series of nines)

For example:
$\frac{1}{17} = 0.\dot{0}5882352\,/\,9411764\dot{7}$

Now (7 last digit of the denominator) when multiplied with the last digit of recurring i.e. 7 gives 49 i.e. 9 ending and also all nines are resulted when the recurring value is

multiplied by 17. Now this enables us to determine, before hand the last digit of recurring decimal equivalent of a given vulgar fraction.

For example:

$\frac{1}{457}$ Will have a recurring decimal equivalent ending in 7 and so on.

Several methods are described by Swamiji to work out the recurring decimal of vulgar fractions under a classification of 3 ending; 7 ending; 9 ending and 1 ending using different sutras such as

Anurupyena

Ekadhikena

Ekanyunena etc

Certain relations exist among the components of the recurring decimals of vulgar fractions such as

(i) Geometrical Progressions

(ii) The relations between quotient digits and remainders

(iii) Complements / Summations etc

The methods suggested by Swamiji have a striking symmetry in working out only half of the answer (quotients) while the second half can be simply written down as complements from 9.

For example:

$$\frac{1}{17} = 0.05882352$$

$$\begin{array}{l} 94117647 \rightarrow \text{complements} \\ \hline 99999999 \end{array}$$

$$\therefore \frac{1}{17} = 0.\dot{0}5882352/9411764\dot{7}$$

In general points like where one has to stop working and take complements of 9 from then onwards is explained.

Using Ekadhika sutram one can work out up to $\frac{D-N}{2}$ digits (as first half) which are then followed by the same number of digits as the complements from 9 of the first half.

Swamiji has introduced a number of methods for the working of recurring decimals.

i) Using Geometrical Progression relationship. Thus leads into a clear ratio between one remainder and another (or dividend).

ii) By using the sutram Seshani Ankena Ceramena each remainder multiplied with last digit.

iii) Ekadhika process from L $\rightarrow$ R (by division) and R $\rightarrow$ L (by multiplication)

iv) Using Vedic code language.

v) Worked out a number of relations between quotients and remainders.

vi) Writing down the multiples of vulgar fractions by considering the original fraction and numbering them in ascending order.
vii) By using Adyam Adyena
viii) By using Antyam Antyena
ix) and An independent method.

Explanation of the given methods:
(i) <u>Geometric progression method</u> (Anurupyena)

If the relationship between remainder or predecessor or successor is known then one knows all the rest of the remainders, if the former is in the G.P. with its successor.

This method can be demonstrated as follows.
Consider the case $\frac{1}{7}$

The first remainder (rem) is 1, the second is 3, the successive remainders are considered to be in G. P i.e. applying "Anurupyena sutram" for writing down the successive remainders followed by castings of the multiples of divisors one get the following results.
The first 2 remainders are in the ratio 1: 3 to write down the 3 rd remainder, we have to consider $3 \times 3 = 9$, $\because 9 > 7$ we cast out $1 \times 7 = 7$ from 9 which results in 2. So 2 happens to be the next remainder.
Similar procedure is adopted for obtaining remaining remainders until the recurring occurs.
The following is the result:

Rem x G.P	Casting Divisor	Successive remainder
		3
$3 \times 3 = 9$	$1 \times 7 = 7; 9 - 7$ -	2
$2 \times 3 = 6$	0	6
$6 \times 3 = 18$	$2 \times 7 = 14; 18 - 14$	4
$4 \times 3 = 12$	$1 \times 7 = 7; 12 - 7$	5
$5 \times 3 = 15$	$2 \times 7 = 14; 15 - 14$	1

7)1.0 (0.142857
7
30
28
20
14
60
56
40
35
50
49
1

(1) If the denominator of fraction contains prime numbers as factors then it gives recurring decimals

Ex: $\frac{1}{7} = 0.\dot{1}4285\dot{7}$

(2) A denominators which has factors as 2 or 5 then non recurring decimal results

Ex: $\frac{1}{4} = 0.25.$

(3) If the denominator is partly of first type and partly of second type this gives a result partly value which is partly recurring and partly non recurring

Ex: $\frac{1}{28} = 0.035\dot{7}1428\dot{5}$

The individual contributions are worked out with the denominators of 1,3,7,9. Endorsing $\frac{1}{3}, \frac{1}{9}$ have only one recurring digit, $\frac{1}{11}$ gives 2 digits, $\frac{1}{7}$ gives 6 digits as recurring and so on.

In every non recurring decimal (with numerator (N) =1) the product of the last digit of denominator and the last digit of equivalent decimal is zero, where as in the recurring decimal the product is 9. Hence one can estimate the last digit of recurring decimal.

For example $\frac{1}{7}$ ends in 7, $\frac{1}{23}$ ends in 3, $\frac{1}{31}$ ends in 9 and so on.

When one works out for 1/7 remainders are metered to be in geometric progression. For example1, 3, (9 casting out 7, remainder is 2) 2, 6, (18 casting out 2×7, remainder is 4) 4. (12 casting out 7' remainder is 5) 5, (15 casting out $2\times7=14$, remainder is 1) 1 $\therefore$ The remainders are 1, 3, 2, 6, 4, 5. This process is called Anurupyena. The remainders give dividends by placing zero by its right side and quotients are obtained by dividing each one by 7.

For example 1 3 2 6 4 5

These dividends 10 30 20 60 40 50 when divided by 7 gives the result

as the quotients 1 4 2 8 5 7

Thus the value of the recurring decimal is $0.\dot{1}4285\dot{7}$

Much simpler method – starting from the G.P method of obtaining remainders.

That is 3, 2, 6, 4, 5,1 and multiplying each one by 7 and considering only the last digits

Then one gets the quotients as $0.\dot{1}\ 4\ 2\ 8\ 5\ \dot{7}$.

This is as per the sutrum Seshani Ankena Ceramena.

It is noticed that one half of the quotients are complements from the 9 for the next half.

It is further noticed that when the difference between Denominator (D) and Numerator (N) is reached as remainder, then one can stop and write down the complements from 9 of the former set.

Let us consider $\frac{1}{13}$ in the same process as obtaining remainders from G.P. The remainders are 1, 10, 9, 12, 3, 4, they give dividends by placing zeros by its right side and quotients are obtained by dividing each one by 13

For example	1	10	9	12	3	4	
These dividends	10	100	90	120	30	40	when divided by 13 gives
the quotients as	0	7	6	9	2	3	

Thus the value of the recurring decimal is $0.\dot{0}7692\dot{3}$

Here we notice that one half of the quotients are the complements from 9 for the next half.

We shall consider 9 ending denominator of the vulgar fraction type $\frac{1}{19}$.

Similarly applying the procedure of G.P for the remainders we get the total set of remainders. Now consider 10 as the geometric ratio to obtain other remainders as

$[(10\times10)-(19\times5)=5]$ 5, $[(5\times10)-(19\times2)=12]$ 12, _ _ _ _ _

$\therefore$ The set of remainders are 1, 10, 5, 12, 6, 3, 11, 15, 17, 18, 9, 14, 7, 13, 16, 8, 4, 2.

We get dividends by placing zeros by right side of the remainders

10, 100, 50, 120, 60, 30, 110, 150, 170, 180, 90, 140, 70, 130, 160, 80, 40, 20 $\rightarrow$ dividends

0, 5, 2, 6, 3, 1, 5, 7, 8, 9, 4, 7, 3, 6, 8, 4, 2, 1 $\rightarrow$ Q set

The quotient digits are "Q set" obtained on dividing the dividends by 19.

Thus the value of recurring decimal is $0.\dot{0}5263157894736842\dot{1}$.

Here we notice that one half of the quotients are the complements from 9 for the next half. When the value D – N occurs as remainder or is the same as the remainder then one can stop working further, consider the complements from 9 with reference to quotients.

i.e. 1,10,5,12,6,3,11,15,17,18. D-N being 18 one can stop working at this stage, instead one can write the quotients for the corresponding remainders using the method specified previously.

Thus the quotient digits are .0 5 2 6 3 1 5 7 8.The next step is to write down the complements of the set, which completes the recurring decimal.

$\therefore \frac{1}{19} = 0.\dot{0}52631578/94736842\dot{1}.$

(ii) <u>Seshani Ankena Ceramena</u>:

The second method is to apply the sutram Seshani Ankena Ceramena the working details of which are as follows. This sutram is applicable for remainders.

For example consider $\left(\frac{1}{7}\right)$.

The remainders are 3, 2, 6, 4, 5, 1, multiply these remainders by 7 (here ceramanka is 7) we get $2\underline{1}$, $1\underline{4}$, $4\underline{2}$, $2\underline{8}$, $3\underline{5}$, $\underline{7}$. Considering the right hand side digits or last digits (Ceramanka) from each product

We get $\frac{1}{7} = 0.\dot{1}4285\dot{7}$

In case of $\frac{1}{13}$ the remainders are 10, 9, 12, 3, 4, 1, multiply these remainders by 3 (here Ceramanka is 3) we get $3\underline{0}, 2\underline{7}, 3\underline{6}, \underline{9}, 1\underline{2}, \underline{3}$.

Considering the last digits of each products we get $\frac{1}{13} = .\dot{0}7692\dot{3}$.

In case $\frac{1}{19}$ the remainders are 10,5,12,6,3,11,15,17,18,9,14,7,13,16,8,4,2,1.

In case of 9 ending denominator of vulgar fraction the last digits of remainders taken in order will be the recurring decimal.

$\therefore \frac{1}{19} = 0.\dot{0}5263157894736842\dot{1}$

But it is to be noted that in case of 9 ending the ekadhikena sutram is the best applicable.

Let us consider the case $\frac{1}{13}$.

We know that the remainders are 10, 9, 12, 3, 4, 1. Here we can multiply the successive remainder by 10, cast out 13's from the result to get the next remainder. This process is continued to get the complete set of remainders. This process can be stopped when D – N is reached for remainder, from which onwards the rest of the remainders are obtained by writing down the complements from the divisor/ denominator to get the complete set.

$$10 \times 9 = 90 - 78 = 12$$

Now this is equal to D – N and hence we get the remainders as complements from 13 which are 3, 4, and 1.

Now set of remainders are 10, 9, 12, 3, 4, and 1.In order to get the quotient digits multiply all the remainders with the last digit of the denominator 3(of 13) the result is $3\underline{0}$, $2\underline{7}$, $3\underline{6}$, $\underline{9}$, $1\underline{2}$, $\underline{3}$.

Hence $\frac{1}{13} = 0.\dot{0}7692\dot{3}$.

Consider $\frac{1}{19}$

We know that the set of remainders are 10, 5, 12, 6, 3, 11, 15, 17,18, 9, 14, 7, 13, 16, 8, 4, 2, 1 now let the first remainder 10, be the geometric ratio. Also we know that 5 is next remainder.

Multiply 10 with 5 and cast out 19's from it to get successive remainders.

$10 \times 10 = 100 - 95 = 5$

i.e. $10 \times 5 = 50 - 38 = 12$ Thus 3'rd remainder is obtained.

Similarly

$10 \times 12 = 112 - 114 = 6$

$10 \times 6 = 60 - 57 = 3$

$10 \times 3 = 30 - 19 = 11$

$10 \times 11 = 110 - 95 = 15$

-- -- -- -- -- -- -- -- -- --

-- -- -- -- -- -- -- -- -- --

-- -- -- -- -- -- -- -- -- --8

$10 \times 8 = 80 - 76 = 4$

$10 \times 4 = 40 - 38 = 2$

$10 \times 2 = 20 - 19 = 1$

Thus by considering the remainder 10 as Geometrical ratio between the successive remainder we get all the other remainders. One can consider any remainder as Geometrical ratio to obtain the set of the remainder.

Now let us consider the second remainder '5' as geometrical ratio. Then multiply each preceding group of 2 remainders ($\because$ 5 is the second remainder) by 5 and cast out 19's from it to get the next remainders.

$10 \times 5 = 50 - 38 = 12$

$5 \times 5 = 25 - 19 = 6$

$\therefore$ The next set of 2 remainders 12, 6 is attained.

Similarly	$12 \times 5 = 60 - 57 = 3$ $6 \times 5 = 30 - 19 = 11$	3, 11
	$3 \times 5 = 15$ $11 \times 5 = 55 - 38 = 17$	15, 17
	$15 \times 5 = 75 - 57 = 18$ $17 \times 5 = 85 - 76 = 9$	18, 9
	$18 \times 5 = 90 - 76 = 14$ $9 \times 5 = 45 - 38 = 7$	14, 7
	-- --	13, 16
	-- --	8, 4
	$8 \times 5 = 40 - 38 = 2$ $4 \times 5 = 20 - 19 = 1$	2, 1

Thus the remainder set is obtained.

The remainders are multiplied with the 'Ceramankas' of the denominators in the case of 3 or 7 ending and take the last digits of the results to get the quotient digits, but in case of 9 ending the ceramanka of remainders themselves are the quotient digits.
Since 19 is a 9 ending denominator the quotient digits are the last digits.
$9\underline{0}, 4\underline{5}, 1\underline{2}, \underline{6}, \underline{3}, 1\underline{1}, 1\underline{5}, 1\underline{7}, 1\underline{8}, \underline{9}, 1\underline{4}, \underline{7}, 1\underline{3}, 1\underline{6}, \underline{8}, \underline{4}, \underline{2}, \underline{1}$

$$\frac{1}{19} = 0.\dot{0}5263157894736842\dot{1}$$

Now consider $\frac{1}{39}$

We know that the remainders are 10, 22,25,16,4,1

Let us consider first remainder 10 to be the geometric ratio

$$10 \times 10 = 100 - 78 = 22$$
$$22 \times 10 = 220 - 195 = 25$$
$$25 \times 10 = 250 - 234 = 16$$
$$16 \times 10 = 160 - 156 = 4$$
$$4 \times 10 = 40 - 39 = 1$$

$\therefore$ The remainders are 10, 22,25,16,4,1.

$$\frac{1}{39} = 0.\dot{0}2564\dot{1}$$

Hence the remainders when considering the last digit we get the quotient digits.

In the G.P. method one can simply write down the remainders when once the first remainder is known. This set is achieved

(i) By simply multiplying the first remainder by itself and

(ii) Casting the n times denominators ($n \in z^+$) from it to arrive at a value less than the denominator – this is the successive remainder.

(iii) Continuing this procedure with the successive remainders till you arrive at '1' or the numerator.

Examples:

7 ending example:

$\frac{1}{37}$ First remainder is 10; multiplied by itself

$10 \times 10 = 100 - 74 = 26 \rightarrow 2^{nd}$ remainder
$26 \times 10 = 260 - 259 = 1 \rightarrow 3^{rd}$ remainder

$\therefore$ The remainder set is

$$\begin{array}{r} 10,\ 26,\ 1 \\ \times \qquad\quad 7 \\ \hline 7\underline{0},\ 18\underline{2},\ \underline{7} \end{array}$$

$\therefore \frac{1}{37} = 0.\dot{0}2\dot{7}$

1 Ending example:

$\frac{1}{31}$

First remainder = 10

$10 \times 10 = 100 - 93 = 7 \times 9 = 6\underline{3}$
$7 \times 10 = 70 - 62 = 8 \times 9 = 7\underline{2}$
$8 \times 10 = 80 - 62 = 18 \times 9 = 16\underline{2}$
$18 \times 10 = 180 - 155 = 25 \times 9 = 22\underline{5}$
$25 \times 10 = 250 - 248 = 2 \times 9 = 1\underline{8}$
$2 \times 10 = 20 - 0 = 20 \times 9 = 18\underline{0}$
$20 \times 10 = 200 - 186 = 14 \times 9 = 12\underline{6}$
$14 \times 10 = 140 - 124 = 16 \times 9 = 14\underline{4}$
$16 \times 10 = 160 - 155 = 5 \times 9 = 4\underline{5}$
$5 \times 10 = 50 - 31 = 19 \times 9 = 17\underline{1}$
$19 \times 10 = 190 - 186 = 4 \times 9 = 3\underline{6}$
$4 \times 10 = 40 - 31 = 9 \times 9 = 8\underline{1}$
$9 \times 10 = 90 - 62 = 28 \times 9 = 25\underline{2}$
$28 \times 10 = 280 - 279 = 1 \times 9 = \underline{9}$

$\therefore \frac{1}{31} = 0.\dot{0}3225806451612\dot{9}$

3 ending example:

$\frac{1}{23}$ We know that the first remainder is 10 and adopting the above method we get the set of remainders as 10, 8, 11, 18, 19, 6, 14, 2, 20, 16, 22, 13, 15, 12, 5, 4, 17, 9, 21, 3, 7, 1.
Now multiply them with 3 and consider the last digits.
i.e. 30, 24, 33, 54, 57, 18, 42, 6, 60, 48, 66, 39, 45, 36, 15, 12, 51, 27, 63, 9, 21, 3.

$\therefore \frac{1}{23} = 0.04347826086 / 95652173913.$

There are certain other methods which are based on very interesting principles, and characteristic features of the conversion of vulgar fractions into decimals with regards to the remainders and the quotients.
The significant points are:
With reference to the remainder when D – N comes as a remainder all the other remainders are complements from the divisors i.e. the denominator.
This means that the quotient digits already obtained and quotients to be obtained are complementary from 9.
Considering any remainder and multiplying it with the Ceramanka of denominator then the last digit of the product is actually the quotient at that step. (This is not applicable in case of a division by a number ending with 1,9)(But applicable for 3 and 7)

For example $\frac{1}{29}$

The remainders are

10 13 14 24 8 22 17 25 18 6 2 0 26 28
19 16 15 5 21 7 12 4 11 23 27 29 3 1 Complements of remainder from divisor.
29 29 29 29 29 29 29 29 29 29 29 29

$\frac{1}{29} = 0.0\ 3\ 4\ 4\ 8\ 2\ 7\ 5\ 8\ 6\ 2\ 0\ 6\ 8 \rightarrow \frac{D-N}{2} = 14$ digits

9 6 5 5 1 7 2 4 1 3 7 9 3 1

It is interesting to note that the quotients are the last digits of the remainder.
(If there is only 1 digit in remainder then that itself is the quotient.)

To arrive at the remaining 14 digits consider the complements from 9 of the set to count the final result.

For example consider $\frac{1}{27}$

The remainders are 10,19,1 when multiplied by 7 we get 70, 133, and 7 leaving the left side digits (taking only Ceramankas) we have

$\frac{1}{27} = 0.\dot{0}3\dot{7}$

Similarly $\frac{1}{7}$, $\frac{1}{17}$ and $\frac{1}{27}$ and so on.
This will also give the confirmation of the result.

(iii) Ekadhika sutra (Process):

One can get an idea of last digit of the recurring decimal as 1,3,7,9 when the denominator of the vulgar fraction ends in 9, 3, 7 and 1 respectively, (if 9 should be the result of multiplying the last digit of the denominator with the expected last digit of the recurring decimal).
From this rule one knows the last digit of the recurring decimal before hand.
Applying ekadhikena sutram i.e. one in excess to the digit or digits preceding the last digit of denominator.
For example:

Number	Ekadhikena
19	2
209	21
1519	152 .... and so on.

In the case of any other ending such as 3, 7 Ekadhika sutram is applicable when one converts the vulgar fraction to its equivalent with 9 ending in the denominator.

For example:
$\frac{1}{23} = \frac{3}{69} = 3 \times \frac{1}{69}$

In this case the recurring decimal ends with 1.
$\frac{3}{69} = 3 \times \frac{1}{69}$ applying Ekadhika for $\frac{1}{69}$ and multiplying with 3. One gets the final result
Ekadhikena L $\rightarrow$ R (by division)
$\frac{1}{29} = 3 \rightarrow$ here is Ekadhika

Here we divide numerator 1 with 3 then 0 will be the quotient and 1 will be the remainder and considering 1 with 0 as 10 as dividend and dividing it with 3 we get 3 as quotient and 1 as remainder, again considering is with zero as dividend and the process is continued till you get repetitions

 0 3 4 4 8 2 7 5 8 6 2 0 6 8 9 6 5 5 1 7 2 4 1 3 7 9 3 1
1 1 1 2 0 2 1 2 1 0 0 2 2 2 1 1 1 0 2 0 1 0 1 2 2 0 0

$\therefore \frac{1}{29} = 0.0344827586206896551724137931$

We can also have complements from 9 as soon as D-N turns up

i.e. $\frac{1}{29} = .0\,3\,4\,4\,8\,2\,7\,5\,8\,6\,2\,0\,6\,8$
$\quad\quad 9\,6\,5\,5\,1\,7\,2\,4\,1\,3\,7\,9\,3\,1$

Numbers ending in 9 :

$\frac{1}{9} = 0.\dot{1}$

$\frac{1}{19} = \cdot\dot{0}\ 5\ 2\ 6\ 3\ 1\ 5\ 7\ 8$ It has18 digit, with complements of 9 from 9^{th} digit onwards

$9\ 4\ 7\ 3\ 6\ 8\ 4\ 2\ \dot{1}$

$\frac{1}{29} = \cdot\dot{0}3\ 4\ 4\ 8\ 2\ 7\ 5\ 8\ 6\ 2\ 0\ 6\ 8$ It has 28 digits, with complements of 9 from 14^{th} digit onwards

$9\ 6\ 5\ 5\ 1\ 7\ 2\ 4\ 1\ 3\ 7\ 9\ 3\ \dot{1}$

$\frac{1}{39} = \cdot\dot{0}25/64\dot{1} = \frac{1}{3\times13}$ factorable and satisfies $\frac{1}{13}$ rule

$\frac{1}{13} = \cdot\dot{0}7692\dot{3},\ \frac{1}{39} = \frac{0.076923}{3} = 0\cdot\dot{0}2564\dot{1}$

$\frac{1}{49} = \cdot\dot{0}2\ 0\ 4\ 0\ 8\ 1\ 6\ 3\ 2\ 6\ 5\ 3\ 0\ 6\ 1\ 2\ 2\ 4\ 4\ 8$ $\frac{1}{49} = \frac{1}{7\times7}$ factorsable

$9\ 7\ 9\ 5\ 9\ 1\ 8\ 3\ 6\ 7\ 3\ 4\ 6\ 9\ 3\ 8\ 7\ 7\ 5\ 5\ \dot{1}$

When $\frac{1}{7} = 0\cdot\dot{1}4285\dot{7}$ is divided by 7 then the recurring nature appears after repeated division of $\frac{1}{7}$, for 7 times i.e. 49 - 7 = 42 is the no. of digits complements of 9 occur from 21^{st} digit onwards.

$\frac{1}{59} = .\dot{0}1\ 6\ 9\ 4\ 9\ 1\ 5\ 2\ 5\ 4\ 2\ 3\ 7\ 2\ 8\ 8\ 1\ 3\ 5\ 5\ 9\ 3\ 2\ 2\ 0\ 3\ 3\ 8$

$9\ 8\ 3\ 0\ 5\ 0\ 8\ 4\ 7\ 4\ 5\ 7\ 6\ 2\ 7\ 1\ 1\ 8\ 6\ 4\ 4\ 0\ 6\ 7\ 7\ 9\ 6\ 6\ \dot{1}$

58 digits complements of 9 occur from 29^{th} digit onwards.

$\frac{1}{69} = \cdot\dot{0}\ 144927536231884057971$

$\frac{1}{3}\times\frac{1}{23}$ Satisfies the rule of $\frac{1}{23}$

$\frac{1}{23} = 0.0434282608695657173913$ on division by 3. We will get the recurring decimal of $\frac{1}{69}$ having 22 digits without complements.

$\frac{1}{79} = \cdot\dot{0}\ 12658227848\dot{1}$

Ekadhikena is no.of digits is not equal to 79 - 1. Sub multiple of 78 i.e. 13(factors)

<u>Numbers ending in 7 :</u>

1) $1/7 = 0.\dot{1}42 / 85\dot{7}$
   Ekhadhikena applicable.
   D-N = 6 No. of digits.
   Complements of 9 from D-N/2 (3) digits.

2) $1/17 = 0.\dot{0}5882352 / 9411764\dot{7}$
   Ekhadhikena applicable.
   D-N = 16 No. of digits.
   Complements of 9 from D-N/2 (8) digits.

3) $1/27 = 0.\dot{0}3\dot{7}$ SP. CASE $\because$ 27x 37 = 999
   No. of digits = 3.
   Ekhadhikena not applicable.
   It has factors 9x3.
   $1/9 \times 1/3 = 0.\dot{1} / 3 = 0.\dot{0}3\dot{7}$
   $1/3 \times 1/9 = 0.\dot{3} / 9 = 0.\dot{0}3\dot{7}$

4) $1/37 = 0.\dot{0}2\dot{7}$ SP. CASE $\because$ 37x 27 = 999
   No. of digits = 3.
   Ekhadhikena not applicable.

5) 1/47 = 7/329 = 1/329 x 7
   It has 46 digits D-N.

```
   7 1 1 2 4 6 2 0 0 6 0 7 9 0 2 7 3 5 5 6 2 3 1
 3  4  8  15 20  6 - 2  20 2  26  29 - 9 24  11  18 18  20  7 10  3
                                                              X 7
 ------------------------------------------------------------------
 9 8 7 2 3 4   0 4   2 5 5 3 1   9 1 4   8   9 3 6 1 7
 0 1 2 7 6 5   9 5   7 4 4 6 8   0 8 5   1   0 6 3 8 2
 ------------------------------------------------------------------
 9 9 9 9 9 9   9 9   9 9 9 9 9   9 9 9   9   9 9 9 9 9
```

   Ekhadhikena applicable. After converting it as 9ending.
   Complements are seen from 23 digits (D-N/2 Digits)

6) 1/57 = 1/19x3
   It has 18 digits (D-N).
   In acc with the factor 19.
   Complements are seen from D-N/2 digits.

7) 1/67 67-1/2 = 66/2 = 33.
   It has (D-N/2) digits i.e., 33 digits.
   No complements as it has odd digits.

8) 1/87 = 1/29x3
$\therefore$ It has (29-1) 28 digits in acc to the factor 29.
Complements occur from 14 digits.
Ekhadhikena applicable after conversion.

9) 1/97 =
Has 96 digits.
Complements from 43 degits.
Ekhadhikena applicable after conversion.

Numbers ending in 3 :

1) 1/13 = 3/39 = 3 x1/39
Has no factors.
$1/39 = .\dot{0}2564\dot{1}$
1221
$3/39 = 3 \times .\dot{0}2564\dot{1} = 0.\dot{0}7692\dot{3}$
$\therefore$ 1/13 = 0.076/923
No. of digits = (D-N/2) = 12/2 = 6 DIGITS.
Complements from 9 are obtained from D-N/4 = 3^{rd} digit onwards.

2) 1/23 = 3/69 = 3 x 1/69
No. of digits is 22 (D-N).
Complements from 9 are seen from
It has no factors.
D-N/2 = 11^{th} digits onwards.

0 14 4 9 2 7 5 3 6 2 3 1 8 8 4 0 5 7 9 7 1
1 3 3 6 1 5 3 2 4 1 2 1 6 5 2 -4 5 6 4 = 1/69

1/69 x 3 = 3/69 =1/29

$.\dot{0}4347826086 / 9565217391\dot{3}$

3) 1/43 = 3/129. It has no factors.
1/129 x 3

1/129 = 2 / 3 9 5 1 9 3 7 9 8 4 4 9 6 1 2 4 0 3 1
4 3 / 12 6 4 12 4 10 12 10 5 6 12 7 1 3 5 4 1

1/129 = 0.007751937984496124031

1/129 x 3
1/43 = 0.023255813953488372093

1/43 have no 21 digits.
(D-N/2) digits.
No complements since it have odd no. of digits.
2 times the recurring gives D-N = 42.

4) 1/53 = 3/159. Has no factors.
Ekadhikena is no of digits is not equal to 53-1,submultiple of 52 is13 (factor)
1/159 = 0.0062893081761
1/159 x 3 = 0.0188679245283
$\therefore$ 1/53 = 0.0188679245283
Has 13 digits.
(D-N/2) digits.
No complements since odd.
4 times the recurring is D-N=52.

5) 1/63 has factors. 21 x 3; 7 x 9
$\therefore$ 1/21 x 1/3 = 1/63
1/7 x 1/9 = $0.\dot{1}428\dot{7}$ / 9 = $0.\dot{0}1587\dot{3}$
1/9 x 1/7 = $0.\dot{1}$ / 7 = $0.\dot{0}15873$
1/63 = $0.\dot{0}1587\dot{3}$
No. of digits = 6
In accordance with the factor '7'.
The complements are seen from 8.

6) 1/73 = It is a special case.

7) 1/83 = 3/249. It has 41 digits.
Eka-25.
Without complements.
1/249 x 3 => D-N/2 = No. of digits.

7 7 5 1 0 0 0 4 0 1 6 0 6 4 2 5 7 0 2 8 1
18 12 2 0 1 10 0 4 15 1 16 10 6 14 17 - 7 20 2 3
1 2 4 4 9 7 9 9 1 9 6 7 8 7 1 4 8 5 9 4 3 7 7 5 1
6 11 12 24 19 24 22 4 24 16 19 21 17 3 12 21 14 23 10 9 19 18 12 2

<u>Numbers ending in 1</u>

1) $1/11 = 0.\dot{0}\dot{9}$
   Has 2 digits.

2) $1/21 = 1/7 \times 1/3 = 0.\dot{1}4285\dot{7}/3 = 0.\dot{0}4761\dot{9}$
   $1/3 \times 1/7 = 0.\dot{3}/7 = 0.\dot{0}4761\dot{9}$
   Have 6 digits in accordance with its factor 7.

3) $1/31 = 0.\dot{0}3225806451612\dot{9}$ this comes under sub multiple rule.
   D-N = 30. 2 X15 = 30

4) 1/41 Special case as 41 x 271 = 11111 Sub multiple rule.
   $1/41 = 0.\dot{0}243\dot{9}$ 5 x 8 = 40

5) 1/51 = 1/17 x 3
   It has 16 digits in accordance with its factor 17.
   1/17 = .0588235294117647
   (1/17) 3 = .0196078431372549

6) 1/61 = No factors.
   Has 60 digits (D-N)
   Complements are present from 30 (D-N).

7) 1/71 Sub multiple rule.
   Has 35 digits D-N/2 digits.
   Complements absent. ($\because$ Odd 35)

8) $1/81 = 0.037/3 = 0.\dot{0}1234567\dot{9}$
   $= 1/3 \times 1/27 = 0.\dot{3}/27 = 0.\dot{0}1234567\dot{9}$
   It has 9 digits.
   No complements.

9) 1/91 = 1/13 x 7
   => $1/13 \times 1/7$ => $0.\dot{0}7692\dot{3}/7 = 0.\dot{0}10/98\dot{9}$
   => $1/7 \times 1/13$ => $0.\dot{1}4285\dot{7}/13 = 0.\dot{0}10/98\dot{9}$

SPECIAL CASES

1) 1/27

27 X 37 = 999.

27 = 999/37

1/27 = 37/999

=> 1/27 = $0.\dot{0}3\dot{7}$

2) 1/37

37x27 = 999

37 = 999/27

1/37 = $0.\dot{0}2\dot{7}$.

3) 1/41

41 x 271 = 1111

∴ 1/41 = 271/11111 = 271 x 9/99999 = $0.\dot{0}243\dot{9}$

Conversely 1/271 = 41/11111 = 41 x 9/99999 = 00369/99999 = $0.\dot{0}36\dot{9}$

(Or) it can be worked out as

1/41 = 9/369 = 9 x 1/369 => 7 2 0
7 3 32

1 0 0 2 7 1
1 10 26 3

.00271 x 9 = - .02439

4) 1 /73 = 73 X 137 = 10001

10001 X 9999 = 99999999 ∴ 1/73 = 137/10001 = 137 X 9999/99999999

$0.\dot{0}136/986\dot{3}$

(or)

it can be worked out as

1/73 = 3/219 = 3 x 1/219

1 / 0 0 4 5 6 6 2 1
/ 1 10 12 14 13 4 2

.00456621 x 3

= 0.01369863

5) 1/91

Ekhadhanikani.

COMPARISION

Ekhanyuna sutra (1/73) ∵137 x 73 = 10001 10001 x 9999 = 99999999 1/73 = 137/10001 = 137 x 9999/99999999 = 0.0136/9863 Complements from 9.	Ekadhika sutra 1/73 = 3/219 = 3 x 1/219 Apply ekhadhanikena rule to 1/219 1/219 = 0.0045 / 0 0 4 5 6 6 2 1 6621 / 1 10 12 14 13 4 2 1/219 x 3 = 0.00456621 x 3 = $0.\dot{0}136986\dot{3}$
1/41 ∵41 x 271 = 11111 11111 x 9 = 99999 ∴1/41 = 271/11111 = 271 x 9/99999 = $0.\dot{0}243\dot{9}$	1/41 = 9/369 = 9 x 1/369 Apply ekhadhanikena rule to 1/369 1/369 = $.\dot{0}02\dot{7}\dot{1}$ 1 10 26 3 9 x 1/369 = 9x$0.\dot{0}027\dot{1}$ = $0.\dot{0}243\dot{9}$
1/27 and 1/37 27 x 37 = 999 1/27 = 37/999 = $0.\dot{0}3\dot{7}$ 1/37 = 27/999 = $0.\dot{0}2\dot{7}$	

<u>OTHER ENDINGS</u> (With out factors in D)

Now consider the fractions ending with 3 in D's but having no factors.

For example $\frac{1}{3}, \frac{1}{13}, \frac{1}{23}, \frac{1}{43}, \frac{1}{53}, \frac{1}{73}, \frac{1}{83}$ - - - -

$\frac{1}{3} = 0.\dot{3}$

In case of $\frac{1}{13}$ the number of digits in recurring decimal is 6 also equal to $\frac{D-N}{2}$ and complements occur from $\frac{D-N}{4}$. The answer is in two halves one set is complementary to the other from 9. The end digit is 3. Ekadhika sutra is not applicable to it.

$\frac{1}{13} = \cdot\dot{0}76/92\dot{3}$

In order to apply Ekadhikena one has to write $\frac{1}{13}$ as $\frac{3}{39}$ for which the Ekadhika is 4 on application of Ekadhikena to $\frac{1}{39}$ get its recurring decimal value which ends with '1'. We write down

$\frac{1}{39} = .0\ 2\ 5\ 6\ 4\ 1$

2 2 1

Now $\frac{3}{39} = 3 \times \frac{1}{39} = 3 \times 0.025641 = \cdot\dot{0}7692\dot{3}$

$\therefore \frac{1}{13} = \cdot\dot{0}7692\dot{3}$

In case $\frac{1}{23}$ now again convert it as $\frac{3}{69}$

Applying Ekadhika to $\frac{1}{69}$ and multiply it with 3 to get the value of $\frac{3}{69}$ which is nothing but the value of $\frac{1}{23}$

Now applying Ekadhika sutram to $\frac{3}{69}$ whose Ekadhika is 7. We get 22 digits of recurring decimal.

$\frac{1}{69} = .0\ 1\ 4\ 4\ 9\ 2\ 7\ 5\ 3\ 6\ 2\ 3\ 1\ 8\ 8\ 4\ 0\ 5\ 7\ 9\ 7\ 1$

1 3 3 6 1 5 3 2 4 1 2 1 6 5 2 4 5 6 4

$\frac{3}{69}$ = it ends in '3' since it has factors 3 x 13

$\therefore \frac{1}{69} = 3 \times \frac{1}{69}$

One can stop when they arrive at $\frac{D-N}{2}$ of $\frac{1}{23}$ i.e. $\frac{22}{2} = 11$ digits and can write complements from 9 to get the full set of recurring.

$\frac{3}{69} = .0\ 4\ 3\ 4\ 7\ 8\ 2\ 6\ 0\ 8\ 6 \rightarrow$ complement set

$9\ 5\ 6\ 5\ 2\ 1\ 7\ 3\ 9\ 1\ 3 \rightarrow$ derived set

$\therefore \frac{3}{69} = \cdot\dot{0}4347826086 / 9565217391\dot{3}$

$\frac{3}{69} = 0.04347826086/95652173913$

$\therefore \frac{1}{23} = 0.04347826086/95652173913$

It has D-N (22) digits and it has complements from 11 digits i.e. $\frac{D-N}{2}$

Now $\frac{1}{43}$ This is converted to $\frac{3}{129}$ and Ekadhika sutram is applied. The Ekadhika is 13 and the last digit of the recurring decimal is 3.

$\therefore \frac{3}{129} = \cdot\dot{0}\ 2\ 3\ 2\ 5\ 5\ 8\ 1\ 3\ 9\ 5\ 3\ 4\ 8\ 8\ 3\ 7\ 2\ 0\ 9\ \dot{3}$

$3\ 4\ 3\ 7\ 7\ 10\ 15\ 12\ 6\ 4\ 6\ 11\ 10\ 4\ 9\ 2\ 1\ 12\ 3$

$\therefore \frac{1}{43}$ Have 21 digits as it is odd number and also a $2 \times 21 = 22$

Which is $\frac{D-N}{2} = \frac{42}{2} = 21$

Has no complement.

Now $\frac{1}{53}$ this is also as the above case converting into $\frac{3}{159}$ and applying Ekadhika sutram whose ekadhikena is 16. Therefore we get $\frac{1}{53}$. But the no. of digits is 13.

$\frac{1}{53} = \frac{3}{159} = \cdot\dot{0}\ 1\ 8\ 8\ 6\ 7\ 9\ 2\ 4\ 5\ 2\ 8\ \dot{3}$

$3\ 14\ 13\ 10\ 12\ 14\ 3\ 7\ 8\ 4\ 13\ 4$

D-N = 52 and 4 times of 13 gives us 52

No complement as it is an odd number.

$\frac{1}{83}$ Can be done similarly and so on _ _ _

Vulgar fractions having factors in the denominator.

Consider $\frac{1}{39}$

$\frac{1}{39} = \frac{1}{3 \times 13}, \frac{1}{49} = \frac{1}{7 \times 7}, \frac{1}{69} = \frac{1}{23 \times 3}, \frac{1}{99} = \frac{1}{9 \times 11}$ etc.

$\frac{1}{39} = \frac{1}{13} \times \frac{1}{3}$ here we consider $\frac{1}{13}$ value and we divide the recurring decimal of $\frac{1}{13}$ by 3.

i.e. $\frac{0.076923}{3} = 0.025641$

$\frac{1}{39}$ Has the same no. of digits as $\frac{1}{13}$.

$\frac{1}{39}$ can also be obtained by applying ekadhikena sutram but the no. of digits in the recurring is in accordance with the method derived as above in case of factors.

$\frac{1}{49} = \frac{1}{7} \times \frac{1}{7}$

i) The total no. of digits in the recurring decimal is not 48 but (49-7) = 42

ii) For $\frac{1}{49}$ ekadhikena is applicable but the complements occur from 21 digits.

This comes under the case of similar factors.

Here we can add $\frac{1}{169} = \frac{1}{13 \times 13}$ is a similar case to the above.

It has $\frac{169-13}{2} = \frac{156}{2} = 78$ total no. of digits in recurring and after 39 digits complements from 9 can be observed.

One can be observes a similar pattern in case of $\frac{1}{289} = \frac{1}{17 \times 17}$

it has 272 total no. of digits and complements occur from 136 digits and so on_ _ _

Now $\frac{1}{69} = \frac{1}{3 \times 23}$ is obtained by taking the recurring decimal as $\frac{1}{23}$ and divide it by 3.

$\frac{1}{69} = \frac{1}{23 \times 3} = 0.04347826086/95652173913 \times \frac{1}{3}$

$\frac{1}{69} = 0.0144927536231884057971$

$\therefore$ It has 22 digits as accordance with its factor 23.

Now the case of 7 denominators with factors

$\frac{1}{27} = \frac{1}{3 \times 9}; \frac{1}{57} = \frac{1}{3 \times 19}$

$\frac{1}{27} = \frac{1}{3} \times \frac{1}{9} \Rightarrow \frac{0.\dot{3}}{9} = 0.\dot{0}3\dot{7}$ or $\frac{1}{9} \times \frac{1}{3} \Rightarrow \frac{0.\dot{1}}{3} = 0.\dot{0}3\dot{7}$

$\frac{1}{27}$ is a special case where the no. of recurring digits is 3.

Refer under special case section.

$\frac{1}{57} = \frac{1}{3 \times 19}$ Here we consider the value of $\frac{1}{19}$ and dividing it by 3 to get the value of $\frac{1}{57}$.

$\therefore \frac{1}{57} = \frac{1}{19} \times \frac{1}{3} \Rightarrow \frac{0.052631578947368421}{3}$

$\frac{1}{57} = 0.017543859649122807$

The no. of digits in the recurring is 18. In accordance with the factor 19.

The Code Language at work:

Swamiji explains the part of the unlocking of a few specimens in which the code in conjunction with ekanuena sutra can be utilized, for the purpose of postulating mental one-line answers. Infact the Vedic sutras which explain rapid process of mental arithmetic have been already demonstrated by Swamiji. But still the master key for unlocking the other portals have been illustrated only in a few cases. For example, " Kevalaiah Saptakam Gunyat where Saptakam means seven.

From the Vedic numerical code (given in appendix as taken from Swamiji book under the chapter of Vedic numerical code).Kevaliah represents 143. In case of kevaliah saptakam, for 7, multiplicand should be 143.

7 x 143 gives 1001

i.e 7 x 143 = 1001

Now this 1001 can be written as equal to $\frac{999999}{999}$

$$\Rightarrow 7 \times 143 = \frac{999999}{999}$$

Now $\quad \frac{1}{7} = \frac{143 \times 999}{999999}$

Applying Ekanyunena sutram it becomes $\frac{142/857}{999999}$

This is simply .142857

While applying ekanyunena sutram, considering 143 x 999 one can immediately write the answer as 142/857

Similarly another sutram read as kalau ksudrasasaiah from the code given in appendix kalau stands for 13 and ksudrasasiah represents 077. The application of ekanyunena. In case of 13, 077x999 = 076 923 i.e. on application of ekanyunena 077 becomes 076 when multiplied with 999 the answer is 076923.& 923 being complements of 076 from 9's.

There are 6 decimal digits in the case of $\frac{1}{13}$.

13 x 077 gives 1001 => 13 x 1007 = 13091

and 1001 can be written as $\frac{999999}{999}$

$$\Rightarrow 13 \times 077 = 1001 = \frac{999999}{999}$$

$$\frac{1}{13} = \frac{077 \times 999}{999999}; \ \frac{1}{13} = \frac{076923}{999999}$$

$$\frac{1}{13} = 0.076923$$

The third case where the Vedic sutram kamse ksamadaha-khalairmaliah kamse. Kamse as read from the code = 17 and means 05882353, this multiplied by 99999999 can be worked out by applying ekanyunena sutram as the first part of the answer and = 05882352/941117647 the results can also be represented as

17 x 05882353

17 x ———— gives 10000 ---1 => 17x05882353 = 100000001 and 1000 ------ 01 can be written as

$$\frac{1}{17} = \frac{05882353 \times 99999999}{9999999999999999}$$

$$17 \times 0588235294117647 = 9999999999999999$$

$$\frac{1}{17} = 0.0588235294117647$$

Remainder quotient complements cycle:

This deals with rules governing remainders.

Let us consider $\frac{1}{7}$ the successive remainders are 3, 2, 6, 4, 5, 1 on reaching D-N i.e. 6 then half of the work is completed the second half are nothing but complements from 7 of the first half.

i.e.

$$\begin{array}{ccc} 3 & 2 & 6 \\ \underline{4} & \underline{5} & \underline{1} \\ 7 & 7 & 7 \end{array} \quad \begin{array}{l} \rightarrow \text{first part} \\ \rightarrow \text{second part (Complements)} \\ \end{array}$$

One can try for $\frac{1}{17}$

The successive remainders are:

$$\begin{array}{cccccccc} 10 & 15 & 14 & 4 & 6 & 9 & 5 & 16/7 \quad 2 \quad 3' \quad 13 \quad 11 \quad 8 \quad 12 \quad 1 \\ 7 & 2 & 3 & 13 & 11 & 8 & 12 & 1 \\ \hline 17 & 17 & 17 & 17 & 17 & 17 & 17 & 17 \end{array}$$

The last 8 remainders are thus complements – from 17 – of the first eight ones.

Now let us try for $\frac{1}{13}$ the remainders are 10, 9, and 12 i.e. (D-N). Hence the rest of remainder will be 3, 4, 1.

$$\begin{array}{cccl} 10 & 9 & 12 & \text{--f.r} \\ 3 & 4 & 1 & \text{--s.p ©} \\ \hline 13 & 13 & 13 & \end{array}$$

This is also applicable in case of 19.the remainders are

$$\begin{array}{ccccccccc} 10 & 5 & 12 & 6 & 3 & 11 & 15 & 17 & 18 \\ 9 & 14 & 7 & 13 & 16 & 8 & 4 & 2 & 1 \\ \hline 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 \end{array}$$

Here again the first nine remainders, when added successively to the next nine, give 19 each time.

We observe that in every case the remainders – halves are complements from the individual divisor and it is also clear that quotient- halves are complements from 9. Infact this reduces our working time for remainders.

1) Now in the case of 7 ending denominators with no factors. Regarding quotients the recurring decimal ends with 7. The division shows. (no. of digits is D-N=6)

7) 10 (0.1 4 2 8 5 7 $\qquad \frac{1}{7} = 0.142857$

$$\begin{array}{r} \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 1 \end{array}$$

The D-N value is 6. Hence there are 6 digits. After $\frac{D-N}{2}$ digits are covered one can stop and take the complements from 9.

For $\frac{1}{17}$ Ekadhika rule is not applicable. But in order to apply Ekadhika, the vulgar fraction must be converted to a fraction having a D ending in 9.

For example $\frac{1}{17} = \frac{1 \times 7}{7 \times 17} = \frac{7}{119}$

Here it can be written as $7 \times \frac{1}{119}$ and $\frac{1}{119}$ can be evaluated by Ekadhika sutra which is 12.

But we know that the recurring decimal of 1/17 has 16 digits and $\therefore$ one can evaluate $\frac{1}{119}$ only up to 16 digits by applying Ekadhika as12. This is invariably 1. Multiply the last 8 digits by 7 and then take complements of 9 to get the remaining 8. Thus the complete answer of $\frac{1}{17}$ is in the following steps. Apply ekadhikena for $\frac{1}{119}$ to get 16 digits.

a) $\frac{1}{119}$ = .7 2 2 6 8 9 0 7 5 6 3 0 2 5 2 1

b) Multiply the 8 digits of (a) by 7 from the last digit. $\qquad$ 9 4 1 1 7 6 4 7

c) Take complements of (b) from 9 to get the complete value of $\frac{1}{17}$

$\cdot\dot{0}5882352/9411764\dot{7}$

$\frac{1}{37}$ Is exception and is derived from code.

$\frac{1}{47}$ Has 46 digits after working for 23 digits one can get the remaining digits by writing the complements from 9.

$\frac{1}{67}$ Have only 33 digits in its recurring form as $\frac{D-N}{2}$ is odd.

$\frac{1}{97}$ Have 96 digits in its recurring form, so after working 48 digits one can write down complements from 9.

(VI) Multiples of basic fractions:

First method:

i) Writing down the multiples of vulgar fractions by considering the original fraction and numbering them in ascending order.

Let us consider $\frac{1}{7}$

$\frac{1}{7} = 0.142857$ now numbering the digits of the remaining decimal in ascending order

$$\frac{1}{7} = 0.\underset{(1)}{1}\ \underset{(3)}{4}\ \underset{(2)}{2}\ \underset{(6)}{8}\ \underset{(4)}{5}\ \underset{(5)}{7}$$

The Cycle for $\frac{1}{7}$ starts with 1, as 1 is first in numbering and then moves in clockwise for all the six digits. Next $\frac{2}{7}$

$\frac{2}{7} = 0.\dot{2}8571\dot{4}$

2 being second in numbering, $\frac{2}{7}$ starts with 2 and the similar method is adopted.

$\frac{3}{7} = 0.\dot{4}2857\dot{1}$

4 being the third in numbering, $\frac{3}{7}$ starts with 4 and the similar method is adopted.

$\frac{4}{7} = 0.\dot{5}7142\dot{8}$

5 being fourth in numbering, $\frac{4}{7}$ starts with 5.

$\frac{5}{7} = 0.\dot{7}1428\dot{5}$

7 being fifth in numbering, $\frac{5}{7}$ starts with 7.

$\frac{6}{7} = 0.\dot{8}5714\dot{2}$

8 being sixth in numbering, $\frac{6}{7}$ starts with 8.

Thus the cycle is performed.
Now in few cases, some digits are found more than once.
For example:

$\frac{1}{17} = 0.\dot{0}58823529411764\dot{7}$

$= 0.\dot{0}\ \ 5\ \ 8\ \ 8\ \ 2\ \ 3\ \ 5\ \ 2\ \ 9\ \ 4\ \ 1\ \ 1\ \ 7\ \ 6\ \ 4\ \ \dot{7}$

(1) (10) (15) (14) (4) (6) (9) (5) (16) (7) (2) (3) (13) (11) (8) (12)

Here we find the digits 1, 2, 4, 5, 7, 8 are repeating, and then one has to consider the very right digit of the repeated digits for ascending order and accordingly should be numbered. Consider one occurring twice in the result of recurring decimal of $\frac{1}{17}$ i.e. 11 and 17 then the numbering should be in accordance with the increasing value. The first preference should be given to the recurring number in a smaller value i.e. 1 from 11 should be numbered first and next 1 in 17, a similar procedure is adopted whenever the repetition occurs.

Thus $\frac{1}{17}$ starts with 0, and the rest of the digits are followed, as it is in the above case.

$\frac{2}{7}$ Can be written as starting with first 1 followed by 1, 7, 6.....etc. in cyclic order.

Thus $\frac{2}{17} = 0.\dot{1}17647058823529\dot{4}$

Similarly $\frac{3}{17}$ is 0.1764... in cyclic order. Thus one can easily write down $\frac{15}{17}$ as $\frac{15}{17}$ = 0,8823529411.... completing the cyclic order. (The values are written according to the numbering)

In cases where the number of possible multiples are much less than the digits in the decimal equivalent then the values are obtained by the simple multiplication. This is the case for $\frac{1}{13}$.

For example:

$\frac{1}{13} = 0.\dot{0}7692\dot{3}$

$\frac{2}{13}$ is obtained by multiplying $\frac{1}{13}$ with 2.

$\therefore \frac{2}{13} = 2 \times \frac{1}{13} = 2 \times 0.076923 = 0.153846$ and so on.

Second Method (Adyam Adyena Sutram):

This method removes the ambiguity met with the numbering method.

Consider $\frac{1}{17} = 0.\dot{0}58823529411764\dot{7}$

<u>First step:</u>

Multiply the opening digit or digits 0.0588 by the required numerator for example when we want the answer for $\frac{2}{17}$ i.e. $2 \times 0.0588 = 0.1176$.

$\therefore \frac{2}{17}$ Should start with 1176. Then one can locate by inspection in $\frac{1}{17}$ occurrence of points 1176 and then write down answer starting with 1176 in cyclic order.

$\frac{2}{17} = 0.\dot{1}17647058823529\dot{4}$.

Similarly $\frac{3}{17}$ starts $3 \times 0.0588 = 0.1764$ on locating this value in $\frac{1}{17}$ one can write down with 0.1764 as starting point and complete in cyclic order.

$\frac{3}{17} = 0.\dot{1}76470588235294\dot{1}$

Similarly one can write down from $\frac{1}{17}$ value, the value of $\frac{15}{17}$ as follows.

$\therefore 15 \times 0.0588 = 0.882$

On locating this value in $\frac{1}{17}$, the value for $\frac{15}{17}$ can be written down starting with 0.882… in cyclic order. $\frac{15}{17} = 0.\dot{8}82352941176470\dot{5}$

<u>Third method (Antyam Antyena sutram):</u>

This method makes use of the multiplication of the required multiple with the last digit of the denominator. Then one has to consider the last digit of the result and locate in the value of $\frac{1}{17}$ and then follow the cyclic order.

For example, to arrive the value of $\frac{5}{17}$ from $\frac{1}{17}$ one has to adopt the following procedure

$5 \times 7 = 35$ so $\frac{5}{17}$ should end with 5.

$\therefore \frac{5}{17} = 0.2941176470588235$

If the repetition of the digit occur then has to consider the phenomenon adopted in numbering the digits in ascending order as in the first method.

Independent Method:

When one wants to get $\frac{3}{7}$; this can be written as $\frac{3\times 7}{7\times 7}=\frac{21}{49}$, using Ekadhikena sutram for $\frac{21}{49}$ whose Ekadhik purna is 5 and adopting the division process one gets the required result. One can stop working either when (D – N) or the last digit of (D – N) is attained, from then Onwards the complements from 9 can be written.

Consider the case of $\frac{1}{7}$

We know that $\frac{1}{7}=0.\dot{1}4285\dot{7}$

$\frac{1}{7}=\frac{7}{49}$ $\Rightarrow$ Ekadhika purva is 5

Straight division by 5 gives 0. 1 4 2 / 8 5 7
2 1 4

We can stop here 42 has arrived (D – N = 49 – 7 = 42)
Hence take its complement.

$\frac{2}{7}=\frac{2\times 7}{7\times 7}=\frac{14}{49} \Rightarrow \frac{14}{5}$ $D-N=49-14=\underline{35}$

Straight division by 5 gives 0. 2 8 5 / 7 1 4
4 2 3

$\therefore \frac{2}{7}=0.\dot{2}8571\dot{4}$

$\frac{3}{7}=\frac{3\times 7}{7\times 7}=\frac{21}{49} \Rightarrow \frac{21}{5}$ $D-N=49-21=28$

Straight division by 5 gives 0. 4 2 8 / 5 7 1
1 4 2

$\therefore \frac{3}{7}=0.\dot{4}2857\dot{1}$

$\frac{4}{7}=\frac{4\times 7}{7\times 7}=\frac{28}{49} \Rightarrow \frac{28}{5}$ $D-N=49-28=21$

Straight division by 5 gives 0. 5 7 1 / 4 2 8
3 0 2

$\therefore \frac{4}{7}=0.\dot{5}7142\dot{8}$

$\frac{5}{7}=\frac{5\times 7}{7\times 7}=\frac{35}{49} \Rightarrow \frac{35}{5}$ $D-N=49-35=14$

Straight division by 5 gives 0. 7 1 4 / 2 8 5
0 2 1 $\therefore \frac{5}{7}=0.\dot{7}1428\dot{5}$

$\frac{6}{7}=\frac{6\times 7}{7\times 7}=\frac{42}{49} \Rightarrow \frac{42}{5}$ $D-N=49-42=7$

Straight division by 5 gives 0. 8 5 7 / 1 4 2
2 3 0 $\therefore \frac{6}{7}=0.\dot{8}5714\dot{2}$

Recapitulation (and Supplementation)

One can derive the same result by considering any G. P taken from the remainder set. While doing so one has to consider the problem as group wise set of remainders.

For example let us consider the problem 1/13 the answer for this is $0.\dot{0}7692\dot{3}$

The method of obtaining set of remainders on the basis of G. P among the remainders and also applying the Seshani Ankena Ceramena is as follows:

1/13 gives the first two remainders as 1 and 10 on the basis of G. P among the remainder one can consider 10 as G. P

With this as starting point we can derive the complete set of remainder.

Starting with 10 the next remainder is obtained 10 x 10 = 100 casting of multiples of 13 from 100.

100-91 = 9.

Next consider 9 and multiply by 10 (9 x 10 = 90) and cast out multiples of 13 i.e. (90 - 13 x 6 = 12) gives you 12.

This process is continued until the recurring nature appear

12 x 10 = 120 - 117 = 3
3 x 10 = 30 - 26 = 4
4 x 10 = 40 - 39 = 1

∴ The remainder set is 10, 9, 12, 3, 4, 1. From remainder set one obtains the quotient by multiplying each remainder with divisor or denominator and consider the ceramankas.

(Application of Seshani Ankena Ceramena) thus one gets

10 x 13 , 9 x 13, 12 x 13 , 3 x 13 , 4 x 13 , 1 x 13
13(0), 11(7), 15(6), 3(9), 5(2), 1(3)
$= 0.\dot{0}7692\dot{3}$

One can obtain the same result by considering any of the remainders as G.P from the set. The procedure adopted is grouping of remainders if the 2nd remainder 9 is considered as G.P group the two remainders as a single unit.

$$\begin{array}{rr} \boxed{10} & \boxed{9} \\ \times 9 & \times 9 \\ \hline 90 & 81 \\ -78 & -78 \\ \hline \boxed{12} & \boxed{3} \\ \times 9 & \times 9 \\ \hline 108 & 27 \\ -104 & -26 \\ \hline \boxed{4} & \boxed{1} \end{array}$$

The remainder set is 10 9 12 3 4 1 - same set as derived in case of G. P = 10

Let us consider the 3rd remainder 12 as G. P. Hence one has to consider or group the three remainders as a single unit.

$\boxed{10}$	$\boxed{9}$	$\boxed{12}$
x 12	x 12	x 12
120	108	144
- 117	- 104	143
$\boxed{3}$	$\boxed{4}$	$\boxed{1}$

The remainder set is 10, 9, 12, 3, 4, 1 which is again same as the previous set.

Let us consider the 4th remainder 3 as G. P. Hence one has to consider or group the 4remainders as a single unit.

$\boxed{10}$	$\boxed{9}$	$\boxed{12}$	$\boxed{3}$
x 3	x 3	x 3	x 3
30	27	36	9
- 26	26	- 26	0
$\boxed{4}$	$\boxed{1}$	10	9

The remainders set is 10, 9, 12, 3, 4, 1

Any repetition of remainders need not be considered as an additional result.

Consider 5th remainder 4 as G.P. Hence the group comprises as 5 remainders as a single unit.

$\boxed{10}$	$\boxed{9}$	$\boxed{12}$	$\boxed{3}$	$\boxed{4}$
x 4	x 4	x 4	x 4	x 4
40	36	48	12	16
- 39	-26	39	0	-13
1	10	9	12	3

Hence the remainders set is 10, 9, 12, 3, 4, 1.

Still another method:

Starting from first 2 remainders and starting with that as geometrical ratio one can simply write down the set of remainders.

For example, let us consider $\frac{1}{19}$ the first 2 remainders are 10, 5 giving geometrical ratios as 2 the remainder 2.5 being not an integer is modified by adding 19 to it and divides G.R 2, so that the third remainder becomes $\left(19+5=\frac{24}{2}=12\right)$. 12 this procedure is continued to get the set of all remainders. As soon as D-N is reached in the remainder set one can write down the rest of the remainders as complements from 19(denominator).

$$\begin{array}{ccccccccc} 10 & 5 & 12 & 6 & 3 & 11 & 15 & 17 & 18 \\ \underline{9} & \underline{14} & \underline{7} & \underline{13} & \underline{16} & \underline{8} & \underline{4} & \underline{2} & \underline{1} \\ \underline{19} & \underline{19} & \underline{19} & \underline{19} & \underline{19} & \underline{19} & \underline{19} & \underline{19} & \underline{19} \end{array}$$

Thus the remainder set obtained is

10 5 12 6 3 11 15 17 18 9 14 7 13 16 8 4 2 1

We multiply the ceramankas of the remainder by one. Then we get quotient digits automatically.

$1/19 = 0.\dot{0}\,5\,2\,6\,3\,1\,5\,7\,8\,9\,4\,7\,3\,6\,8\,4\,2\,\dot{1}$

The ratio is determined by considering not only the first two but any two of them,

that is at any stage of the work.

Now in a case where the ratio of the remainder is $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ ---- then the procedure is as follows.

Let us consider $\frac{1}{17}$ the first 2 remainders are 10 & 15. The G.R is $1\frac{1}{2}$ as such, one has to adopt the following procedure.

From the second remainder 15 how to write down the next? Whenever an odd number crops by, say for example 15. Its successor is $\frac{15+17}{2}=16$ this is added to 15+17 giving the result as 48 (15+17+16) Now as 48 is greater than the divisor 17, one can cast off 2 x 17 from 48 resulting in the remainder 14. Now 14 being even number and G.R being $1\frac{1}{2}$ the next value will be 14 + 7 = 21 from which the remainder is obtained after casting of 1 x 17 giving 4 as remainder -1. The next remainder is 4 + 2 which are 6. This process is continued in a similar manner bearing in mind the odd and even nature of remainder and also coupled with result whether greater than remainder or otherwise and after D-N turns up one get the rest of remainders as complements from 17.

Thus the remainder set of $\frac{1}{17}$ is

$$\begin{array}{cccccccc} 10 & 15 & 14 & 4 & 6 & 9 & 5 & 16 \\ 7 & 2 & 3 & 13 & 11 & 8 & 12 & 1 \\ \hline 17 & 17 & 17 & 17 & 17 & 17 & 17 & 17 \end{array}$$

Thus the remainder set obtained is 10, 15, 14, 4, 6, 9, 5, 16, 7, 2, 3, 13, 17, 8, 12, 1
Now multiplying the ceramankas of remainders with ceramankas of divisor 17 (i.e. 7). One gets the quotient set (considering the ceramankas of individual results)

$\therefore \frac{1}{17} = 0.\dot{0}58823529411764\dot{7}$

No. Of decimal places

The no. of decimal places which on division, the decimal equivalent of a vulgar fraction can be known in advance without working the full division.

With the help of successive remainders, we can work out all the forth coming remainders. (Reference methods (ii), (iii)) Application of geometrical progression and sutram Sesani Ankena Ceramena. This tabulation can be prepared successfully at any moment. The method can be summarized as follows,

i. As soon as one reaches the starting point for example $\frac{1}{7}, \frac{1}{13}, \frac{1}{17}, \frac{1}{19}$ etc. the no. of decimal places , remaining a hand can be easily known
ii.
iii. As soon as one reaches the difference between the denominator and numerator (D - N), One can stop working and complete the remaining by considering the complements of denominator.
iv. As soon as one reaches the fairly small and manageable. Remainder one can easily expect the remaining.

Consider $\frac{1}{7}$

The first remainder is 3, this is taken as the successive multiplier together with the provision for the casting out of sevens (7)

3, 2, 6 / 4, 5, 1 → this is the set of remainder as '1' is reached

(or)

Consider the remainders up to the value of D - N i.e. (7-1) which is 6 is reached as the remainder and hence one can stop at this stage and take the complements of 7 i.e. 4, 5, 1 as the rest of three successive remainders

(or)

From the consideration of quotients i.e. the first two quotients are 1, 4 at this stage the remainder is 2 multiplying the two quotient digits (1 4) as a group by 2. We get 2 8 as the second group of quotients. At this stage the remainder is 4 i.e. 2 x 2 i.e. two times the first remainder. Now furthering the process the group 28 is multiplied by 2 to get 56 as the third group following the above argument the remainder is 4 x 2 = 8 we get 8 > 7 hence cast of 7 we get 1, now we have to add this to the group 56 which results in 57 and the final remainder is 1.

So the answer for $\frac{1}{7}$ is $\cdot\dot{1}4285\dot{7}$.

Similarly we can work out for $\frac{1}{17}$

one has to consider the division to see until the first minimum remainder is obtained.

This is achieved after four decimals. Which are .0588 multiplying the quotient (.0588) by remainder 4 i.e. 0.0588 x 4 = .2352 which is the second quotient group. At this stage remainder is 4 x 4 i.e. 16. We can stop working as we reached $\frac{D-N}{2}$ and simply write down the remaining 8 as complements from 9. Hence $\frac{1}{17}$ = 0.058823594117647

It is noticed that the geometrical series concept 1, r, r^2_ _ _ _ is used in helping us to predetermine the no. of decimals of vulgar fraction thus the algebrical principle is utilized.

For example in case of $\frac{1}{17}$ the remainders are 4, 4^2_ _ _ _ $\frac{1}{7}$ 2, 2^2 ------

In summary we can say that Ekadhika sutra works more superior to other methods, any and every vulgar fraction can be tackled and converted to the corresponding recurring decimal and working out the decimal places in advance. Now these methods can be worked out for all possible denominators and numerator leading to the no. of decimals in any vulgar fraction.

Converse operation:

The converse process is the one in which the conversion of decimals into its vulgar fraction is worked out.

The working details of one example is given below this is based on

the proposition $.9 = \frac{9}{9} = 1$

Similarly $.\dot{9}\dot{9} = \frac{99}{99} = 1$ and

$.\dot{9}9\dot{9} = \frac{999}{999} = 1$ and add infitinum

Knowing a recurring decimal one should find out by what number the former is multiplied so that one gets all 9's as the result. The given recurring decimal is 0.076923

Step1:
This is multiplied first by 3

```
 0.076923
       x3
 0.230769
```

(From the point of view that the last digit of decimal is 3) and to get the result in all 9

Step2:
One should choose next the digit in such a way that, when the given decimal is multiplied by it, the last digit of the result when added to 6 again gives 9.
This procedure is continued until one gets the result all in 9's.

General explanation:
For example one can work out similarly

```
   4 6 1 2 3 2 6 2 1 5 4 3 4
 0.0 5 8 8 2 3 5 2 9 4 1 1 7 6 4 7̇
                            x 17
   4 1 1 7 6 4 7 0 5 8 8 2 3 5 2 9
   5 8 8 2 3 5 2 9 4 1 1 7 6 4 7
   9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9
```

Consider the recurring decimal
$0.\dot{0}58823529411764\dot{7}$. When this is multiplied by 7 we get the last digit of the result as 9. In order to convert 2 to 9 one should multiply the recurring decimal and place it underneath the digit 2. Thus the entire result is in all 9's.
Hence 0.0588235294117647
Consider another example
The recurring decimals.
0.0144927536231884057971. When this is multiplied by 7. We get the last digit of the result as 9. In order to convert to 9 one should multiply the recurring decimal and place it underneath the digit. Thus the result is in all 9's.

```
     4 4 8 2 6 4 5 5 2 2 1 7 7 3     7 8 6
 0.0̇1 4 4 9 2 7 5 3 6 2 3 1 8 8 4 0 5 7 9 7 1̇
                                        x 69
   1 3 0 4 3 4 7 8 2 6 0 8 6 9 5 6 5 2 1 7 3 9
   8 6 9 5 6 5 2 1 7 3 9 1 3 0 4 3 4 7 8 2 6 x
   9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9
```

$\therefore 0.0144927536231884057971 = \frac{1}{69}$

Significant points(some characteristic features):

Significant points noted in the evaluation of the recurring decimals in case of vulgar fractions $\left(\frac{1}{D}\right)$ where denominator is ending with 9.

1. Ekadhikena sutram is applicable in all cases.
2. The number of digits in the recurring decimal equivalent is (Denominator (D) - Numerator (N)).

i.e., (D - N) → (D - 1) (Refer table item (a))

(In all cases where the Denominator has no factors)

But if the Denominator has factors, the (D - N) rule will be different and it will be dealt under similar factors and dissimilar factors separately. However the Ekadhikena sutram is applicable and the final result is thus obtained from it.

Recurring decimals of the fractions with the Denominator having similar factors.

When Ekadhika sutra is applied, the answer for $\frac{1}{49}$

The number of digits of the recurring decimal equivalent is not (49 - 1). But it is 42. This can be explained in the following manner:

Let us consider $\frac{1}{7} = 0.\dot{1}4285\dot{7}$ and divide it by 7. This gives the value of $\frac{1}{49}$.

The division when carried will be complete when 7 times the recurring decimal of $\frac{1}{7}$ is considered successively that amounts to 42 digits. Hence (49 - 7) = 42, instead of (49 - 1) = 48 is the correct answer.

Under this category one can explain the recurring decimal of

$\frac{1}{169} = \frac{1}{13 \times 13}; \frac{1}{289} = \frac{1}{17 \times 17}$

Recurring decimals of the fractions with the denominators having dissimilar factors:

$\frac{1}{39} = \frac{1}{13 \times 3}; \frac{1}{69} = \frac{1}{23 \times 3}; \frac{1}{129} = \frac{1}{43 \times 3}$

Fractions with no factors in denominators $\frac{1}{79}, \frac{1}{89}, \frac{1}{139}$

When Ekadhika sutra is applied, the answer shows the following details regarding the number of digits which follows the sub-multiple rule, i.e., for example in the case of $\frac{1}{79}$, (D - N) = 78. The sub-multiples of 78 are 2, 3, 6, 13, 26, and 39.

The number of digits as obtained for $\frac{1}{79}$ by using Ekadhika sutra will be 13 digits which is sub-multiple of 78.

Similarly, in the case of $\frac{1}{89}$, (D - N) = 88.
Number of digits in the recurring decimal equivalent is 44, which is sub-multiple of 88.

Similarly, in the case of $\frac{1}{139}$, (D - N) = 138.
Number of digits in the recurring decimal equivalent is 46, which is also a sub-multiple of 138.

When Ekadhika is applied to $\frac{1}{39} = \left(\frac{1}{13\times 3}\right)$, it gives the recurring decimal consisting of only 6 digits (instead of 38). $.\dot{0}2564\dot{1}$ This is equal to $\frac{1}{3}$ of $\frac{1}{13}$, i.e., $\frac{1}{3}$ of $(.\dot{0}7692\dot{3})$. $\frac{1}{3}$ gives 6 digits as its recurring value. Thus the final number of digits in $\frac{1}{39}$ is thus equals to 6 digits.

Similarly, when Ekadhika is applied to $\frac{1}{69} = \frac{1}{23\times 3}$, it gives the recurring decimal equivalent consisting of only 22 digits (not 68 digits). $\dot{0}14492753623188405797\dot{1}$. This is equal to $\frac{1}{3}$ of $\frac{1}{23}$, i.e., $\frac{1}{3}$ of $(.\dot{0}43428260869565717391\dot{3})$. $\frac{1}{23}$ Gives 22 digits as its recurring value. Thus the final number of digits in $\frac{1}{69}$ is 22.

Similarly, when Ekadhika is applied to $\frac{1}{129}$, it gives the recurring decimal equivalent consisting of only 21 digits (not 128 digits) i.e. $(.\dot{0}0775193798449612403\dot{1})$. This is equal to $\frac{1}{3}$ of $\frac{1}{43}$, i.e., $\frac{1}{3}$ of $(.\dot{0}2325581395348837209\dot{3})$. $\frac{1}{43}$ Gives 21 digits as its recurring value. Thus the final number of digits in the recurring decimal value of $\frac{1}{129}$ is 21.

Similarly, when Ekadhika is applied to $\frac{1}{119}$, it gives the recurring decimal equivalent consisting of only 48 digits (but not 118 digits).

$.\dot{0}0840336134453781512605042016806722689075630252\dot{1}$
This is equal to $\frac{1}{7}$ of $\frac{1}{17}$, i.e., $\frac{1}{7}$ of (0.0588235294117647)

$\frac{1}{17}$ gives 16 digits as its recurring values only when it is noticed that 3 times of repeated division of it by 7 successively, the value of the recurring decimal of $\frac{1}{17}$ is arrived at this final number of digits in the recurring decimal value of $\frac{1}{119}$ is 48.

Similarly, when Ekadhikena is applied to $\frac{1}{189} = (\frac{1}{27} \times \frac{1}{7})$ gives the recurring decimal equivalent consisting of only 6 digits but not (188 digits). $\dot{0}0529\dot{1}$. This is equal to $\frac{1}{27}$ of $\frac{1}{7}$, i.e. $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{7}$.

$\frac{1}{7}$ Gives 6 digits as its recurring value and division of $\frac{1}{7}$ by three times gives the value of $\frac{1}{189}$. The final number of digits in the recurring decimal value of $\frac{1}{189}$ is 6.

Regarding the recurring decimal of the vulgar fractions with its denominator ending in 1, 3, 7. The details are given in that sections dealing with them.

Recurring decimal of fractions whose denominator ends with 1, 3, 7. The common procedure is to convert them to 9 ending and then applying the method described in the previous section for the 9 ending cases.

For example, $\frac{1}{11} = \frac{9}{99} = 9 \times \frac{1}{99}$. Applying ekadhikena for $\frac{1}{99} = 0.\dot{0}\dot{1}$

$$\frac{1}{11} = 9 \times \frac{1}{99} = 9 \times (0.\dot{0}\dot{1}) = 0.\dot{0}\dot{9}$$

$\frac{1}{99}$ Does not satisfy the (D -N) rule, but it satisfies the sub-multiple rule $2 \times 49 = 98$

$\frac{1}{21} = \frac{9}{189}$.

$\frac{9}{189}$ is already dealt with and gives a recurring decimal of 6 digits $0.\dot{0}0529\dot{1}$. Hence $\frac{1}{21}$ is exactly 9 times of $(0.\dot{0}0529\dot{1})$

$\frac{1}{21}$ can also be written as $\frac{1}{3} \times \frac{1}{7}$. It is workable starting from $\frac{1}{3}$ and dividing by 7 (or) starting from $\frac{1}{7}$ and dividing by 3.

(a) Starting from $\frac{1}{3}$ i.e., $0.\dot{3}$ and dividing by 7 successively 6 times one gets the recurring nature for $\frac{1}{21}$ as $0.\dot{0}4761\dot{9}$.

The number of digits in recurring decimal is in accordance with the factors rule i.e., same as that of the highest factor.

(b) starting from $\frac{1}{7}$ i.e., $0.\dot{1}4285\dot{7}$ and dividing by 3 one gets the value for $\frac{1}{21}$ as $0.\dot{0}4761\dot{9}$

The number of digits in the recurring decimal is not 20 but in accordance with factors rule.

$\frac{1}{31}$ is obtained by converting it into a 9 ending decimal as $\frac{9}{279}$ and applying ekadhikena.

$\frac{1}{41}$ is a special case as $41 \times 271 = 11111$. The value can also be obtained by applying Ekadhika to $\frac{9}{369}$. The recurring decimal equivalent of $\frac{1}{41} = 0.\dot{0}243\dot{9}$.

$\frac{1}{41} = \frac{271}{11111} = \frac{271 \times 9}{99999} = \frac{2439}{99999} = 0.\dot{0}243\dot{9}$ $(\because 99999 = 9 \times 11111)$

The number of digits in the recurring decimal equivalent is 5 (but not 40). This comes under sub-multiple rule as $5 \times 8 = 40$.

$\frac{1}{51} = \frac{1}{17 \times 3}$ The recurring decimal equivalent of $\frac{1}{51}$ is 16 digits but not 50.

$\frac{1}{51} = 0.\dot{0}19607843137254\dot{9}$

$\frac{1}{51}$ is workable starting from $\frac{1}{3}$ and dividing by 17 (or) starting from $\frac{1}{17}$ and dividing by 3.

(a) Starting from $\frac{1}{3}$ i.e., $(0.\dot{3})$ and dividing it by 17 successively 16 times one gets the recurring nature for $\frac{1}{51}$.

(b) Starting from $\frac{1}{17}$ i.e., (0.0588235294117647) and dividing it by 3 one gets the value of $\frac{1}{51}$ the number of digits in the recurring decimal equivalent is 16 which are in accordance with the factor $\frac{1}{17}$.

<u>3 Ending</u>:

$\frac{1}{3} = 0.\dot{3}$

$\frac{1}{13}$ Can be converted into $\frac{3}{39}$ applying ekadhikena to $\frac{1}{39}$ and multiplying it with 3, we get the value of $\frac{1}{13}$ as 0.076923.

$\frac{1}{13} = \frac{3}{39} = 3 \times \frac{1}{39} = 0.\dot{0}2564\dot{1} \times 3 = 0.\dot{0}7692\dot{3}$

The number of digits is not 12 but 6, which is a sub-multiple of 12.

$\frac{1}{23}$ Can be converted into $\frac{3}{39}$, applying Ekadhika to $\frac{1}{69}$ and multiplying it with 3, we get the value of $\frac{1}{23}$.

$\frac{1}{23} = \frac{3}{69} = 3 \times \frac{1}{69} = 3 \times (.0144927536231884057971)$

$\frac{1}{23} = .\dot{0}4347826086/9565217391\dot{3}$ It has 22 digits.

$\frac{1}{33}$ Can be converted as $\frac{3}{99}$ applying Ekadhika to $\frac{1}{99}$ and multiplying with 3, one gets the value of $\frac{1}{33}$.

$\frac{1}{33} = \frac{3}{99} = 3 \times \frac{1}{99} = 3 \times 0.\dot{0}\dot{1} = 0.\dot{0}\dot{3}$

The number of digits in the recurring decimal equivalent is 2 but not 32. We notice 2 is sub-multiple of 32.

$\frac{1}{43}$ is obtained by converting it as $\frac{3}{129}$. Similar method is adopted as in previous refer $\frac{1}{129}$.

All the 3 ending cases can be converted to 9 ending and then Ekadhika can be applied (or) if there are factors then factors rule is applied.

<u>7 Ending</u>:

$\frac{1}{7} = 0.\dot{1}4285\dot{7}$.

This can also be written as $\frac{7}{49}$ i.e., $7 \times \frac{1}{49}$ (Refer to $\frac{1}{49}$)

$\frac{1}{17}$ Can be converted to 9 ending as $\frac{7}{119}$ (Refer to $\frac{1}{119}$)

$\frac{1}{27}$ is $\frac{1}{3 \times 3 \times 3}$.

$\frac{1}{3}$ is $0.\dot{3}$, when this is divided by 3, we get $0.\dot{1}$ now when again divided by 3 we get $0.\dot{0}3\dot{7}$.

This can also be dealt with special cases.

$\frac{1}{37}$ is a special case ---------- (Refer to its page).

Section-2

AUXILIARY FRACTIONS

In the usual division in case of vulgar fractions & decimal fractions we generally consider the usual method of divisions by powers of ten and the resulting number is later considered for the actual division.

For example:

1) $\frac{84}{90} = \frac{8.4}{9}$ 2) $\frac{73657}{10000} = 7.3657$

3) $\frac{73657}{100000} = .73657$ and so on

Swamiji has developed a simpler method by introducing what are called sahayaks "Auxiliary fractions", and the work appears to be well facilitated.

First Type:

Use is made of Ekadhika purna sutram by means of which the last digit of the denominator is dropped out and the penultimate is increased by one.

This operation is applicable in case of denominators ending in 9.

This is known as Ekadhika operation for example the (AF) auxiliary fractions for various fractions are as follows.

1) $\frac{3}{19}$ the AF is $\frac{3}{2}$

2) $\frac{48}{29}$ the AF is $\frac{4.8}{3}$

3) $\frac{51}{59}$ the AF is $\frac{5.1}{6}$

4) $\frac{73}{109}$ the AF is $\frac{7.3}{11}$

5) $\frac{1}{129}$ the AF is $\frac{0.01}{13}$

6) $\frac{471}{1499}$ the AF is $\frac{4.71}{15}$

7) $\frac{257}{18999}$ the AF is $\frac{.257}{19}$

8) $\frac{23}{299999}$ the AF is $\frac{0.00023}{3}$

9) $\frac{1538}{189999999}$ the AF is $\frac{0.0001538}{9}$

10) $\frac{32403}{69999999999}$ the AF is $\frac{0.0000032403}{7}$

Modus Operandi (first type):

In case of single nine ending the Ekadhika means operation resulting in removal of 9 followed by adding 1 to the previous digit.
In case of number ending in two nines one has to write down the numerator consisting of two decimals and applying Ekadhika to the denominator after removal of two nines.

For example in the problem (6), 15 is the Ekadhika of the denominator of

$\therefore \frac{471}{1499}$ in the AF as $\frac{4.71}{15}$

This principle is continued for the rest of number 9 ending denominator as shown in examples.

To work out completely let us consider the example $\frac{23}{299999}$

$\frac{0.00023}{3}$ is the AF of $\frac{23}{299999}$

Now we write down prefining the remainder '2' to the first quotient group 2: 00007 And divide the new dividend 200007 by 3 the second quotient group is 66669 with zero remainder.

The dividend 66669 by 3 we get 22223 which is the 3rd quotient group with remainder zero hence the next dividend is 22223 and dividing it by 3 we get 7407 with 2 as remainder.

$\therefore$ The dividend is 27407 and so on

This can be shown as following $\frac{23}{299999}$ the AF is $\frac{0.00023}{3}$

$$\frac{3)0.00023}{Q=.00007-R=2}$$

.00007: .66669: .22223: .7407
2 : 0 : 0 : 2

Or considering a simpler problem

$\frac{51}{59}$ the AF is $\frac{5.1}{6}$

$$\frac{6)5.1}{Q=.8-R=3}$$

.8: .6: .4: .4: .06:
3 : 2 : 2 : 0 : 4 :

$\therefore \frac{51}{59}$=0.864406 and so on

The proof there of is

$: 5.1$

$6: \frac{3}{.8}$

$\therefore$ Ans = 0.864406 and so on

This method is also applicable to the denominator ending in 3 because it can be converted to a AF of denominator ending in 9.

For example:

For $\frac{14}{23}$ it can be written as $\frac{42}{69}$

The AF of $\frac{42}{69}$ is $\frac{4.2}{7}$

$\therefore$ F=0.608695652

For $\frac{21}{73}=\frac{63}{219}$ the AF $\frac{6.3}{22}$

$\therefore$ F= 0.287671232

Denominators ending in 9:

(1) For $\frac{1}{19}$, Auxiliary Fraction = $\frac{0.1}{2}$

(2) For $\frac{1}{49}$, A. F = $\frac{0.1}{5}$

(3) For $\frac{1}{79}$, A.F = $\frac{0.1}{8}$

(4) For $\frac{8}{69}$, A. F = $\frac{0.8}{7}$

(5) For $\frac{59}{899}$, A. F = $\frac{0.59}{9}$

(6) For $\frac{1}{129}$, A. F = $\frac{0.1}{13}$

(7) For $\frac{82}{199}$, A. F = $\frac{0.82}{2}$

(8) For $\frac{0.6}{399}$, A. F = $\frac{0.006}{4}$

(9) For $\frac{50}{299}$, A .F = $\frac{0.5}{3}$

(10) For $\frac{432}{6999}$, A.F = $\frac{0.432}{7}$

(11) For $\frac{5}{79999}$, A. F = $\frac{0.0005}{8}$

(12) For $\frac{600}{599999}$, A.F = $\frac{0.006}{6}$

1) $\frac{1}{19}$

<u>Current Method</u>

19) 1.00 (·052631578947368421

$\underline{95}$

50

$\underline{38}$

120

$\underline{114}$

60

$\underline{57}$

30

$\underline{19}$

110

$\underline{95}$

150

$\underline{133}$

170

$\underline{152}$

180

$\underline{171}$

90

$\underline{76}$

140

$\underline{133}$

70

$\underline{57}$

130

$\underline{114}$

160

$\underline{152}$

80

$\underline{76}$

40

$\underline{38}$

20

$\underline{19}$

$\underline{1}$

$\therefore \frac{1}{19} = 0.\dot{0}52631578947368421\dot{1}$

Vedic Method

$\frac{1}{19}$. A.F = $\frac{0.1}{2}$

$2\overline{)0.1}(0.05$ Q

$\underline{0.1}$

0 R

0.05 2 6 3 1 5 7 8 / 9 4 7 3 6 8 4 2 1

0 1 0 0 1 1 1 1 / 0 1 0 1 1 0 0 0 0

$\therefore \frac{1}{19} = 0.\dot{0}52631578947368421\dot{1}$

(2) $\frac{1}{49}$

Current Method

49) 1.00 (.020408163265306122448979591836734693877551

$\underline{98}$
200
$\underline{196}$
400
$\underline{392}$
80
$\underline{49}$
310
$\underline{294}$
160
$\underline{147}$
130
$\underline{98}$
320
$\underline{294}$
260
$\underline{245}$
150
$\underline{147}$
300
$\underline{294}$
60
$\underline{49}$
110
$\underline{98}$
120
$\underline{98}$
220
$\underline{196}$

240
<u>196</u>
440
392 ....

$$\therefore \frac{1}{49} = 0.\dot{0}2040816326530612244897959183673469387755\dot{1}$$

<u>Vedic Method</u>

$\frac{1}{49}$ A.F = $\frac{0.1}{5}$ 5)0.1 (0.02

<u>0.1</u>
<u>0</u>

0.02	0	4	0	8	1	6	3	2	6	5	3	0	6	1	2	2	4	4	8
0	2	0	4	0	3	1	1	3	2	1	0	3	0	1	1	2	2	4	4

9	7	9	5	9	1	8	3	6	7	3	4	6	9	3	8	7	7	5	5	1
3	4	2	4	0	4	1	3	3	1	2	3	4	1	4	3	3	2	2	0	0

$$\therefore \frac{1}{49} = 0.\dot{0}2040816326530612244897959183673469387755\dot{1}$$

(3) $\frac{1}{79}$

<u>Current Method</u>

79) 1.00(0.126582278481
<u>79</u>
210
<u>158</u>
520
<u>474</u>
460
<u>395</u>
650
<u>632</u>
180
<u>158</u>
220
<u>158</u>
620
<u>553</u>
670
<u>632</u>
380
<u>316</u>
640
<u>632</u>
80
<u>79</u>

$\therefore \frac{1}{79} = 0.\dot{0}12658227848\dot{1}$

<u>Vedic Method</u>

$\frac{1}{79}$ A.F $= \frac{0.1}{8}$

```
8)0.01(0.01          8)21(2
  0.08                 16
  0.02                  5
```

```
      0.01   2   6   5   8   2   2   7   8   4   8   1
  0.02     5   4   6   1   2   6   6   3   6   0   0
```

$\therefore \frac{1}{79} = 0.\dot{0}12658227848\dot{1}$

(4) $\frac{8}{69}$

<u>Current Method</u>

```
69)8.0(·11594202898550724637 68
    6 9
    110
     69
     410
     345
      650
      621
       290
       276
        140
        138
         200
         138
          620
          552
           680
           621
            590
            552
             380
             345
              350
```

345
500
483
170
138
320
276
440
414
260
207
530
483
470
414
560
552
8

$\therefore \frac{8}{69} = 0.\dot{1}159420289855072463768\dot{8}$

Vedic Method

$\frac{8}{69}$ A.F $= \frac{0.8}{7}$ 7)0.8 (0.1

0.7
.1

0.1 1 5 9 4 2 0 2 8 9 8 5 5 0 7 2 4 6 3 7 6 8

.1 4 6 2 1 0 2 6 6 5 3 3 0 5 1 3 4 2 5 4 5 0

$\therefore \frac{8}{69} = 0.\dot{1}159420289855072463768\dot{8}$

(5) $\frac{59}{899}$

Current Method

899)59.00 (.06562847608453837597
53 94
5060
4495
5650
5394
2560
1798
7620
7192
4280

$$\begin{array}{r}
\underline{3596} \\
6840 \\
\underline{6293} \\
5470 \\
\underline{5394} \\
7600 \\
\underline{7192} \\
4080 \\
\underline{3596} \\
4840 \\
\underline{4495} \\
3450 \\
\underline{2697} \\
7530 \\
\underline{7192} \\
3380 \\
\underline{2697} \\
6830 \\
\underline{6293} \\
5370 \\
\underline{4495} \\
8750 \\
\underline{8091} \\
6590 \\
\underline{6293} \\
\underline{297}
\end{array}$$

$\therefore \frac{59}{899} = 0.0656284760845383 7597$

<u>Vedic Method</u>

$\frac{59}{899}$ A.F $= \frac{0.59}{9}$

$9)0.59(0.06$

$\underline{0.54}$

$\underline{.05}$

$$\begin{array}{ccccccccccc}
 & 0.06 & 56 & 28 & 47 & 60 & 84 & 53 & 83 & 75 & 97 \ldots\ldots\ldots \\
5 & 2 & 4 & 5 & 7 & 4 & 7 & 6 & 8 & 2 &
\end{array}$$

$\therefore \frac{59}{899} = 0.06562847608453837597$

(6) $\frac{1}{129}$

Current Method

$$
\begin{array}{l}
129)\,1.000\,(\,.00775193798449 \\
\underline{\;\;903} \\
\;\;\;970 \\
\;\;\;\underline{903} \\
\;\;\;\;\;670 \\
\;\;\;\;\;\underline{645} \\
\;\;\;\;\;\;\;250 \\
\;\;\;\;\;\;\;\underline{129} \\
\;\;\;\;\;\;\;1210 \\
\;\;\;\;\;\;\;\underline{1161} \\
\;\;\;\;\;\;\;\;\;\;490 \\
\;\;\;\;\;\;\;\;\;\;\underline{387} \\
\;\;\;\;\;\;\;\;\;\;1030 \\
\;\;\;\;\;\;\;\;\;\;\underline{\;\;903} \\
\;\;\;\;\;\;\;\;\;\;\;\;1270 \\
\;\;\;\;\;\;\;\;\;\;\;\;\underline{1161} \\
\;\;\;\;\;\;\;\;\;\;\;\;\;\;1090 \\
\;\;\;\;\;\;\;\;\;\;\;\;\;\;\underline{1032} \\
\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;580 \\
\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\underline{516} \\
\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;640 \\
\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\underline{516} \\
\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;1240 \\
\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\underline{1161} \\
\;\underline{\;\;79}
\end{array}
$$

$\therefore \frac{1}{129} = 0.00775193798449$

Vedic Method

$\frac{1}{129}$ A.F = $\frac{0.1}{13}$

$$
\begin{array}{l}
13)\,0.1\,(\,0.0 \\
\;\;\;\;\;\;\underline{0.0} \\
\;\;\;\;\;\;\underline{\;\;\;1}
\end{array}
$$

0.0	0	7	7	5	1	9	3	7	9	8	4	4	9....
1	10	9	6	2	12	4	10	12	10	5	6	12	

$\therefore \frac{1}{129} = 0.00775193798449$

(7) $\frac{82}{199}$

<u>Current Method</u>

```
199)82.0(·41206030150753768844221 1
    796
     240
     199
      410
      398
       1200
       1194
          600
          597
           300
           199
           1010
            995
             1500
             1393
              1070
               995
                750
                597
                1530
                1393
                 1370
                 1194
                  1760
                  1592
                   1680
                   1592
                     880
                     796
                      840
                      796
                       440
                       398
                        420
                        398
                         220
                         199
                          210
                          199
                           11
```

$\therefore \frac{82}{199} = 0.41206030150753768844221 1$

<u>Vedic Method</u>

$\frac{82}{199}$ A.F $= \frac{0.82}{2}$

2)0.82(0.41	0.41	20	60	30	15	07	53	76	88	44	22	11 ......
<u>0.82</u>	0	1	0	0	0	1	1	1	0	0	0	0
<u>0</u>												

$\therefore \frac{82}{199} = 0.412060301507537688442211\ldots$

(8) $\frac{0.6}{399}$

<u>Current Method</u>

$\frac{0.6}{399} = \frac{6}{3990}$

3990)6.000(·0015037593984962406015

<u>3990</u>
20100
<u>19950</u>
15000
<u>11970</u>
30300
<u>27930</u>
23700
<u>19950</u>
37500
<u>35910</u>
15900
<u>11970</u>
39300
<u>35910</u>
33900
<u>31920</u>
19800
<u>15960</u>
38400
<u>35910</u>
24900
<u>23940</u>
9600
<u>7980</u>
16200
<u>15960</u>
24000
<u>23940</u>
6000

$$\begin{array}{l}\underline{3990}\\20100\\\underline{19950}\\\underline{\ \ 150}\end{array}$$

$\therefore \frac{0.6}{399} = 0.0015037593984962406015$

Vedic Method

$\frac{0.6}{399}$ A.F $= \frac{0.006}{4}$

0.00	15	03	75	93	98	49	62	40	60	15 …..
6	0	3	3	3	1	2	1	2	0	0

$\therefore \frac{0.6}{399} = 0.0015037593984962406015$

(9) $\frac{50}{299}$

Current Method

299) 50.0 (0.167224080267558528428093

$$\begin{array}{l}\underline{299}\\2010\\\underline{1794}\\\ \ 2160\\\ \ \underline{2093}\\\ \ \ \ 670\\\ \ \ \ \underline{598}\\\ \ \ \ \ 720\\\ \ \ \ \ \underline{598}\\\ \ \ \ \ 1220\\\ \ \ \ \ \underline{1196}\\\ \ \ \ \ \ 2400\\\ \ \ \ \ \ \underline{2392}\\\ \ \ \ \ \ \ \ 800\\\ \ \ \ \ \ \ \ \underline{598}\\\ \ \ \ \ \ \ \ 2020\\\ \ \ \ \ \ \ \ \underline{1794}\\\ \ \ \ \ \ \ \ \ 2260\\\ \ \ \ \ \ \ \ \ \underline{2093}\\\ \ \ \ \ \ \ \ \ \ 1670\\\ \ \ \ \ \ \ \ \ \ \underline{1495}\\\ \ \ \ \ \ \ \ \ \ \ 1750\\\ \ \ \ \ \ \ \ \ \ \ \underline{1495}\\\ \ \ \ \ \ \ \ \ \ \ \ 2550\\\ \ \ \ \ \ \ \ \ \ \ \ \underline{2392}\\\ \ \ \ \ \ \ \ \ \ \ \ \ 1580\\\ \ \ \ \ \ \ \ \ \ \ \ \ \underline{1495}\\\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ 850\\\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \underline{598}\end{array}$$

2520
2392
1280
1196
840
598
2420
2392
2800
2691
1090
897
193

$\therefore \frac{50}{299} = 0.167224080267558528428093$

Vedic Method

$\frac{50}{299}$ A.F = $\frac{0.5}{3}$

0.16	72	24	08	02	67	55	85	28	42	80	93 ……
2	0	0	0	2	1	2	0	1	2	2	1

$\therefore \frac{50}{299} = 0.167224080267558528428093$

(10) $\frac{432}{6999}$

Current Method

6999)432.00(.061723103300471495927
41994
12060
6999
50610
48993
16170
13998
21720
20997
7230
6999
23100
20997
21030
20997
33000
27996
50040

48993
10470
6999
34710
27996
67140
62991
41490
34995
64950
62991
19590
13998
55920
48993
6927

$\therefore \frac{432}{6999} = 0.0617231033004714959 27$

Vedic Method

$\frac{432}{6999}$ A.F $= \frac{0.432}{7}$

0.061	723	103	300	471	495	927……
5	0	2	3	3	6	6

$\therefore \frac{432}{6999} = 0.0617231033004714959 27$

(11) $\frac{5}{79999}$

Current Method

79999)5.00000(·000062500781259765747071
4 79994
200060
159998
400620
399995
625000
559993
650070
639992
100780
79999
207810
159998
478120

```
399995
 781250
 719991
  612590
  559993
   525970
   479994
    459760
    399995
     597650
     559993
      376570
      319996
       565740
       559993
        574700
        559993
         147070
          79999
          67071
```

$\therefore \frac{5}{79999} = 0.0000625007812597657 47071$

Vedic Method

$\frac{5}{7999}$ A.F $= \frac{0.0005}{8}$

0.0000	6250	0781	2597	6574	7071
5	0	2	5	5	6

$\therefore \frac{5}{79999} = 0.0000625007812597657 47071$

(12) $\frac{600}{599999}$

Current Method

```
599999)600.000(.0010000016666669444449074
       599 999
          1000000
           599999
           4000010
           3599994
            4000160
            3599994
             4001660
             3599994
              4016660
```

$$\begin{array}{r}
\underline{3599994} \\
4166660 \\
\underline{3599994} \\
5666660 \\
\underline{5399991} \\
2666690 \\
\underline{2399996} \\
2666940 \\
\underline{2399996} \\
2669440 \\
\underline{2399996} \\
2694440 \\
\underline{2399996} \\
2944440 \\
\underline{2399996} \\
5444440 \\
\underline{5399991} \\
4444900 \\
\underline{4199993} \\
2449070 \\
\underline{2399996} \\
\underline{49074}
\end{array}$$

$$\therefore \frac{600}{599999} = 0.0010000016666694444449074$$

<u>Vedic Method</u>

$$\frac{600}{599999} \quad \text{A.F} = \frac{0.006}{6}$$

$$\begin{array}{cccccc}
0.00100 & 00016 & 66669 & 44444 & 90740 & \ldots\ldots\ldots \\
0 & 4 & 2 & 5 & 4 &
\end{array}$$

$$\therefore \frac{600}{599999} = 0.0010000016666694444490740$$

```
399995
 781250
 719991
  612590
  559993
   525970
   479994
    459760
    399995
     597650
     559993
      376570
      319996
       565740
       559993
        574700
        559993
         147070
          79999
          67071
```

$\therefore \frac{5}{79999} = 0.0000625007812597657 47071$

Vedic Method

$\frac{5}{7999}$ A.F $= \frac{0.0005}{8}$

0.0000	6250	0781	2597	6574	7071
5	0	2	5	5	6

$\therefore \frac{5}{79999} = 0.00006250078125976574 7071$

(12) $\frac{600}{599999}$

Current Method

```
599999)600.000(·0010000016666694444449074
       599 999
          1000000
           599999
           4000010
           3599994
            4000160
            3599994
             4001660
             3599994
              4016660
```

3599994
4166660
3599994
5666660
5399991
2666690
2399996
2666940
2399996
2669440
2399996
2694440
2399996
2944440
2399996
5444440
5399991
4444900
4199993
2449070
2399996
49074

$\therefore \frac{600}{599999} = 0.001000001666669444449074$

Vedic Method

$\frac{600}{599999}$ A.F $= \frac{0.006}{6}$

0.00100	00016	66669	44444	90740 ....
0	4	2	5	4

$\therefore \frac{600}{599999} = 0.0010000016666694444490740$

Second Type:

If the denominator of the given fraction is ending in 1 then drop the 1 and decrease the numerator by 1.

Example:

1) $\frac{3}{41}$ AF = $\frac{2}{40} = \frac{0.2}{4}$

2) $\frac{44}{61}$ AF = $\frac{43}{60} = \frac{4.3}{6}$

3) $\frac{71}{31}$ AF = $\frac{70}{30} = \frac{7}{3}$

4) $\frac{2861}{8001}$ AF = $\frac{2860}{8000} = \frac{2.860}{8}$

Modus Operandi:

The principles of the prefixing etc. and other details are the same as is in the previous Ekadhika Auxiliary fractions; but the procedure is different in particulars. At the divisible that is after the first division we prefix the remainder not to each quotient digit but to <u>its complement</u> from nine and carry out the division as usual.

For example:

$\frac{13}{41}$ $\therefore$ AF = $\frac{12}{40} = \frac{1.2}{4}$

i. We divide 1.2 by 4 and set 3 down as first quotient digit and 0 as the first remainder.

ii. We then divide not 03 but 0, 6 (the complement of 3 from 9) by 4 and put 1 and 2 as the second quotient digit and second remainder respectively.

$\therefore$ We have

$$\begin{array}{cc} .3 & 1 \\ 0 & 2 \end{array}$$

iii. We now take 28 as dividend instead of 21 dividing it by 4 we get

$$\begin{array}{ccc} .3 & 1 & 7 \\ 0 & 2 & 0 \end{array}$$

Thus now we divide 02 by 4 so on we get

$$F = \begin{array}{cccccc} .3 & 1 & 7 & 0 & 7 & 3 \\ 0 & 2 & 0 & 2 & 1 & 0 \end{array}$$

Some more illustrative examples are: -

(1) $\frac{33}{51}$, AF = $\frac{3.2}{50} \Rightarrow \frac{3.2}{5}$

$$= \begin{array}{cccccccc} 0. & 6 & 4 & 7 & 0 & 5 & 8 & 8 \\ & 2 & 3 & 0 & 2 & 4 & 4 & 1 \end{array}$$

(2) $\frac{4}{131}$, AF = $\frac{3}{130} = \frac{0.3}{13}$

$$\begin{array}{l} = 0.0\,3\,0\,5\,3\,4\,3 \\ \quad\;\; 3\,0\,6\,4\,5\,4\,6 \end{array}$$

(3) $\frac{32}{701} = \frac{31}{700}$ AF = $\frac{0.31}{7}$

$$\begin{array}{lll} 0.04 & 56 & 49 \\ \quad\;3 & \;3 & \;0 \end{array}$$

$\therefore \frac{32}{701} = 0.045649$

(4) $\frac{1424}{3001} = \frac{1423}{3000}$ AF = $\frac{1.423}{3}$

$\therefore \frac{1.423}{3} = 0.474508497167$

$\therefore \frac{1424}{3001} = 0.474508497167$

(5) $\frac{186}{27} = \frac{558}{81}$ AF = $\frac{557}{80} = \frac{55.7}{8} = 6.96629$

iv. In case the given fraction is not vulgar fraction then convert it into a mixed fraction

Ex:

$\frac{186}{27} = 6 + \frac{24}{27}$ (vulgar fraction)

Apply the rule to the vulgar fraction part thus

$\frac{2.4}{27} = \frac{72}{81} \Rightarrow \frac{7.1}{8} = 0.8888$

$\therefore \frac{186}{27} = 6 + 0.8888$

(6) $\frac{346}{23} = 15 + \frac{1}{23}$

$\frac{1}{23} = \frac{7}{161}$ AF = $\frac{6}{160} = \frac{0.6}{16}$

$$\begin{array}{l} = 0.0\,4\,3\,4\,7 \\ \;\;\, 0.6\,5\,7\,1\,2\,1\,3 \end{array}$$

$\therefore \frac{346}{23} = 15.04347$

Second Type:

Denominators ending in 1

(1) For $\frac{2}{31}$ Auxiliary Fraction = $\frac{1}{30} = \frac{0.1}{3}$

(2) For $\frac{12}{51}$ A.F = $\frac{11}{50} = \frac{1.1}{5}$

(3) For $\frac{54}{91}$ A.F = $\frac{53}{90} = \frac{5.3}{9}$

(4) For $\frac{5}{321}$ A.F = $\frac{4}{320} = \frac{0.4}{32}$

(5) For $\frac{13}{171}$ A.F = $\frac{12}{170} = \frac{1.2}{17}$

(6) For $\frac{2}{601}$ A.F = $\frac{1}{600} = \frac{0.01}{6}$

(7) For $\frac{25}{1501}$ A.F = $\frac{24}{1500} = \frac{0.24}{15}$

(8) For $\frac{13}{23001}$ A.F = $\frac{12}{23000} = \frac{0.012}{23}$

(9) For $\frac{235}{1400001}$ A.F = $\frac{234}{1400000} = \frac{0.00234}{14}$

(10) For $\frac{1}{600000001}$ A.F = $\frac{0}{600000000} = \frac{0.00000000}{6}$

(1) $\frac{2}{31}$

<u>Current Method</u>

```
31)2.00(·064516129032258
    1 86
      140
      124
       160
       155
         50
         31
         190
         186
           40
           31
```

$$
\begin{array}{r}
90 \\
\underline{62} \\
280 \\
\underline{279} \\
100 \\
\underline{93} \\
70 \\
\underline{62} \\
80 \\
\underline{62} \\
180 \\
\underline{155} \\
250 \\
\underline{248} \\
\underline{2}
\end{array}
$$

$\therefore \dfrac{2}{31} = 0.\dot{0}64516129032225\dot{8}$

<u>Vedic Method</u>

$\dfrac{2}{31}$ $\quad$ A.F $= \dfrac{1}{30} = \dfrac{0.1}{3}$

$$
\begin{array}{ccccccccccccccccc}
0.0 & 6 & 4 & 5 & 1 & 6 & 1 & 2 & 9 & 0 & 3 & 2 & 2 & 5 & 8 & / & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & / & 1
\end{array}
$$

$\therefore \dfrac{2}{31} = 0.\dot{0}64516129032225\dot{8}$

(2) $\dfrac{12}{51}$

<u>Current Method</u>

$$
\begin{array}{l}
51)\,12.0\,(\cdot 2352941176470588 \\
\quad \underline{10\,2} \\
\quad\ 1\,80 \\
\quad\ \underline{1\,53} \\
\qquad 270 \\
\qquad \underline{255} \\
\qquad\ 150 \\
\qquad\ \underline{102} \\
\qquad\quad 480 \\
\qquad\quad \underline{459} \\
\qquad\quad\ 210 \\
\qquad\quad\ \underline{204} \\
\qquad\qquad 60 \\
\qquad\qquad \underline{51} \\
\qquad\qquad\ 90 \\
\qquad\qquad\ \underline{51}
\end{array}
$$

```
390
357
 330
 306
  240
  204
   360
   357
    300
    255
     450
     408
      420
      408
       12
```

$\therefore \frac{12}{51} = 0.\dot{2}35294117647058\dot{8}$

Vedic Method

$\frac{12}{51}$ A.F $= \frac{11}{50} = \frac{1.1}{5}$

```
0.2  3  5  2  9  4  1  1  7  6  4  7  0  5  8  8 / 2
1    2  1  4  2  0  0  3  3  2  3  0  2  4  4  1 / 1
```

$\therefore \frac{12}{51} = 0.\dot{2}35294117647058\dot{8}$

(3) $\frac{54}{91}$

Current Method

```
91)54.0(.593406
   45 5
    8 50
    8 19
      310
      273
       370
       364
         600
         546
          54
```

$\therefore \frac{54}{91} = 0.59340\dot{6}$

Vedic Method

$A.F = \frac{53}{90} = \frac{5.3}{9}$

0.5 9 3 4 0 6 / 5

8 3 3 0 5 5 / 8

$\therefore \frac{54}{91} = 0.\dot{5}9340\dot{6}$

(4) $\frac{5}{321}$

Current Method

321)5.00(·01557632398

3 21

1 790

1 605

1850

1605

2450

2247

2030

1926

1040

963

770

642

1280

963

3170

2889

2810

2568

2420

2247

173

$\therefore \frac{5}{321} = 0.015576323987$

Vedic Method

$A.F = \frac{4}{320} = \frac{0.4}{32}$

0.0 1 5 5 7 6 3 2 3 9 8 7 .....

4 17 18 24 20 10 7 12 31 28 25 27

$\therefore \frac{5}{321} = 0.015576323987$

(5) $\frac{13}{171}$

<u>Current Method</u>

$$
\begin{array}{l}
171)13.00(\cdot 076023391812865497 \\
\quad\;\; \underline{11\ 97} \\
\quad\;\;\; 1\ 030 \\
\quad\;\;\; \underline{1\ 026} \\
\qquad\quad 400 \\
\qquad\quad \underline{342} \\
\qquad\quad\; 580 \\
\qquad\quad\; \underline{513} \\
\qquad\qquad 670 \\
\qquad\qquad \underline{513} \\
\qquad\qquad 1570 \\
\qquad\qquad \underline{1539} \\
\qquad\qquad\;\; 310 \\
\qquad\qquad\;\; \underline{171} \\
\qquad\qquad\;\; 1390 \\
\qquad\qquad\;\; \underline{1368} \\
\qquad\qquad\qquad 220 \\
\qquad\qquad\qquad \underline{171} \\
\qquad\qquad\qquad\; 490 \\
\qquad\qquad\qquad\; \underline{342} \\
\qquad\qquad\qquad\; 1480 \\
\qquad\qquad\qquad\; \underline{1368} \\
\qquad\qquad\qquad\quad 1120 \\
\qquad\qquad\qquad\quad \underline{1026} \\
\qquad\qquad\qquad\qquad 940 \\
\qquad\qquad\qquad\qquad \underline{855} \\
\qquad\qquad\qquad\qquad\; 850 \\
\qquad\qquad\qquad\qquad\; \underline{684} \\
\qquad\qquad\qquad\qquad\; 1660 \\
\qquad\qquad\qquad\qquad\; \underline{1539} \\
\qquad\qquad\qquad\qquad\quad 1210 \\
\qquad\qquad\qquad\qquad\quad \underline{1197} \\
\qquad\qquad\qquad\qquad\quad \underline{\quad 13}
\end{array}
$$

$0.\therefore \frac{13}{171} = 0.\dot{0}7602339181286549\dot{7}$

<u>Vedic Method</u>

$A.F = \frac{12}{170} = \frac{1.2}{17}$

$$
\begin{array}{cccccccccccccccccccc}
 & 0.0 & 7 & 6 & 0 & 2 & 3 & 3 & 9 & 1 & 8 & 1 & 2 & 8 & 6 & 5 & 4 & 9 & 7 / & 0 \\
12 & & 10 & 0 & 3 & 5 & 6 & 15 & 3 & 13 & 2 & 4 & 14 & 11 & 9 & 8 & 16 & 12 & 1 / & 12
\end{array}
$$

$\therefore \frac{13}{171} = 0.\dot{0}7602339181286549\dot{7}$

(6) $\frac{2}{601}$

<u>Current Method</u>

$601\overline{)2.000}$(.00332778702163061564

$\underline{1\ 803}$
1970
$\underline{1803}$
1670
$\underline{1202}$
4680
$\underline{4207}$
4730
$\underline{4207}$
5230
$\underline{4808}$
4220
$\underline{4207}$
1300
$\underline{1202}$
980
$\underline{601}$
3790
$\underline{3606}$
1840
$\underline{1803}$
3700
$\underline{3606}$
940
$\underline{601}$
3390
$\underline{3005}$
3850
$\underline{3606}$
2440
$\underline{2404}$
$\underline{\ 36}$

$\therefore \frac{2}{601} = 0.00332778702163061564$

<u>Vedic Method</u>

$A.F = \frac{1}{600} = \frac{0.01}{6}$

0.00	33	27	78	70	21	63	06	15	64 .....
1	1	4	4	1	3	0	0	3	0

$\therefore \frac{2}{601} = 0.00332778702163061564$

(7) $\frac{25}{1501}$

Current Method

```
1501) 25.00 (.016655562958027981
        15 01
         9 990
         9 006
            9840
            9006
             8340
             7505
              8350
              7505
               8450
               7505
                9450
                9006
                 4440
                 3002
                 14380
                 13509
                   8710
                   7505
                   12050
                   12008
                      4200
                      3002
                      11980
                      10507
                       14730
                       13509
                        12210
                        12008
                          2020
                          1501
                           519
```

$\therefore \frac{25}{1501} = 0.016655562958027981$

Vedic Method

$A.F = \frac{24}{1500} = \frac{0.24}{15}$

0.01	66	55	56	29	58	02	79	81 ....
9	8	8	4	8	0	11	12	5

$\therefore \frac{25}{1501} = 0.016655562958027981$

(8) $\frac{13}{23001}$

Current Method

```
23001)13.0000(.000565192817703578
        11 5005
         1 49950
         1 38006
           119440
           115005
             44350
             23001
             213490
             207009
               64810
               46002
               188080
               184008
                 40720
                 23001
                 177190
                 161007
                  161830
                  161007
                     82300
                     69003
                     132970
                     115005
                      179650
                      161007
                       186430
                       184008
                         2422
```

$\therefore \frac{13}{23001} = 0.00056519817703578$

Vedic Method

$A.F = \frac{12}{23000} = \frac{0.012}{23}$

0.000	565	192	817	703	578 ....
12	4	18	16	13	2

$\therefore \frac{13}{23001} = 0.000565192817703578$

(9) $\frac{235}{1400001}$

<u>Current Method</u>

```
1400001)235.0000(·00016785702295926931
         140 0001
          94 99990
          84 00006
          10 999840
           9 800007
           1 1998330
           1 1200008
              7983220
              7000005
               9832150
               9800007
                3214300
                2800002
                 4142980
                 2800002
                 13429780
                 12600009
                   8297710
                   7000005
                   12977050
                   12600009
                     3770410
                     2800002
                      9704080
                      8400006
                      13040740
                      12600009
                        4407310
                        4200003
                         2073070
                         1400001
                          673069
```

$\therefore \frac{235}{1400001} = 0.00016785702295926931$

<u>Vedic Method</u>

$A.F = \frac{234}{1400000} = \frac{0.00234}{14}$

$$\begin{array}{llll} 0.00016 & 78570 & 22959 & 26931\ \ldots\ldots \\ \quad 10 & \ \ 3 & \ \ 3 & \ \ 6 \end{array}$$

$\therefore \frac{235}{1400001} = 0.00016785702295926931$

(10) $\frac{1}{600000001}$

Current Method

```
600000001)1.000000000(·0000000016666666663888888
           600000001
           3999999990
           3600000006
            3999999840
            3600000006
             3999998340
             3600000006
              3999983340
              3600000006
               3999833340
               3600000006
               3998333340
               3600000006
                3983333340
                3600000006
                 3833333340
                 3600000006
                  2333333340
                  1800000003
                   5333333370
                   4800000008
                    5333333620
                    4800000008
                     5333336120
                     4800000008
                      5333361120
                      4800000008
                       5333611120
                       4800000008
                        5336111120
                        4800000008
                         536111112
```

$\therefore \frac{1}{600000001} = 0.0000000016666666663888888$

Vedic Method

$\frac{1}{600000001}$ A.F = $\frac{0}{600000000} = \frac{0.00000000}{6}$

$$\begin{array}{ccc} 0.00000000 & 16666666 & 63888888 \\ 0 & 3 & 5 \end{array}$$

$$\therefore \frac{1}{600000001} = 0.000000001666666663888888$$

Other endings:

(a) <u>7 and 3</u>

(1) $\frac{3}{37}$

<u>Current Method</u>

```
37)3.00(·081
   2 96
     40
     37
      3
```

$\therefore \frac{3}{37} = 0.\dot{0}8\dot{1}$

<u>Vedic Method</u>

$\frac{3}{37} = \frac{9}{111}$ (7 ending converted to 1 ending)

$A.F = \frac{8}{110} = \frac{0.8}{11}$

$$\begin{array}{cccccccc} & 0.0 & & 8 & & 1 & / & 0 \\ 8 & & 1 & & 0 & & / & 8 \end{array}$$

$\therefore \frac{3}{37} = 0.\dot{0}8\dot{1}$

(2) $\frac{23}{167}$

<u>Current Method</u>

```
167)23.0(·13772455089820359281
    16 7
     6 30
     5 01
     1 290
     1 169
       1210
       1169
         410
         334
          760
          668
           920
```

0.000000000835 16666666 63888888 ......
0 850 3 4
835

$\therefore \frac{1}{600000001} = 0.00000000166666666663888888$

1500
1336
1640
1503
1370
1336
340
334
600
501
990
835
1550
1503
470
334
1360
1336
240
167
73

Other endings:

(a) 7 and 3

(1) $\frac{3}{37}$

Current Method

37) 3.00 (.081
2.96
40
37
3

$\therefore \frac{3}{37} = 0.\dot{0}8\dot{1}$

Vedic Method

$\frac{3}{37} = \frac{9}{111}$ (7 ending converted to 1 ending)

$A.F = \frac{8}{110} = \frac{0.8}{11}$

$\therefore \frac{23}{167} = 0.1377245508982035928\dot{1}$

$\frac{3}{37} = 0.\dot{0}8\dot{1}$

Vedic Method

(2) $\frac{23}{167} = \frac{69}{501}$

$A.F = \frac{68}{500} = \frac{0.68}{5}$

Current Method

167) 23.0 (.1377245508982035928
0.13 77 24 55 08 98 20 35 92 81 ......
3 1 2 0 4 1 1 4 4 2
16 7
6 30
5 01
1 290
1 169
1210
1169
410
334
760
668
920

$\therefore \frac{23}{167} = 0.1377245508982035928\dot{1}$

(3) $\frac{1}{33}$

<u>Current Method</u>

$33)1.00(\cdot 03$
$\underline{99}$
$\underline{1}$

$\therefore \frac{1}{33} = 0.\dot{0}\dot{3}$

<u>Vedic Method</u>

$\frac{1}{33} = \frac{3}{99}$

A.F = $\frac{0.3}{10}$

$3^{0.0}$ 0 $^{3}/_{3}$ 0

$\therefore \frac{1}{33} = 0.\dot{0}\dot{3}$

(4) $\frac{71}{533}$

<u>Current Method</u>

533) 71.0(·1332082551594746
53 3
17 70
15 99
1 710
1 599
1110
1066
4400
4264
1360
1066
2940
2665
2750
2665
850
533
3170
2665
5050
4797

2530
2132
3980
3731
2490
2132
3580
3198
382

$\therefore \frac{71}{533} = 0.1332082551594746$

Vedic Method

$\frac{71}{533} = \frac{213}{1599}$

$A.F = \frac{2.13}{16}$

0.13	32	08	25	51	59	47	46	....
5	1	4	8	9	7	7	11	

$\therefore \frac{71}{533} = 0.1332082551594746$

Ending in 8:

(1) $\frac{3}{38}$

Current Method

38)3.00(·0789473684210526
2 66
340
304
360
342
180
152
280
266
140
114
260
228
320
304

160
<u>152</u>
80
<u>76</u>
40
<u>38</u>
200
<u>190</u>
100
<u>76</u>
240
<u>228</u>
<u>12</u>

$\therefore \frac{3}{38} = 0.0789473684210526$

<u>Vedic Method</u>

Problem is $\frac{3}{38}$

In this context (denominator ending with 8) first convert it to Auxiliary Fraction which is $\frac{0.3}{4}$. Now the devisor is 4. We shall have to prefix the remainder (R) to the quotient (Q) and read it as RQ to get the result. Then the new dividend is got by adding the quotient to the result first obtained. (RQ+1Q).

Step1: 0 is the quotient (Q) 3 is the remainder(R)
Step2: Prefix the remainder3 and write 30 in the order RQ.
Step3: Adding the quotient (Q) 0 to the first result i.e (30 + 0 =30).
Step4: As quotient is zero (step1) then this processes is continued i.e divide 30 by 4.
Step5: 7 is the quotient and 2 is the remainder.
Step6: Prefix the remainder 2, write the quotient in the order RQ i.e 27.Then add the quotient to this, which is 27+7=34 as the dividend for further division by 4 and so on.

0.0	7	8	9	4	7	3	6	8	4	2	1	0	5	2	6......
3	2	2	0	2	0	2	2	0	0	0	0	2	0	2	0

$\therefore \frac{3}{38} = 0.0789473684210526$

(2) $\frac{17}{798}$

<u>Current Method</u>

```
798) 17.00 (.021303258145363408
      15 96
       1 040
         798
         2420
         2394
          2600
          2394
           2060
           1596
            4640
            3990
             6500
             6384
              1160
               798
              3620
              3192
               4280
               3990
                2900
                2394
                 5060
                 4788
                  2720
                  2394
                   3260
                   3192
                    6800
                    6384
                     416
```

$\therefore \frac{17}{798} = 0.021303258145363408$

<u>Vedic Method</u>

$\frac{17}{798}$ A.F = $\frac{0.17}{8}$

0.02	13	03	25	81	45	36	34	08	
1	0	2	6	2	2	2	0	4	

$\therefore \frac{17}{798} = 0.021303258145363408$

Ending in 7:

(1) $\frac{5}{27}$

<u>Current Method</u>

$$\begin{array}{r l}
27)\,5.0\,(\cdot 185 & \\
\underline{2\,7} & \\
2\,30 & \\
\underline{2\,16} & \\
140 & \\
\underline{135} & \\
\underline{\;\;5} &
\end{array}$$

$\therefore \frac{5}{27} = 0.\dot{1}8\dot{5}$

<u>Vedic Method</u>

Problem is $\frac{5}{27}$

In this context (denominator ending with 7) first convert it to Auxiliary fraction which is $\frac{0.5}{3}$. Now the devisor is 3. We shall have to prefix the remainder and add to the quotient (Q) to get the result. Then the new dividend is got by adding twice the quotient to the result first obtained. (RQ+2Q).

Step1: 0.1 is the quotient (Q) 2 is the remainder.

Step2: Prefix the remainder 2 and write quotient in the order RQ i.e 21

Step3: Adding twice the quotient Q (2*1=2) to the first result i.e21+2=23 as dividend for further dividing by 3 and so on.

$$\begin{array}{cccccccc}
 & 0.1 & & 8 \;/ & & 5 & & 1 \\
2 & & \bar{1} & / & \bar{1} & & 2 &
\end{array}$$

$\therefore \frac{5}{27} = 0.\dot{1}8\dot{5}$

(2) $\frac{135}{7997}$

<u>Current Method</u>

$$\begin{array}{r}
7997)\,135.00\,(\cdot 016881330498937101 \\
\underline{79\,97} \\
55\,030 \\
\underline{47\,982} \\
7\,0480 \\
\underline{6\,3976} \\
65040 \\
\underline{63976} \\
10640 \\
\underline{7997} \\
26430 \\
\underline{23991}
\end{array}$$

24390
<u>23991</u>
39900
<u>31988</u>
79120
<u>71973</u>
71470
<u>63976</u>
74940
<u>71973</u>
29670
<u>23991</u>
56790
<u>55979</u>
8110
<u>7997</u>
11300
<u>7997</u>
<u>3303</u>

$$\therefore \frac{135}{7997} = 0.016881330498937101$$

<u>Vedic Method</u>

$\frac{135}{7997}$ A.F = $\frac{0.135}{8}$

0.016	881	330	498	936	1101
7	0	3	6	6	0

$$\therefore \frac{135}{7997} = 0.016881330498937101$$

Ending in 6:

(1) $\frac{23}{96}$

Current Method

```
96)23.0(·239583
    19 2
     3 80
     2 88
       920
       864
        560
        480
         800
         768
          320
          288
           32
```

$\therefore \frac{23}{96} = 0.23958\dot{3}$

Vedic Method

Problem is $\frac{23}{96}$

In this context (denominator ending with 6) first convert it to Auxiliary Fraction which is $\frac{2.3}{10}$. Now the devisor is 10. We shall have to prefix the remainder to the quotient (Q) to get the result. Then the new dividend is got by adding thrice the quotient to the result first obtained. (RQ+3Q).

For example 23 divide with 10

Step1: 2 is the quotient (Q) 3 is the remainder

Step2: Prefix the remainder3 and write quotient in the order RQ i.e32

Step3: Adding the thrice the quotient Q (3*2=6) to the first result i.e 32 + 6 =38, as dividend for further division by 10 and so on.

```
   0.2   3   9   5   8   3   3
 3     8   2   6   0   2   2
```

$\therefore \frac{23}{96} = 0.23958\dot{3}$

(2) $\dfrac{46}{39996}$

Current Method

39996)46.000(·0011501
39 996
6 0040
3 9996
2 00440
1 99980
46000
39996
6004

$\therefore \dfrac{46}{39996} = 0.0\dot{0}11\dot{5}$

Vedic Method

$\dfrac{46}{39996}$ A.F = $\dfrac{0.0046}{4}$

$_{2}0.0011 \quad _{0}5011 \quad _{0}5011 \quad \ldots\ldots$

$\therefore \dfrac{46}{39996} = 0.0\dot{0}11\dot{5}$

Ending in 2:

(1) $\dfrac{23}{5002}$

Current Method

5002)23.000(·00458160735705717
20 008
2 9920
2 5010
49100
45018
40820
40016
8040
5002
30380
30012
36800
35014
17860
15004

28540
<u>25010</u>
35300
<u>35014</u>
28600
<u>25010</u>
35900
<u>35014</u>
8860
<u>5002</u>
38580
<u>35014</u>
<u>3566</u>

$\therefore \frac{23}{5002} = 0.004598160735705717$

<u>Vedic Method</u>

Problem is $\frac{23}{5002}$

In this context (denominator ending with 2) first convert it to Auxiliary fraction, which is $\frac{0.022}{5}$. Now the devisor is 5, we shall have to prefix the remainder (R) and write the 9's complement to the quotient (Q) and subtract the quotient in each step. Put it as R (9's complement of Q) then subtract Q from it to get the dividend.

Step1: 0.004 is the quotient (Q) 2 is the remainder

Step2: Prefix the remainder2 and write 9's complement to the quotient (004) i.e. 995 writing in the order RQ which is 2995.

Step3: Then subtract the quotient (Q) from 2995. One gets 2991, as the dividend for further division by 5.

5)0.022(0.004
<u>0.020</u>
<u>2</u>

ND = 2995
<u>– 004</u>
<u>2991</u>

0.004	598	160	735	705	717 ....
2	1	3	4	4	4

$\therefore \frac{23}{5002} = 0.004598160735705717$

(2) $\frac{3}{1200002}$

<u>Current Method</u>

```
1200002)3.000000(·000002499995833334027
         2 400004
           5999960
           4800008
           11999520
           10800018
            11995020
            10800018
             11950020
             10800018
              11500020
              10800018
               70000020
               60000010
               10000100
                9600016
                4000840
                3600006
                 4008340
                 3600006
                  4083340
                  3600006
                   4833340
                   4800008
                     3333200
                     2400006
                      9331940
                      8400014
                       931926
```

$\therefore \frac{3}{1200002} = 0.000002499995833334027$

<u>Vedic Method</u>

$A.F = \frac{2}{1200000} = \frac{0.00002}{12}$

$$\begin{array}{llll} 0.00000 & 24999 & 95833 & 34027 \ldots\ldots \\ \quad 2 & \quad 11 & \quad 5 & \quad 9 \end{array}$$

$\therefore \frac{3}{1200002} = 0.000002499995833334027$

Ending in 3:

(1) $\frac{53}{8003}$

Current Method

```
8003)53.000(·006622516556291390
      48 018
       4 9820
       4 8018
         18020
         16006
          20140
          16006
           41340
           40015
            13250
             8003
             52470
             48018
              44520
              40015
               45050
               40015
                50350
                48018
                 23320
                 16006
                  73140
                  72027
                   11130
                    8003
                   31270
                   24009
                    72610
                    72027
                     58300
```

$\therefore \frac{53}{8003} = 0.00662251655629390$

Vedic Method

Problem is $\frac{53}{8003}$

In this context (denominator ending with 3) first convert it to Auxiliary Fraction which is $\frac{52}{8000} = \frac{0.052}{8}$. Now the devisor is 8. We shall have to prefix the remainder (R) and write the 9's complement to the quotient (Q) and subtract twice the quotient in each step. Put it as R(9's complement of Q) then subtract 2Q from it to get the dividend.

Step1: 0.006 is the quotient (Q) 4 is the remainder

Step2: Prefix the remainder 4 and write 9's complement to the quotient (006) i.e. 993 write in the order RQ which is 4993.

Step3: Then subtract twice the quotient (2Q=2*0.006) 012 from 4993; one gets 4993-012=4981, as the dividend for further division by 8.

0.006	622	516	556	291	390 ......
4	5	5	3	3	6

$$\therefore \frac{53}{8003} = 0.006622516556291390$$

(2) $\frac{7}{6000003}$

<u>Current Method</u>

```
6000003) 7.000000(.0000011666660833333624999
          6 000003
            9999970
            6000003
            39999670
            36000018
             39996520
             36000018
              39965020
              36000018
               39650020
               36000018
                36500020
                36000018
                  50000200
                  48000024
                   20001760
                   18000009
                    20017510
                    18000009
                     20175010
                     18000009
                      21750010
                      18000009
                       37500010
                       36000018
                        14999920
                        12000006
                         29999140
                         24000012
```

Step2: Prefix the remainder2 and write 9's complement to the quotient (0) i.e. 9 write in the order RQ i.e 29
Step3: Then subtract thrice the quotient (3Q=0*3=0) from 29 , one gets 29-0=29 as the dividend for further division by 7.

$$ {}_{2}0.{}_{1}0\;{}_{3}4\;{}_{4}0\;5 \;/\; {}_{1}4\;{}_{3}0\;{}_{4}5 $$

$\therefore \frac{3}{74} = 0.0\dot{4}0\dot{5}$

2) $\frac{25}{16004}$

Current Method

16004)25.000(.001562109472631842
16 004
8 9960
8 0020
99400
96024
33760
32008
17520
16004
151600
144036
75640
64016
116240
112028
42120
32008
101120
96024
50960
48012
29480
16004
134760
128032
67280
64016
32630
32008
622

$\therefore \frac{25}{16004} = 0.001562109472631842$

<u>Vedic Method</u>

$\frac{25}{16004}$ A.F $= \frac{24}{16000} = \frac{0.024}{16}$

0.001	562	109	472	631	842 ....
8	3	7	11	15	3

$\therefore \frac{25}{16004} = 0.001562109472631842$

Astounding Application

In the case the numbers are neither immediately below nor above a ten-power base or a multiple, the Anurupya sutra is applied as follows

For example let us consider $\frac{27}{78}$

$AF = \frac{2.7}{8}$

<u>Step 1</u>:

Dividing 2.7 by 8 we get the first quotient as 3 and remainder as 3 as shown below $\frac{2.7}{8}$

.3
3

i.e., prefixing the remainder on the left hand side of the quotient.

<u>Step 2</u>:

The dividend now will not be 33 but it will be 33 + 3 which is 36 dividing this by 8 we get the next quotient as 4 and the remainder as 4.

Which is shown as .3 4
3 4

<u>Step 3</u>:

Now the dividend is not 44 but 44 + 4 = 48.
This is divided by 8 gives 6 quotients and 0 remainder.

0.3 4 6
3 4 0

<u>Step 4</u>:

Now the dividend is not 6 but 6 + 6 = 12.
Dividing 12 by 8 we get 1, 4 as quotient and remainder respectively.

0.3 4 6 1
3 4 0 4

Step 5:

Now dividing 41 + 1 = 42 by 8 we get 5 as a quotient and 2 as remainder.

0.3 4 6 1 5
3 4 0 4 2

We can continue this procedure to any required decimal of our choice

∴ F = 0.3 4 6 1 5 3 8 4 6 1 5
3 4 0 4 2 6 2 4 0 4 2

Let us see the conventional method.

```
78) 27.0(0.346153846153-----etc
    234
     360
     312
      480
      468
       120
        78
        420
        390
         300
         234
          660
          624
           360
           312
            480
            468
             120
              78
              420
              390
               30
```

A few more illustrative examples

1) $\frac{78}{117}$

Conventional method

```
117)870(0.7435897
     819
      510
      468
       420
       351
```

```
  690
  585
   1050
    936
   1140
    1053
      870
      819
```

Vedic Method

$\frac{78}{117}$, AF = $\frac{8.7}{12}$

F = 0.7 4 3 5 8 9 7 4 3 5
      3 3 6 9 9 6 3 3 0 9

Here we add twice the quotient digit (since 117 is 2 less from 119).
We also observe that this is a recurring decimal of 0.743589

i.e. $\frac{78}{117} = 0.\dot{7}4358\dot{9}$

2) $\frac{123}{497}$ AF = $\frac{1.23}{5}$

Here we group 2 digits at a time as 497 is nearer to 499. Add twice the 2 digit at every step.

Vedic Method: -

$\frac{123}{497} \Rightarrow \frac{1.23}{5}$

.24	74	84	90	94
3	2	2	2	0

Conventional Method:

```
447) 1230(0.24 | 74 | 84 | 90 | 94
      994
      2360
      1988
       3720
       3479
        2410
        1988
         4220
```

```
 3976
 ----
 2440
 1998
 ----
 4520
 4473
 ----
 4700
 4473
 ----
 2270
 1988
 ----
  282
```

3) $\frac{5432}{6997}$ AF = $\frac{5.432}{7}$

F = $\underset{0}{0.776} \Big|\ \underset{4}{332} \Big|\ \underset{5}{713} \Big|$

```
6997) 54320 (0.776 | 332 | 71 |
      48979
      -----
       54310
       48979
       -----
        44310
        41982
        -----
         23280
         20991
         -----
          22890
          20991
          -----
           18990
           13994
           -----
            49960
            48979
            -----
             9810
             6997
             -----
             28130
```

Section-3

DIVISIBILITY AND SIMPLE OSCULATORS

There are certain general rules for finding out the divisibility of a number by 2, 5, 10, 3, 6, 9, 18, 11, 12 and 22 and so on. These are well known and understood. The advanced part of the subject is dealt by Swamiji with the introduction of the term 'Vestanas' means osculators. The operation of which is a tool for divisibility and is dealt with elaborately. The classification of these are positive osculator, negative osculator and complex multiplex osculators. The explanation and modus operandi is detailed below.

The Ekadhika purvena sutram is useful in determining the divisibility of certain given dividend by the given divisor.

The Ekadhika for seven (7) is derived from $7\times7 = 49$ $\therefore 5$ is the Ekadhika of 7 We have understood in the chapter on recurring decimals such as 1/19, 1/29, 1/39 etc. The Ekadhika is 2,3,4 etc. So in order to get the recurring decimal of 1/7 one can convert it to 7/49 which is equal to $7\times 1/49$ thus we can understand that we can take the Ekadhika of 7 as 5. The process by means of which one can identify the Ekadhika for a number and this process is called Vestana or osculators.

Let us consider a number 21. If we want to know whether the number, is divisible by 7 or not.

The following are the steps.

Step1: Consider the osculator of 7 which is 5.

Step2: Multiply the last digit of given number with the osculator.

Step3: The result is added to the remaining number. The process is called osculation. If the result of osculation is equal to the divisor or a repetition of previous result we say that the original number is divisible by 7.

A few examples to show the divisibility by 7 are given below.

14; $4\times5+1=21$ and $1\times5+2=7$ $\therefore$ yes.

63; $3\times5+6=21$ (already done)

868; $8\times5+86=126$; $6\times5+12=42$

$2\times5+4=14$

3983; $3\times5+398=413$; $3\times5+41=56$

$6 \times 5 + 5 = 35$

$5 \times 5 + 3 = 28$

$5 \times 8 + 2 = 42$

3984; $20 + 398 = 418$; $8 \times 5 + 41 = 81$

$5 \times 1 + 8 = 13$

We know that 13 being 1 less than 14 is not divisible.

1815436; $30 + 181543 = 181573$

$15 + 18157 = 18172$

$10 + 1817 = 1827$

$35 + 182 = 217$

$35 + 21 = 56$ (Divisible)

The table for the divisibility of a number by 17

The osculator for 17 is 12

(Conversion of 17 to 9 ending gives $7 \times 17 = 119$ and when Ekadhika considered is 12

$\therefore$ Osculator is 12)

17; $7 \times 12 + 1 = 85$

$5 \times 12 + 8 = 68$

$8 \times 12 + 6 = 102$

$2 \times 12 + 10 = 34$

$4 \times 12 + 3 = 51$

$1 \times 12 + 5 = 17$

34; already dealt; yes divisible

51; already dealt; Yes

68; already dealt; Yes

85; already dealt; Yes

102; already dealt; Yes

119; $9 \times 12 + 11 = 119$ (Itself $\therefore$ divisible)

3978; $8 \times 12 + 397 = 493$

$3 \times 12 + 49 = 85$ (already dealt) $\therefore$ Yes

3763; $3 \times 12 + 376 = 412$

$2 \times 12 + 41 = 65$ ($\therefore$ 65 is 3 less 68 it is not divisible) $\therefore$ No

Examples of the osculation procedure (Vestana):

17 Continuously osculated by 12 gives 85, 68, 102, 34, 51, 17 etc

9 (by 5) gives 45,29,47,39,48,44,24,22,12,11 etc

64 (by 7) gives 34, 31,10,8 etc

82 (by 3) gives 20,2,6,18,25,17,22,8,24 etc

231 (by 15) gives 38,123,57,110,26 etc

4364 (by 12) gives 484,96,81,20,2 etc

91974 (by 11) gives 9241, 935, 148, 102, 32, 25 etc

The Process can be continued as soon as we reach a comparatively smaller number which gives the clue for the divisibility or otherwise by the divisor whose Ekadhika is the osculator.

An important role is thus played by Ekadhika in the problem on divisibility.

Rules for Ekadhikas:

<u>(1) 9 ending</u>: If the divisor is 9 ending then its Ekadhika is 1+ the preceding number.

For example 239, 23+1 = 24 is Ekadhika of 239

<u>(2) 3 ending</u>: If the divisor is 3 ending then convert it into 9 ending by multiplying it with 3 and then apply rule (1).

For example 123, 369 (123×3)

36+1 = 37 is Ekadhika of 1 2 3

<u>(3) 7 ending</u>: If the divisor is 7 ending then convert it into 9 ending by multiplying it with 7 and then apply rule (1).

For example 47, $47\times7 = 329$

32+1 = 33 is Ekadhika

<u>(4) Ending</u>: If the divisor is 1 ending then convert it into a 9 ending by multiplying it with 9 and then apply rule (1).

For example 71,639 (71×9)

63+1= 64 is Ekadhika

Osculation by own Ekadhika:

It's interesting to note that Osculation of any number by its own Ekadhika will give the number itself or a multiple.

Let us consider a number 63

63 osculated by its Ekadhika 19 gives 63

Similarly

126 Osculated by 19 gives 126

252 Osculated by 19 gives 63

189 Osculated by 19 gives 189

Operation of Osculation:

Let us consider the number 701119 to find out whether this number is divisible by 19 or not.

Put down the given number 701119

Consider the Ekadhika of the divisor, which is 2.

Multiply the last digit of the given number by 2 and then add the result to the previous digit 1. The result is $9\times2+1=19$.

Place the result under the second digit from right hand side.

7	0	1	1	1	9
				19	

Multiply 19 by 2 and add the third digit (1). The result is $19\times2+1=39$

Now cast down 19's from 39. Which gives the remainder as '1'.

This remainder is to be put under the third digit. Now $2\times1+1$ gives 3.

3 is to be placed under the 4th digit

7	0	1	1	1	9
19	6	3	39	19	
			(1)		

$2\times3+0=6$. This is placed under zero.

$2\times6+7=19$. When we get either 19 or multiple of 19 the given number is divisible by19.

(or)

The same result is obtained by osculating (by itself). The procedure is as follows.

The first three steps adopted in the first method are followed.

We reach the step of 19 which is put under the 2nd digit from right

7	0	1	1	1	9
				19	

Now osculating the result 19 itself and casting the 19's from it we get 1

i.e. $(9\times2+1)+1=20-19=1$.

Now put down 1 under 3rd digit from right

7	0	1	1	1	9
			1	19	

Now osculating the result '1'
Itself and adding the previous digit we get
$2\times 1 = 2+1 = 3$

Now put the result 3 under the 4th digit

7	0	1	1	1	9
		3	1	19	

Osculating the result 3
And adding the previous
Digit '0' of the number
We get $2\times 3+0 = 6$

Put the 6 under the 5th digit

7	0	1	1	1	9
	6	3	1	19	

Now osculating 6 by '2'
And adding 7 the previous digit we get 19
i.e. $2\times 6+7 = 12+7 = 19$

Now put down 19 under 7

7	0	1	1	1	9
19	6	3	1	19	

As 19 is the last digit
The given number is divisible by 19.

Let us consider another example.

Is 8759975764 divisible by 2489.

The Ekadhika is 249

Step1:

8 7 5 9 9 7 5 7 6 4

$4\times 249 = 996+6 = 1002$

Step2:

8 7 5 9 9 7 5 7 6 4

605 1002

$2\times 249+100 = 598+7 = 605$

Step3:

8 7 5 9 9 7 5 7 6 4

1310 605 1002

$5 \times 249 + 60 = 1305 + 5 = 1310$

Step4:

8 7 5 9 9 7 5 7 6 4

138 1310 605 1002

$0 \times 249 + 131 = 131 + 7 = 138$

Step5:

8 7 5 9 9 7 5 7 6 4

2014 138 1310 605 1002

$8 \times 249 + 13 = 2005 + 9 = 2014$

Step6:

8 7 5 9 9 7 5 7 6 4

1206 2014 138 1310 605 1002

$4 \times 249 + 201 = 1197 + 9 = 1206$

Step7:

8 7 5 9 9 7 5 7 6 4

1619 1206 2014 138 1310 605 1002

$6 \times 249 + 120 = 1614 + 5 = 1619$

Step8:

8 7 5 9 9 7 5 7 6 4

2409 1619 1206 2014 138 1310 605 1002

$9 \times 249 + 161 = 2402 + 7 = 2409$

Step9:

8 7 5 9 9 7 5 7 6 4

(2489) 2409 1619 1206 2014 138 1310 605 1002

$9 \times 249 + 240 = 2481 + 8 = 2489$

Hence it is divisible.

Is 1 5 2 9 6 1 9 8 8 divisible by 1239

The Ekadhika is 124

1 5 2 9 6 1 9 8 8
1000
(124 × 8 = 992 + 8 = 1000)

1 5 2 9 6 1 9 8 8
109 1000
124 × 0 + 100 = 100 + 9 = 109

1 5 2 9 6 1 9 8 8
1127 109 1000
124 × 9 + 10 = 1126 + 1 = 1127

1 5 2 9 6 1 9 8 8
986 1127 109 1000
124 × 7 + 112 = 980 + 6 = 986

1 5 2 9 6 1 9 8 8
851 986 1127 109 1000
124 × 6 + 98 = 842 + 9 = 851

1 5 2 9 6 1 9 8 8
211 851 986 1127 109 1000
124 × 1 + 85 = 209 + 2 = 211

1 5 2 9 6 1 9 8 8
150 211 851 986 1127 109 1000
124 × 1 + 21 = 145 + 5 = 150

1 5 2 9 6 1 9 8 8
16 150 211 851 986 1127 109 1000

∴ The given number is not divisible by 1239

Some more examples:

(1) Is 2 3 4 4 6 divisible by 19

Osculator = 2

2 3 4 4 6
19 18 17 16 ∴ yes

Since 19 is the final result the given number is divisible by 19

(2) Is 3 5 8 0 0 5 divisible by 29

Osculator is 3

3	5	8	0	0	5	
<u>29</u>	28	27	16	15		∴ yes

∴ The number is divisible by 29

(3) Is 4 8 1 5 5 5 divisible by 39

Osculator is 4

4	8	1	5	5	5	
<u>25</u>	15	31	27	25		∴ No

∴ The number is not divisible by 39

(4) Is 6 0 4 9 5 4 divisible by 49

Osculator is 5

6	0	4	9	5	4	
<u>49</u>	38	37	36	25		∴ Yes

∴ The number is divisible by 49

(5) Is 7 2 8 1 8 3 8 divisible by 59

Osculator is 6

7	2	8	1	8	3	8	
<u>13</u>	60	49	56	19	51		∴ No

∴ The number is not divisible by 59.

(6) Is 6 8 1 3 1 2 0 divisible by 69

Osculator is 7

6	8	1	3	1	2	0	
<u>33</u>	63	67	39	15	2		∴ No

∴ The number is not divisible by 69

(7) Is 4 5 0 1 7 3 6 divisible by 79

Osculator is 8

4	5	0	1	7	3	6	
<u>79</u>	39	24	3	20	51		∴ Yes

∴ The number is divisible by 79

(8) Is 7 8 2 7 9 2 3 8 divisible by 89

Osculator is 9

7	8	2	7	9	2	3	8	
89	19	21	12	50	54	75		∴ Yes

∴ The number is divisible by 89

(9) Is 5 6 2 3 0 3 1 7 divisible by 99

Osculator is 10

5	6	2	3	0	3	1	7	
99	49	34	23	2	20	71		∴ Yes

∴ The given number is divisible by 99

(10) Is 1 4 2 3 7 6 0 7 3 divisible by 159

Osculator is 16

1	4	2	3	7	6	0	7	3	
159	149	19	11	80	94	85	55		∴ Yes

∴ The given number is divisible by 159

Positive Osculator (p)

For any given number, its Ekadhika is the positive Osculator (p)

Consider 19 Ekadhika = positive (Vestana) Osculator = 2

29 P = 3
79 P = 8
23 $23 \times 3 = 69$ P = 7
53 $53 \times 3 = 159$ P = 16
37 $37 \times 7 = 259$ P = 26
17 $17 \times 7 = 119$ P = 12
41 $41 \times 9 = 369$ P = 37
21 $21 \times 9 = 189$ P = 19

Worked examples:

(1) Is 63 Divisible by 7?
Divisor = 7
$7 \times 7 = 49$
∴ P = 5
$3 \times 5 + 6 = 21$
$1 \times 5 + 2 = 7$ ∴ YES

(2) Is 108 Divisible by 9?
Divisor = 9
P = 1
$8 \times 1 + 10 = 18$
$8 \times 1 + 1 = 9$ ∴ YES

(3) Is 252 Divisible by 3?
Divisor = 3
$3 \times 3 = 9$
P = 1
$2 \times 1 + 25 = 27$
$7 \times 1 + 2 = 9$ ∴ YES

(4) Is 79 Divisible by 7?
Divisor = 7
$7 \times 7 = 49$
P = 5
$9 \times 5 + 7 = 52$
$2 \times 5 + 5 = 15$ ∴ NO

(5) Is 608 Divisible by 19?
Divisor = 19
P = 2

6 0	8	
19	16	∴ YES

Or the osculation results are 76, 19.

(6) Is 3828 Divisible by 29?
Divisor = 29
P = 3

3 8	2	8	
29	28	26	∴ YES

(7) Is 4962 Divisible by 39?
Divisor = 39
P = 4

4 9	6	2	
30	26	14	∴ NO

(8) Is 26117 Divisible by 49?
Divisor = 49
P = 5

2 6 1 1 7
49 29 34 36 ∴ YES

(9) Is 9062587 Divisible by 59?
Divisor = 59
P = 6

9 0 6 2 5 8 7
47 26 24 3 10 50 ∴ NO

(10) Is 367425 Divisible by 69?
Divisor = 69
P = 7

3 6 7 4 2 5
69 39 54 56 37 ∴ YES

(11) Is 10098762 Divisible by 79?
Divisor = 79
P = 8

1 0 0 9 8 7 6 2
59 27 33 14 50 25 22 ∴ NO

(12) Is 9427592 Divisible by 89?
Divisor = 89
P = 9

9 4 2 7 5 9 2
89 88 39 14 70 27 ∴ YES

(13) Is 567021361 Divisible by 99?
Divisor = 99
P = 10

5 6 7 0 2 1 3 6 1
49 44 83 67 76 47 64 16 ∴ NO

(14) Is 7128950762 Divisible by 109?
Divisor = 109
P = 11

7 1 2 8 9 5 0 7 6 2
109 39 53 74 115 79 86 97 28 ∴ YES

(15) Is 2198760351 Divisible by 119?

Divisor = 119

P = 12

2	1	9	8	7	6	0	3	5	1	
66	45	83	26	61	64	104	88	17		∴ NO

(16) Is 210023739 Divisible by 129?

Divisor = 129

P = 13

2	1	0	0	2	3	7	3	9	
129	109	48	93	27	121	19	120		∴ YES

(17) Is 81021675 Divisible by 139?

Divisor = 139

P = 14

8	1	0	2	1	6	7	5	
51	13	120	88	26	111	77		∴ NO

(18) Is 3024 Divisible by 21?

Divisor = 21

$21 \times 9 = 189$

P = 19

3	0	2	4	
189	159	78		∴ YES

(19) Is 1657953 Divisible by 33?

Divisor = 33

$33 \times 3 = 99$

P = 10

1	6	5	7	9	5	3	
99	89	38	33	62	35		∴ YES

(20) Is 20978204 Divisible by 17?

Divisor = 17

$17 \times 7 = 119$

P = 12

2	0	9	7	8	2	0	4	
17	31	72	35	42	102	48		∴ YES

(21) Is 412865107 Divisible by 43?

Divisor = 43

$43 \times 3 = 129$

P = 13

4	1	2	8	6	5	1	0	7	
61	54	14	120	88	46	23	91		∴ NO

Procedure of the above examples:

Is 608 Divisible by 19

$8 \times 2 + 60 = 76$

$6 \times 2 + 7 = 19$

∴ YES

Is 3828 Divisible by 29

$8 \times 3 + 382 = 406$

$6 \times 3 + 40 = 58$

$8 \times 3 + 5 = 29$

∴ YES

Is 4962 Divisible by 39

$2 \times 4 + 496 = 504$

$4 \times 4 + 50 = 66$

$6 \times 4 + 6 = 30$

∴ NO

Is 26117 Divisible by 49

$7 \times 5 + 2611 = 2646$

$6 \times 5 + 264 = 294$

$4 \times 5 + 29 = 49$

∴ YES

Is 9062587 Divisible by 59

$7 \times 6 + 906258 = 906300$

$0 \times 6 + 90630 = 90630$

$0 \times 6 + 9063 = 9063$

$3 \times 6 + 906 = 924$

$4 \times 6 + 92 = 116$

$6 \times 6 + 11 = 47$

$7 \times 6 + 4 = 46$

∴ NO

Is 367425 Divisible by 69

$5 \times 7 + 36742 = 36777$
$7 \times 7 + 3677 = 3726$
$6 \times 7 + 372 = 414$
$4 \times 7 + 41 = 69$
$\therefore$ YES

Is 10098762 Divisible by 79

$2 \times 8 + 1009876 = 1009892$
$2 \times 8 + 100989 = 101005$
$5 \times 8 + 10100 = 10140$
$0 \times 8 + 1014 = 1014$
$4 \times 8 + 101 = 133$
$3 \times 8 + 13 = 37$
$\therefore$ NO

Is 9427592 Divisible by 89

$2 \times 9 + 942759 = 942777$
$7 \times 9 + 94277 = 94340$
$0 \times 9 + 9434 = 9434$
$4 \times 9 + 943 = 979$
$9 \times 9 + 97 = 178$
$8 \times 9 + 17 = 89$
$\therefore$ YES

Is 567021361 Divisible by 99

$1 \times 10 + 56702136 = 56702146$
$6 \times 10 + 5670214 = 5670274$
$4 \times 10 + 567027 = 567067$
$7 \times 10 + 56706 = 56776$
$6 \times 10 + 5677 = 5737$
$7 \times 10 + 573 = 643$
$3 \times 10 + 64 = 94$
$\therefore$ NO

Is 7128950762 Divisible by 109

$2 \times 11 + 712895076 = 712895098$
$8 \times 11 + 71289509 = 71289597$
$7 \times 11 + 7128959 = 7129036$
$6 \times 11 + 712903 = 712969$
$9 \times 11 + 71296 = 71395$
$5 \times 11 + 7139 = 7194$
$4 \times 11 + 719 = 763$
$3 \times 11 + 76 = 109$
$\therefore$ YES

Is 2198760351 Divisible by 119

$1 \times 12 + 219876035 = 219876047$
$7 \times 12 + 21987604 = 21987688$
$8 \times 12 + 2198768 = 2198864$
$4 \times 12 + 219886 = 219934$
$4 \times 12 + 21993 = 22041$
$1 \times 12 + 2204 = 2216$
$6 \times 12 + 221 = 293$
$3 \times 12 + 29 = 65$
$\therefore$ NO

Is 210023739 Divisible by 129

$9 \times 13 + 21002373 = 21002490$
$0 \times 13 + 2100249 = 2100249$
$9 \times 13 + 210024 = 210141$
$1 \times 13 + 21014 = 21027$
$7 \times 13 + 2102 = 2193$
$3 \times 13 + 219 = 258$
$8 \times 13 + 25 = 129$
$\therefore$ YES

Is 81021675 Divisible by 139

$5 \times 14 + 8102167 = 8102237$
$7 \times 14 + 810223 = 810321$
$1 \times 14 + 81032 = 81046$
$6 \times 14 + 8104 = 8188$
$8 \times 14 + 818 = 930$
$0 \times 14 + 93 = 93$
$\therefore$ NO

Is 3024 Divisible by 21

$4 \times 19 + 302 = 378$
$8 \times 19 + 37 = 189$
$\therefore$ YES

Is 1657953 Divisible by 33

$3 \times 10 + 165795 = 165825$
$5 \times 10 + 16582 = 16632$
$2 \times 10 + 1663 = 1683$
$3 \times 10 + 168 = 198$
$8 \times 10 + 19 = 99$
$\therefore$ YES

Is 20978204 Divisible by 17

$4 \times 12 + 2097820 = 2097868$
$8 \times 12 + 209786 = 209882$
$2 \times 12 + 20988 = 21012$
$2 \times 12 + 2101 = 2125$
$5 \times 12 + 212 = 272$
$2 \times 12 + 27 = 51$
$1 \times 12 + 5 = 17$
$\therefore$ YES

Is 412865107 Divisible by 43

$7 \times 13 + 41286510 = 41286601$
$1 \times 13 + 4128660 = 4128673$
$3 \times 13 + 412867 = 412906$
$6 \times 13 + 41290 = 41368$
$8 \times 13 + 4136 = 4240$
$0 \times 13 + 424 = 424$
$4 \times 13 + 42 = 94$
$4 \times 13 + 9 = 61$
$1 \times 13 + 6 = 19$
$\therefore$ NO

Negative Osculator (Q)

An Introduction of second type of osculator which is called as negative osculator (Q) as being distinguished from the already worked out osculator which is positive and is designated as (P).

This negative osculator is obtained through the sutra "Paravartya" and is not the addition as in the case of (Ekadhika) Positive osculator but it is a subtraction left ward

Determination of negative osculation:

Negative osculator consists of clauses.

(i) For all divisors ending in 1. Drop 1 and

(ii) In all other cases multiply the divisor by a digit to give the products as one' ending then drop 1.

Ex: If the divisors are 3, 7, 9 multiply them with 7, 3, and 9 to get the products as one Ending respectively.

In case of one ending like 11, 21, 31, 41, 51, 141, 181, 171, 121, 10891 and soon the Q's are 1, 2, 3, 4, 5, 14, 18, 17, 12 and 1089 respectively.

If it is '7' ending such as, 7,27,127,157,1037 and so on then one has to multiply the given divisor by 3 to convert it to '1' ending and then after removing the ending '1' s we get the Q's as 2,8,38,47,311 respectively.

If it is '3' ending one has to multiply the divisor by 7 to convert it to '1' ending then after dropping the last digit '1's we get the negative osculator.

For example 23, 63, 103, 1133, 1243

The Q's are 16, 44, 72, 793, and 870 respectively

If it is '9' ending one has to multiply the divisor with 9 to convert it into the '1' ending and then after dropping the last digit of '1's we get the Q's

For example 19, 29, 79, 159_ _ _ _ _

The Q's are 17, 26, 71, 143 _ _ _ _ _

It is interesting to note that whatever may be the divisor its P and Q when added then gives divisor itself. i.e. P+Q = D

A table demonstrating the above rule is shown.

Table:

Number (D)	Multiple for P	Multiple for Q	P	Q	Total P + Q = D
9	(9)	81	1	8	9
21	189	(21)	19	2	21
73	219	511	22	51	73
81	729	(81)	73	8	81
121	1089	(121)	109	12	121
143	429	1001	43	100	143
277	1939	831	194	83	277
1049	(1049)	9441	105	944	1049
8967	62769	26901	6277	2690	8967
54153	162459	379071	16246	37907	54153

Few similar examples:

(1) For 29 P = 3, Q = 26

(2) For 67 P = 47, Q = 20

(3) For 23 P = 7, Q = 16

(4) For 71 P = 64, Q = 7

(5) For 139 P = 14, Q = 125

(6) For 109 P = 11, Q = 98

(7) For 153 P = 46, Q = 107

(8) For 1241 P = 1117, Q = 124

(9) For 7469, P = 747, Q = 6722

(10) For 14287, P = 10001, Q = 4286

Process of Negative Osculation:

(1) Is 259266 divisible by 21?

Osculator (Negative) is 2(one digit)

Put 6 under the 2nd digit from right (In the process of negative osculation alternative addition and subtraction is adopted) the process is illustrated from the following examples).

(i) $\bar{2}$ 5 $\bar{9}$ 2 $\bar{6}$ 6

6

$2 \times 6 = 12 - 6 = 6$

(ii) $\bar{2}$ 5 $\bar{9}$ 2 $\bar{6}$ 6

14 6

$2 \times 6 + 2 = 14$

(iii) $\bar{2}$ 5 $\bar{9}$ 2 $\bar{6}$ 6

-2 14 6

$2 \times 4 - 1 - 9 = -2$

(iv) $\bar{2}$ 5 $\bar{9}$ 2 $\bar{6}$ 6

0 1 -2 14 6

$\therefore$ The number is divisible by 21

(2) Is 7162818 divisible by 131 osculator is 13(two digits)

(i) $7 \; \bar{1} \; 6 \; \bar{2} \; 8 \; \bar{1} \; 8$

103

$13 \times 8 = 104 - 1 = 103$

(ii) $7 \quad \bar{1} \quad 6 \quad \bar{2} \quad 8 \quad \bar{1} \quad 8$

$0 \quad 70 \quad 76 \quad 86 \quad 37 \quad 103$

$\therefore$ The number is divisible by 131

(3) Is 75338 divisible by 61?

Osculator is 6

$7 \quad \bar{5} \quad 3 \quad \bar{3} \quad 8$

$\underline{45} \quad 47 \quad 29 \quad 45$

$\therefore$ The number is not divisible by 61

In negative osculation the alternative application of positives and negatives is marked by means of Vinculum from right to left on all even place digits.

A number is said to the divisible by divisor only when the final result is '0' or the itself osculator.

(3) Is 1 3 3 7 9 0 3 is divisible by 81

Osculator is (8)

$1 \quad \bar{3} \quad 3 \quad \bar{7} \quad 9 \quad \bar{0} \quad 3$

$\underline{-1} \quad 20 \quad 13 \quad 62 \quad 39 \quad 24$

$\therefore$ No the number is not divisible by 81

(4) Is the number 3 0 9 8 5 9 5 is divisible by 251

Osculator is 25

$3 \quad \bar{0} \quad 9 \quad \bar{8} \quad 5 \quad \bar{9} \quad 5$

$\underline{0} \quad 30 \quad 202 \quad 78 \quad 144 \quad 116$

$\therefore$ The number is divisible by 251

Negative Osculator (Q):

Consider

41	Negative Osculator (Q) = 4	
91	Q = 9	
53	$53\times7=371$	Q = 37
13	$13\times7=91$	Q = 9
17	$17\times3=51$	Q = 5
87	$87\times3=261$	Q = 26
19	$19\times9=171$	Q = 17
29	$29\times9=261$	Q = 26

Worked examples:

(1) Is 99 Divisible by 11?
Divisor = 11
Q = 1
$9-(9\times1)=0$
∴ YES

(2) Is 105 Divisible by 21?
Divisor = 21
Q = 2
$10-(5\times2)=0$
∴ YES

(3) Is 341 Divisible by 31?
Divisor = 31; Q = 3

$$\begin{array}{ccc} 3 & \bar{4} & 1 \\ 0 & -1 & \end{array}$$

∴ YES

(4) Is 25681 Divisible by 41?
Divisor = 41
Q = 4

$$\begin{array}{ccccc} & 2 & \bar{5} & 6 & \bar{8} & 1 \\ -14 & -4 & 10 & -4 & & \end{array}$$

∴ NO

(5) Is 5036862 Divisible by 51?

Divisor = 51

Q = 5

$5 \quad \bar{0} \quad 3 \quad \bar{6} \quad 8 \quad \bar{6} \quad 2$

$0 \quad -1 \quad 10 \quad 32 \quad 28 \quad 4$

∴ YES

(6) Is 965230134 Divisible by 61?

Divisor = 61; Q = 6

$9 \quad \bar{6} \quad 5 \quad \bar{2} \quad 3 \quad \bar{0} \quad 1 \quad \bar{3} \quad 4$

$-35 \quad -48 \quad -7 \quad -2 \quad 0 \quad 30 \quad 5 \quad 21$

∴ NO

(7) Is 8765376 Divisible by 71?

Divisor = 71; Q = 7

$8 \quad \bar{7} \quad 6 \quad \bar{5} \quad 3 \quad \bar{7} \quad 6$

$71 \quad 9 \quad 53 \quad 27 \quad 35 \quad 35$

∴ YES

(8) Is 1205467123 Divisible by 81?

∴ Divisor = 81; Q = 8

$\bar{1} \quad 2 \quad \bar{0} \quad 5 \quad \bar{4} \quad 6 \quad \bar{7} \quad 1 \quad \bar{2} \quad 3$

$68 \quad 39 \quad 35 \quad 55 \quad 67 \quad 19 \quad 32 \quad 15 \quad 22$

∴ NO

(9) Is 1852477102021 Divisible by 91?

Divisor = 91; Q = 9

$1 \quad 8 \quad 5 \quad \bar{2} \quad 4 \quad \bar{7} \quad 7 \quad \bar{1} \quad 0 \quad \bar{2} \quad 0 \quad \bar{2} \quad 1$

$0 \quad 10 \quad 2 \quad -81 \quad -47 \quad -36 \quad -74 \quad -9 \quad 80 \quad 19 \quad 63 \quad 7$

∴ YES

(10) Is 27691605 Divisible by 27?

Divisor = 27

27 × 3 = 81; ∴ Q = 8

$2 \quad 7 \quad \bar{6} \quad 9 \quad \bar{1} \quad 6 \quad \bar{0} \quad 5$

$27 \quad 34 \quad 54 \quad 48 \quad 15 \quad 2 \quad 40$

∴ YES

(11) Is 5268091 Divisible by 57?

Divisor = 57

$57 \times 3 = 171$; Q = 17

5	$\bar{2}$	6	$\bar{8}$	0	$\bar{9}$	1
31	82	15	81	136	8	

∴ NO

(12) Is 15049383 Divisible by 23?

Divisor = 23

$23 \times 7 = 161$; Q = 16

$\bar{1}$	5	$\bar{0}$	4	$\bar{9}$	3	$\bar{8}$	3
-23	-102	-57	-74	–25	–1	40	

∴ YES

(13) Is 3867374 Divisible by 19

Divisor = 19

$19 \times 9 = 171$; Q = 17

3	$\bar{8}$	6	$\bar{7}$	3	$\bar{7}$	4
–133	- 8	0	60	14	61	

∴ YES (133 = 19 × 7)

Procedure of the above examples:

(3) 341

$34 - 1 \times 3 = 31$ ∴ YES

(4) 25681

$2568 - 1 \times 4 = 2564$

$256 - 4 \times 4 = 240$

$24 - 0 \times 4 = 24$

$2 - 4 \times 4 = -14$ ∴ NO

(5) 5036862

$503686 - 2 \times 5 = 503676$

$50367 - 6 \times 5 = 50337$

$5033 - 7 \times 5 = 4998$

$499 - 8 \times 5 = 459$

$45 - 9 \times 5 = 0$ ∴ YES

(4) 965230134

$96523013 - 6\times4 = 96522989$
$9652298 - 9\times4 = 9652262$
$965226 - 2\times4 = 965218$
$96521 - 8\times4 = 96489$
$9648 - 9\times4 = 9612$
$961 - 2\times4 = 953$
$95 - 3\times4 = 83$
$8 - 3\times4 = -4$ $\therefore$ NO

(7) 8765376

$876537 - 6\times7 = 876495$
$87649 - 5\times7 = 87614$
$8761 - 4\times7 = 8733$
$873 - 3\times7 = 852$
$85 - 2\times7 = 71$ $\therefore$ YES

(8) 1205467123

$120546712 - 3\times8 = 120546688$
$12054668 - 8\times8 = 12054604$
$1205460 - 4\times8 = 1205428$
$120542 - 8\times8 = 120478$
$12047 - 8\times8 = 11983$
$1198 - 3\times8 = 1174$
$117 - 4\times8 = 85$ $\therefore$ NO

(9) 1852477102021

$185247710202 - 1\times9 = 185247710193$
$18524771019 - 3\times9 = 18524770992$
$1852477099 - 2\times9 = 1852477081$
$185247708 - 1\times9 = 185247699$
$18524769 - 9\times9 = 18524688$
$1852468 - 8\times9 = 1852396$
$185239 - 6\times9 = 185185$
$18518 - 5\times9 = 18473$
$1847 - 3\times9 = 1820$
$182 - 0\times9 = 182$
$18 - 2\times9 = 0$ $\therefore$ YES

(10) 27691605

2769160 − 5×8 = 2769120
276912 − 0×8 = 276912
27691 − 2×8 = 27675
2767 − 5×8 = 2727
272 − 7×8 = 216
21 − 6×8 = −27 ∴ YES

(11) 5268091

526809 − 1×17 = 526792
52679 − 2×17 = 52645
5264 − 5×17 = 5179
517 − 9×17 = 364
36 − 4×17 = −32 ∴ NO

(12) 15049383

1504938 − 3×16 = 1504890
150489 − 0×16 = 150489
15048 − 9×16 = 14904
1490 − 4×16 = 1426
142 − 6×16 = 46
46 = 2×23 ∴ YES

(13) 3867374

386737 − 4×17 = 386669
38666 − 9×17 = 38513
3851 − 3×17 = 3800
380 − 0×17 = 380
38 − 0×17 = 38
38 = 2×19 ∴ YES

DIVISIBILITY AND COMPLEX MULTIPLE OSCULATORS

Coming to sizeable divisors the osculators also accordingly larger. As such a scheme of groups of digits which can be osculated not as individual digits but in lumps.

Examples of Multiplex Vestana

(1) 123456 Osculated by 4(P) for 2 digits at a time gives 224 + 1234 = 1458

(2) 74070 Osculated by 6(P) for 3 digits at a time gives 420 + 74 = 494

(3) 74070 Osculated by 6(Q) for 3 digits gives +74 - 420 = -346

(4) 21674382156 Osculated by 5(P) for 4 digits gives 10780 + 2167438 = 2178218

(5) 1684324163 Osculated by 8(Q) for 5 digits gives 16843 – 193304 = -176461

(6) 241687 Osculated by 7(p) for 3 digits gives 4809 + 241 = 5050

(7) 84569 Osculated by 8(Q) for 3 digits gives 84 – 4552 = -4468

(8) 2301 Osculated by 4(P) for 5 digits gives 0 + 9204 = 9204

(9) 14886 Osculated by 3(Q) for 6 digits gives 0 – 44658 = -44658

(10) 123456789 Osculated by 7(Q) for 10 digits gives 0 – 864197523 = -864197523

DIFFERENT CATEGORIES OF DIVISERS AND THEIR OSCULATORS

This is in relation to the positive and negative Osculators which suit the divisors.

(i) Those which end in nine or a series of nines come under Ekadhika type Osculator or positive Osculator.

(ii) Those which terminate in 1 or contain series of zeros ending in 1 come under the jurisdiction of negative Osculator or 'Viparita'.

(iii) Those which by suitable multiplication yield a multiple of either of the two sorts described in (i) and (ii)

First type

Let us consider the divisor 399999 its Osculator p is 4 and covers 5 digits symbolically $p_5 = 4$

Few more examples

(i)	33799	$P_2 = 338$
(ii)	279	$P_1 = 28$
(iii)	20001	$Q_4 = 2$

(iv)	476999	P_3=477
(v)	2900001	Q_5=29
(vi)	76001	Q_3=76
(vii)	54999999	P_6=55
(viii)	186999	P_3=187
(ix)	6400000001	Q_8=64
(x)	7469999	P_4=747

Reality of the symbology

We notice that through the Osculation process the final result is the original number itself or a multiple or Zero.

Let us test per it for previous examples

(i) 33799 (with p_2 = 338) gives us 337 + 338 (99) = 33462+337 = 33799

(ii) 279 (with p_1=28) gives us 27+28(9)=27+252 = 279

(iii) 20001(with Q_4=2) 2x1-2=0

(iv) 476999 (with P3=477) gives us 476+477(999) = 476523+476=476999

(v) 2900001(with Q_5=29) gives us 29x1-29=0

(vi) 76001 (with Q_3=76) gives us 76x1-76=0

(vii) 54999999 (with P_6=55) gives us 54+55(999999)=54999945+54=54999999

(ix) 6400000001 (with Q_8=64) 64x1-64=0

(x) 7469999 (with P_4= 747) gives us 746 + 747(9999) = 746+7469253=7469999

Significance of the Symbology and use this will enable us to determine the no: of digits to be taken in each group and the actual Osculator in each individual case

(I) Is 71475344 divisible by 799

Here P2=8 therefore we have to split the number into 2 digit group for Osculation by 8. Thus

71	47	53	44
799	91	405	

∴ Yes

∴ The given number is divisible by 799

The Osculation results are 715105, 7191, 799

(2) Is 123, 334,542 divisible by 999

Here P_3=1

123 334 542

999 876 ∴ yes

∴ The number is divisible by 999 The Osculation results are 123876, 999.

(3) Is 987654321 divisible by 2001?

Here P_3=2

987 654 321

1580 1296 ∴ NO

The Osculation results are 987975, 1962

(4) Is 1132370697 divisible by 6001

Osculator Q_3=6

$\bar{1}$ 323 $\overline{706}$ 97

$\overline{001}$ 132 $\overline{370}$ 697

0 5001 3812 ∴ Yes

∴ The given number is divisible by 6001

The osculation results are 1128188, 0 0 Yes

(5) Is 597296889 divisible by 801

Q_2=8

5 $\overline{97}$ 29 $\overline{68}$ 89

0 500 375 644 ∴ yes

The given numbers is divisible by 801

The Osculation results are 5 972 256, 59274, 0

(6) Is 19665916 Divisible by 159

P_2=16

19 66 59 16

166 309 315 ∴ NO

The given number is not divisible by 159

The osculation results are 196915, 2209 166

(7) Is 6171600433 divisible by 4999

$P_3=5$

6 171 600 433

4999 3998 2765 $\therefore$ Yes

The given no: is divisible by 4999

The osculation results are 6173765, 9998 (4999x2=9998)

(8) Is 259797055 divisible by 2101

$Q_2=21$

2 $\overline{59}$ 79 $\overline{70}$ 55

1189 1057 1854 1085 $\therefore$ NO

The osculation results are 2596815, 25653, and -857

(9) Is 2102241654 is divisible by 17001

$Q_3=17$

$\overline{002}$ 102 $\overline{241}$ 654

0 15001 10877 $\therefore$ yes

The Osculation results are 2091123, 0

(10) Is 2763889344 is divisible by 2799

$P_2=28$

27 63 88 93 44

2799 99 801 1325 $\therefore$ Yes

Osculation results are 27640125, 277101, 2799.

Divisibility and Complex Multiple Osculators

Ending in 9 or a series of nines:

(1) 199 $P_2 = 2$
(2) 99 $P_2 = 1$
(3) 3999 $P_3 = 4$
(4) 49999 $P_4 = 5$
(5) 49 $P_1 = 5$
(6) 799 $P_2 = 8$
(7) 299999 $P_5 = 3$

(1) Is 494215037797 Divisible by 299
Divisor = 299
$P_2 = 3$

49	42	15	03	77	97
299	183	47	210	368	

(Or) ∴ YES

4942150377 + (97×3) = 4942150668
49421506 + (68×3) = 49421710
494217 + (10×3) = 494247
4942 + (47×3) = 5083
50 + (83×3) = 299 ∴ YES

(2) Is 871390442865 Divisibly by 6999?
Divisor = 6999
$P_3 = 7$

871	390	442	865
6999	3875	6497	

∴ YES

871390442 + (865×7) = 871396497
871396 + (497×7) = 874875
874 + (875×7) = 6999 ∴ YES

(3) Is 92601874651 Divisible by 39999?
Divisor = 39999
$P_4 = 4$

926	0187	4651
36091	18791	

∴ NO

9260187 + (4651 × 4) = 9278791
927 + (8791 × 4) = 36091 ∴ NO

(4) Is 27648068409471288 Divisible by 799999?

Divisor = 799999

$P_5 = 8$

27	64806	84094	71288
799999	499996	654398	

(or) ∴ YES

276480684094 + (71288×8) = 276481254398
2764812 + (54398×8) = 3199996
31 + (99996×8) = 799999 ∴ YES

(5) Is 7733933757 Divisible by 99?

Divisor = 99

$P_2 = 1$

77	33	93	37	57
99	121	187	94	

∴ YES

(or)

77339337 + (57×1) = 77339394
773393 + (94×1) = 773487
7734 + (87×1) = 7821
78 + (21×1) = 99 ∴ YES

(or)

77 + 33 + 93 + 37 + 57 = 297
297 = 99×3 ∴ YES

(6) Is 1035468792567 divisible by 999?

Divisor = 999

$P_3 = 1$

1	035	468	792	567
864	863	828	1359	

∴ NO

(or)

1035468792 + (567×1) = 1035469359
1035469 + (359×1) = 1035828
1035 + (828×1) = 1863
+ (863 ×1) = 864 ∴ NO

(or)

$1 + 035 + 468 + 792 + 567 = 1863$

$\therefore$ NO

(7) Is 12345555443211 Divisible by 99999?

Divisor = 99999

$P_5 = 1$

1234	55554	43211
99999	98765	

$\therefore$ YES

(or)

$123455554 + (43211 \times 1) = 123498765$

$1234 + (98765 \times 1) = 99999$ $\therefore$ YES

(or)

$1234 + 55554 + 43211 = 99999$ $\therefore$ YES

ending in 1 or a series of zeroes ending in unity.

SECOND TYPE:

Divisor 2/4/5/6/8/0 Ending

1) Is 335802224 divisible by 272 ?

Factors of 272

136 x 2

68 x 2

34 x 2

17 x 2

$\therefore$ 272 = 17x2x2x2x2

Let us find out the Divisibility of the given no. by 17

$\therefore$ 17 to be converted to 9 ending for / ending

9 ending 119

+ vsc = P_1=12

3	3	5	8	0	2	2	2	4
85	106	78	16	80	86	7	50	

85 is multiple of 17

$\therefore$ The given number is divisible by 272

P_1=12

3	3	5	8	0	2	2	2	3
18	31	42	13	50	24	101	38	

18 is not a multiple of 17

$\therefore$ The given number is not divisible by 272

2. Is 335802224 divisible by 3824

P_1=24

2x1912

2x956

2x478

2x239

3	3	5	8	0	2	2	2	4
101	24	210	138	105	94	203	98	

101 is not multiple / Sub multiple of 3824

Hence it is not divisible.

3. Is 1956680544 divisible by 245

$P_1=4$

245

5 49

6 7 7

1	9	5	6	6	8	0	5	4	4
39	29	44	39	18	42	28	7	20	

245 is not a multiple of 39

∴ The given number is not divisible.

4. Is 1956437184 divisible by 105

$Q_1=5$

105

5 21

$\bar{1}$	9	$\bar{5}$	6	$\bar{4}$	3	$\bar{7}$	1	$\bar{8}$	4
-3	-31	-8	-21	-36	-37	-8	10	12	

105 is not a multiple of 3 (No)

5. 3271604588 divisible by 346

2 2 173

173x3=519

$p_1=51$

3	2	7	1	6	0	4	5	8	8
306	485	249	384	267	65	141	352	416	

∴ No

∴ 306 is not a multiple of 346

6. Is 306172616 divisible by 248?

2x124

$Q_1=3$ 2x62

2x31

3	$\bar{0}$	6	$\bar{1}$	7	$\bar{2}$	6	$\bar{1}$	6
0	-1	10	22	18	14	26	17	

∴ Yes the number is divisible by 248

7. Is 148 148 1360 divisible by 120?

2x60

3x3=9 2x30

∴ $P_1=1$ 2x15

2x5x3

1	4	8	1	4	8	1	3	6	0
9	8	13	5	13	9	10	9	6	

Yes the number is divisible by 120

Few more examples:

(1) Is 11314159388l9 Divisible by 201?

Divisor = 201

$Q_2 = 2$

	1	$\overline{13}$	14	$\overline{15}$	93	$\overline{88}$	19
0	−101	−44	−29	−7	−50		∴ YES

(or)

11314159388 − (19×2) = 11314159350
113141593 − (50×2) = 113141493
1131414 − (93×2) = 1131228
11312 − (28×2) = 11256
112 − (56×2) = 0 ∴ YES

(2) Is 2871640404196231 Divisible by 3001?

Divisor = 3001 ; $Q_3 = 3$

$\overline{2}$ 871 $\overline{640}$ 404 $\overline{196}$ 231

0 1001 2044 1895 497 $\therefore$ YES

(or)

287164044196 – (231×3) = 2871640403503

2871640403 – (503×3) = 2871638894

2871638 – (894×3) = 2868956

2868 – (956×3) = 0 $\therefore$ YES

(3) Is 928450123678 Divisible by 50001?

Divisor = 50001

$Q_4 = 5$

9284 $\overline{5012}$ 3678

26173 13378 $\therefore$ NO

(4) Is 440526927148445566354 Divisible by 600001?

Divisor = 600001

$Q_5 = 6$

$\overline{44052}$ 69271 $\overline{48445}$ 56354

0 607343 289679 $\therefore$ YES

(or) 440526927148445 – (56354×6) = 440526926810321

4405269268 – (10321×6) = 4405207342

44052 – (07342×6) = 0 $\therefore$ YES

(5) Is 918274563820018365372819 Divisible by 1000001?

Divisor = 1000001

$Q_6 = 1$

$\overline{918274}$ 563820 $\overline{018365}$ 372819

0 918274 354454 $\therefore$ YES

(or) 918274563820018365 – (372819×1) = 918274563819645546

918274563819 – (645546×1) = 918273918273

918273 – (918273×1) = 0 $\therefore$ YES

(or) 372819 – 0186365 + 563820 – 918274 = 0 $\therefore$ YES

(6) 1295432065432756891234 Divisible by 70000001?

Divisor = 70000001

$Q_7 = 7$

$\overline{1}$ 2954320 $\overline{6543275}$ 6891234

33752997 14821857 41695363 $\therefore$ NO

(or) 129543206543275 $-(6891234\times 7) = 129543158304637$
12954315 $-(8304637\times 7) = 71410756$ $\therefore$ NO

(7) Is 1036345653641632206543275 Divisible by 800000001?
Divisor = 800000001
$Q_8 = 8$

10363456 $\overline{53641632}$ 06543275
0 -1295432 $\therefore$ YES

(or) 1036345653641632 $-(06543275\times 8) = 1036345601295432$
10363456 $-(01295432\times 8) = 0$ $\therefore$ YES

Other endings:

(1) Is 930031782 Divisible by 123?
$123\times 13 = 1599$
$\therefore P_2 = 16$

9	30	03	17	82
246	1314	480	1329	

$246 = 123\times 2$ $\therefore$ YES

(3) Is 168551886607182 Divisible by 7173?
$7173\times 237 = 1700001$
$Q_5 = 17$

16855 $\overline{18866}$ 07182
71730 103228
$71730 = 7173\times 10$ $\therefore$ YES

(5) Is 4703694296301 Divisible by 381?
$381\times 21 = 8001$
$\therefore Q_3 = 8$

4	$\overline{703}$	694	$\overline{296}$	301
0	4000	1588	2112	

$\therefore$ YES

(7) Is 119570740501599925937883 Divisible by 968523?
$968523\times 413 = 399999999$
$\therefore P_8 = 4$

1195707 40501599 25937883
178208232 144253131
$178208232 = 184\times 968523$
$\therefore$ YES

(9) Is 1304804489509741 Divisible by 567?

$567 \times 3 = 1701$

$\therefore Q_2 = 17$

$\bar{1}$	30	$\overline{48}$	04	$\overline{89}$	50	$\overline{97}$	41
-1134	-667	-41	1001	659	44	600	

$1134 = 567 \times 2$

$\therefore$ YES

(11) Is 3776464125535054029408 Divisible by 369?

$369 \times 271 = 99999$

$\therefore P_5 = 1$

37	76464	12553	50540	29408
69003	168965	92501	79948	

$69003 = 369 \times 187$

$\therefore$ YES

(or) $37 + 76464 + 12553 + 50540 + 29408$

$= 169002$

$169002 = 369 \times 458$

$\therefore$ YES

(13) Is 339159262679121159262679114410 Divisible by 33241?

$33241 \times 36 = 12000001$

$\therefore Q_6 = 12$

339159	$\overline{262679}$	121159	$\overline{262679}$	114410
1130194	5065920	1444050	1110241	

$1130194 = 33241 \times 34$

$\therefore$ YES

(15) Is 28939442941578074547 Divisible by 2407 ?

$2407 \times 457 = 1099999$

$\therefore P_5 = 11$

28939	44294	15780	74547
447702	438069	835797	

$447702 = 2407 \times 186$

$\therefore$ YES

(2) Is 6701245321089254 Divisible by 889?

$889 \times 9 = 8001$

$Q_3 = 8$

$\bar{6}$	701	$\overline{245}$	321	$\overline{089}$	254
7789	5975	6660	7864	1943	

$\therefore$ NO

(4) Is 1275605234123675 Divisible by 421?

$421 \times 19 = 7999$

$P_3 = 8$

1	275	605	234	123	675
1785	8222	3993	4423	5523	

$\therefore$ NO

(6) Is 8756431027952134517 Divisible by 857?

$857 \times 7 = 5999$

$\therefore P_3 = 6$

8	756	431	027	952	134	517
3224	6535	1963	2255	2371	3236	

$\therefore$ NO

(8) Is 498712653012568213 Divisible by 353?

$353 \times 17 = 6001$

$\therefore Q_3 = 6$

$\overline{498}$	712	$\overline{653}$	012	$\overline{568}$	213
2868	6562	975	4272	710	

$\therefore$ NO

(10) Is 6530153456745321078125 Divisible by 989?

$989 \times 91 = 89999$

$\therefore P_4 = 9$

65	3015	3456	7453	2107	8125
23073	42556	44393	54548	75232	

$\therefore$ NO

(12) Is 230156701347658192365401 Divisible by 221?

$221 \times 181 = 40001$

$\therefore Q_4 = 4$

$\overline{2301}$ 5670 $\overline{1347}$ 6581 $\overline{9236}$ 5401
26130 17108 22860 16052 12368
$\therefore$ NO

(14) Is 9126504321568453213 Divisibly by 137?
$137 \times 73 = 10001$
$Q_4 = 1$

912 $\overline{6504}$ 3215 $\overline{6845}$ 3213
-6009 -6921 -417 -3632 $\therefore$ NO

(16) Is 564321972158067934 Divisible by 6923?
$6923 \times 13 = 89999$
$P_4 = 9$

56 4321 9721 5806 7934
54528 46052 74636 77212 $\therefore$ NO

(17) Is 107775622243756233 Divisible by 873?
$873 \times 63 = 54999$
$\therefore P_3 = 55$

107 775 622 243 756 233
13968 1252 37008 31661 13571

$13968 = 873 \times 16$
$\therefore$ YES

(18) Is 7136543217065412856 Divisible by 677?
$677 \times 13 = 8801$
$Q_2 = 88$

$\overline{7}$ 13 $\overline{65}$ 43 $\overline{21}$ 70 $\overline{65}$ 41 $\overline{28}$ 56
-4736 -2354 -927 -1810 -8322 -5995 -769 -8 4900
$\therefore$ NO

(19) Is 31640399122705508024481 Divisible by 751?
$751 \times 249 = 186999$
$P_3 = 187$

31 640 399 122 705 508 024 481
114903 54614 118288 79630 33425 182174 89971

$114903 = 751 \times 153$ $\therefore$ YES

(20) Is 16532156321401561 32 Divisible by 463?

463 × 27 = 12501

Q_2 = 125

$\bar{1}$	65	$\overline{32}$	15	$\overline{63}$	21	$\overline{40}$	15	$\overline{61}$	32
474	2504	6120	9850	4079	10834	6287	4851	3939	

∴ NO

Modus Operandi:

1. Ascertain the P and Q for 127

127	127
37	63
889	381
381	762
4699	8001
P_2=47	Q_3=8

Obviously Q_3=8 is preferable to P_2=47

Test: Is 8128 divisible by 127

(128) 8-8=1024-8=1016 and 1016 is a multiple of 127.

Yes it divisible

2. Find out the P and Q for 149

149	149
51	49
149	1341
745	596
7599	7301
∴ P_1=15	∴Q_2=73

P_1=15 is preferable

Test : Is 894 divisible by 149

15(4) + 89 = 149

Yes it is divisible

3. Ascertain P and Q for 423

423	423
13	87
1269	2961
423	3384
5499	36801

P_2=55 is preferable than Q_2=368

∴ Is 38493 divisible by 423

(93) 55 + 384 = 5115+384=5499

and we know that 5499 is a multiple of 423

4. Ascertain P and Q for 721

721	721
319	81
6489	58401
721	
3163	
229999	Q_2=584
P_4=23	P_4=23 is preferable than Q_2 584

5. Ascertain P and Q for 5533

5533	5533
003	197
16599	38731
	49797
	5533
	1090001

P_2=166 Q_4=109 Q_4 = 109 is preferable than P_2=166

Section-4

PARTIAL FRACTIONS AND APPLICATION TO INTEGRATION

Given expression, in the general form of $\frac{p}{q}$, can be expressed in terms of partial fractions where p and q are functions of one variable. A few forms are considered for solving. These are

(1) $\frac{lx+m}{(x-a)(x-b)}$, $\frac{lx+m}{(ax+b)(cx+d)}$

(With one or more factors in repetition) For example square, cube etc

(2) $\frac{lx^2+mx+n}{(x-a)(x-b)(x-c)}$, $\frac{lx^2+mx+n}{(ax+b)(cx+d)(ex+f)}$

(3) $\frac{lx^2+mx+n}{(x-a)^K}$ and $\frac{lx^2+mx+n}{(ax+b)^K}$

This is helpful in the factorization of the functions we give below the method existing under current system.

(1) $\frac{lx+m}{(x-a)(x-b)}$

Let us consider the example

$\frac{5x+2}{(x+2)(x+4)}$ This can be written as sum of two partial fractions

$\frac{5x+2}{(x+2)(x+4)} = \frac{A}{x+2} + \frac{B}{x+4}$ Hand on the right side

By taking the L.C.M. and equating the corresponding coefficients on both sides,

We get A + B = 5 ———— (1)

4A+2B = 2———— (2)

Solving the above two, A = -4, B = 9

So the given expression is $\frac{-4}{x+2} + \frac{9}{x+4}$

In the Vedic method, the problem is simplified and one can write down the values of A and B by paravartya operation which is explained as follows.

$$\frac{lx+m}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

One can get the values of A and B by equating the each factor's denominator successively to zero which results in x – a = 0 or x – b = 0 i.e. x = a or x = b.

The value of A is obtained by substituting the value x = a in the given expression and omitting (x – a) in the denominator

That is $A = \frac{la+m}{a-b}$

Simultaneously after substituting the value of x = b and omitting (x-b) in the denominator.

$$B = \frac{lb+m}{b-a}$$

Substituting the values of a and b from the given problem as a = –2 and b=–4 as read from the expressions (x+2 = 0, x = -2; x+4 = 0, x = -4) to rewrite the denominator. The l and m are read from the numerator of the given expression as l = 5 and m = 2 and the partial fractions can be simply written down

The values of A and B are obtained as A = –4 and B = 9

Thus the partial fractions can be simply written down mentally as $\frac{-4}{x+2}+\frac{9}{x+4}$

By Vedic Method you can solve by the method of vilokanam as described above, when the expression is in the above form.

(2) Let us consider the resolution of the expression into partial fractions.

The numerator of the given expression is of the Quadratic form.
Let us consider

$$\frac{5x^2+7x-4}{(x-2)(x+1)(x+3)}$$

In the current method the given expression is split into the partial fractions of

$$\frac{A}{x-2}+\frac{B}{x+1}+\frac{C}{x+3}$$

By taking the L.C.M. and collecting the coefficients of powers of x on the RHS and compare them with LHS.

$$5x^2 + 7x - 4 = Ax^2 + 4Ax + 3A + Bx^2 + Bx - 6B + Cx^2 - Cx - 2C$$

We get 3 equations

$$A + B + C = 5 \quad \text{\_\_\_\_\_ (1)}$$
$$4A + B - C = 7 \quad \text{\_\_\_\_\_ (2)}$$
$$3A - 6B - 2C = -4 \quad \text{\_\_\_\_\_ (3)}$$

Solving these 3 equations by suitable combinations we get

(1) + (2) gives $\quad 5A + 2B = 12$ ——— (4)

2(1) + (3) $\quad \underline{5A - 4B = 6}$ ——— (5)

$6B = 6 \Rightarrow B = 1$

$A = 2, C = 2$

In the Vedic Method the expression is similarly written as $\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$

The denominator of each factor in the L.H.S. is equated to zero which leads to $x-a=0$ or $x=a$; $x-b=0$, $x=b$ or $x-c=0$, $x=c$

By paravartya $x=a$ or $x=b$ or $x-c$

When $x=a$ is considered

The constant A can be deduced by substituting the value of x in the numerator and in the denominator leaving $x-a$, and the other two are being substituted.

$$\text{i.e., } A=\frac{la^2+ma+n}{(a-b)(a-c)}$$

$$\text{Similarly } B=\frac{lb^2+mb+n}{(b-a)(b-c)}$$

$$C=\frac{lc^2+mc+n}{(c-a)(c-b)}$$

Now the values of A, B, C are obtained by substituting the values a, b, c and l, m, n as read from numerator in the read from L.H.S. of the given expression.

i.e., $a=2$, $b=-1$, $c=-3$ and l, m, n or 5, 7, 4 respectively.

The given expression is written into partial fractions as $\frac{2}{x-2}+\frac{1}{x+1}+\frac{2}{x+3}$

(3) Resolution of partial fractions of the type. $\frac{lx+m}{(a-x)^2}$

$\frac{2x+3}{(1-x)^2}$ Where the denominator has more factors of repetition

In the Current Method, one can resolve by two methods.

It can be written as $\frac{A}{(1-x)^2}+\frac{B}{(1-x)}$

Then L.C.M. is taken and comparison of x coefficients is carried out which leads to

$2x+3=A+B(l-x)$

$A+B=3$

$\therefore B=-2$

$A=5$

$\therefore$ Given expression is $\frac{5}{(1-x)^2}-\frac{2}{(1-x)}$

This is also arrived at by keeping $(1-x)$ as p $\Rightarrow p=(1\text{-}x)$. and rewriting the given expression in terms of p as

$$\frac{2(1-p)+3}{p^2} = \frac{-2p+5}{p^2} = \frac{-2}{p} + \frac{5}{p^2}$$

Substituting for p, the value $1 - x$

We get the given expression as $\frac{-2}{1-x} + \frac{5}{(1-x)^2}$

In the Vedic Method,

$$\frac{2x+3}{(1-x)^2} = \frac{A}{(1-x)^2} + \frac{B}{(1-x)}$$

This can be simplified as $2x + 3 = A + B\,(1 - x)$

Considering the denominator on the LHS as zero and applying the usual paravartya we get $x = 1$

In general the denominators of partial fractions contain the descending powers of repeating factor.

Substituting the value $x = 1$, $A = 5$

The above equation is true for all values of x and hence we can give any other value say for example 0.

$$3 = A + B$$
$$B = -2$$

Hence given expression $\frac{2x+3}{(1-x)^2}$ can be written as $\frac{5}{(1-x)^2} - \frac{2}{(1-x)}$

In general a combination of 2 or more types of the above form can be solved.

This can be extendable to include other powers of the denominator as well.

Examples

1. Express into Partial Fractions

$$\frac{lx+m}{(x-a)(x-b)} = \frac{5x+2}{(x+2)(x+4)} = \frac{A}{(x+2)} + \frac{B}{(x+4)}$$

Current Method

$$\frac{5x+2}{(x+2)(x+4)} = \frac{A}{x+2} + \frac{B}{x+4}$$

$$= \frac{A(x+4)+B(x+2)}{(x+2)(x+4)}$$

$5x + 2 = Ax + 4A + Bx + 2B$

$= (A + B)\,x + 4A + 2B$

$\therefore\ A + B = 5$ ---------- (1)
$4A + 2B = 2 \Rightarrow 2A + B = 1$ ------ (2)

Subtracting (1) from (2)

$2A + B = 1$
$\underline{A + B = 5}$
$A = -4$

Substituting A in (1), we get B = 9

$$\therefore\ \frac{5x+2}{(x+2)(x+4)} = \frac{9}{x+4} - \frac{4}{x+2}$$

Vedic Method

We can express into partial fractions by using the following formulas:

Suppose the given expression is in the form

$$\frac{lx+m}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

Then $A = \frac{la+m}{a-b}$ $\quad B = \frac{lb+m}{b-a}$

$\therefore$ In the given problem

$$A = \frac{-10+2}{2} = \frac{-8}{2} = -4$$

$$B = \frac{-20+2}{-2} = \frac{-18}{-2} = 9$$

We can calculate A and B mentally by using the above formulas.

2. Express into Partial Fractions:

$$\frac{lx+m}{(x-a)(x-b)}=\frac{11x-1}{(x-3)(x+5)}=\frac{A}{(x-3)}+\frac{B}{(x+5)}$$

Current Method

$$\frac{11x-1}{(x-3)(x+5)}=\frac{A}{x-3}+\frac{B}{x+5}$$

$$=\frac{A(x+5)+B(x-3)}{(x-3)(x+5)}$$

$11x-1=Ax+5A+Bx-3B$

$=(A+B)\,x+5A-3B$

$A+B=11$ ---------- (1)

$5A-3B=-1$ ---------- (2)

Multiplying (1) by (3) and adding with Eq (2)

$$\begin{array}{l} 3A+3B=33 \\ \underline{5A-3B=-1} \\ \underline{8A\qquad=32} \end{array}$$

$A=4$

Substituting A in 1, we get B = 7

$$\therefore \frac{11x-1}{(x-3)(x+5)}=\frac{4}{x-3}+\frac{7}{x+5}$$

Vedic Method

$l=1$, $m=-1$ and $a=3$, $b=-5$

$A=4$, $B=7$

$$\therefore \frac{11x-1}{(x-3)(x+5)}=\frac{4}{x-3}+\frac{7}{x+5}$$

3. Resolve into Partial Fractions

$$\frac{lx^2+mx+n}{(x-a)(x-b)(x-c)} = \frac{5x^2+7x-4}{(x-2)(x+1)(x+3)} = \frac{A}{(x-2)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}$$

Current Method

$$\frac{5x^2+7x-4}{(x-2)(x+1)(x+3)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$\frac{5x^2+7x-4}{(x-2)(x+1)(x+3)} = \frac{A(x^2+4x+3)+B(x^2+x-6)+C(x^2-x-2)}{(x-2)(x+1)(x+3)}$$

$$5x^2+7x-4 = Ax^2+4Ax+3A+Bx^2+Bx-6B+Cx^2-Cx-2C$$
$$= (A+B+C)x^2+(4A+B-C)x+3A-6B-2C$$

$\therefore A+B+C=5$ ———— (1)
$4A+B-C=7$ ———— (2)
$3A-6B-2C=-4$ ———— (3)

Adding 1 and 2, we get

$$\begin{array}{l} A+B+C=5 \\ \underline{4A+B-C=7} \\ 5A+2B \quad =12 \end{array}$$ ———— (4)

Vedic Method

Suppose the given expression is $\frac{lx^2+mx+n}{(x-a)(x-b)(x-c)}$

$l=5, m=7, n=-4; a=2, b=-1, c=-3$

$$A = \frac{la^2+ma+n}{(a-b)(a-c)} = \frac{20+14-4}{(3)(5)} = 2$$

$$B = \frac{lb^2+mb+n}{(b-c)(b-a)} = \frac{5-7-4}{(-3)(2)} = 1$$

$$C = \frac{lc^2+mc+n}{(c-a)(c-b)} = \frac{45-21-4}{(-5)(-2)} = 2$$

$$\therefore \frac{5x^2+7x-4}{(x-2)(x+1)(x+3)} = \frac{2}{x-2} + \frac{1}{x+1} + \frac{2}{x+3}$$

Multiplying 1 by 2 and adding with 3, we get

$$\begin{aligned}
2A + 2B + 2C &= 10 \\
\underline{3A - 6B - 2C} &= \underline{-4} \\
5A - 4B \quad &= 6 \text{ ——— } 5
\end{aligned}$$

Subtracting 5 from 4

$$\begin{aligned}
5A + 2B &= 12 \\
\underline{5A - 4B} &= \underline{6} \\
6B &= 6 \text{ or } B = 1
\end{aligned}$$

Substituting B in 4, we get

$5A + 2 = 12$

$5A = 10 \Rightarrow A = 2$

Substituting A and B in 1

$1 + 2 + C = 5$

$\therefore C = 2$

$$\frac{5x^2 + 7x - 4}{(x-2)(x+1)(x+3)} = \frac{2}{x-2} + \frac{1}{x+1} + \frac{2}{x+3}$$

4. **Resolve into Partial Fractions**

$$\frac{lx^2+mx+n}{(x-a)(x-b)(x-c)} = \frac{3x^2-2x-14}{(x-2)(x-3)(x+4)} = \frac{A}{(x-2)} + \frac{B}{(x-3)} + \frac{C}{(x+4)}$$

Current Method

$$\frac{3x^2-2x-14}{(x-2)(x-3)(x+4)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x+4}$$

$$\frac{3x^2-2x-14}{(x-2)(x-3)(x+4)} = \frac{A(x^2+x-12)+B(x^2+2x-8)+C(x^2-5x+6)}{(x-2)(x-3)(x+4)}$$

$$\begin{aligned} 3x^2-2x-14 &= Ax^2+Ax-12A+Bx^2+2Bx-8B+Cx^2-5Cx+6C \\ &= (A+B+C)x^2+(A+2B-5C)x+6C-12A-8B \end{aligned}$$

Therefore,

$A+B+C=3$ ---------- (1)

$A+2B-5C=-2$ ---------- (2)

$-12A-8B+6C=-14$

$-6A-4B+3C=-7$ ---------- (3)

Vedic Method

$l=3$, $m=-2$, $n=-14$; $a=2$, $b=3$, $c=-4$

$A=1$, $B=1$, $C=1$ (using the formulas)

$$\therefore \frac{3x^2-2x-14}{(x-2)(x-3)(x+4)} = \frac{1}{x-2} + \frac{1}{x-3} + \frac{1}{x+4}$$

Subtracting 1 from 2, we get

$$\begin{array}{l} A + 2B - 5C = -2 \\ \underline{A + B + C = 3} \\ \quad B - 6C = -5 \qquad \text{---------- (4)} \end{array}$$

Multiplying Eq. 1 with 6 and adding with Eq. 3, we get

$$\begin{array}{l} 6A + 6B + 6C = 18 \\ \underline{-6A - 4B + 3C = -7} \\ \quad 2B + 9C = 11 \qquad \text{---------- (5)} \end{array}$$

Multiplying Eq. 4 with 2 and subtracting from Eq. 5, we get

$$\begin{array}{l} 2B + 9C = 11 \\ \underline{2B - 12C = -10} \\ \quad 21C = 21 \Rightarrow C = 1 \end{array}$$

Substituting C in equation 4, we get

$B - 6 = -5$

$B = 1$

Substituting B, C in equation 1

$1 + 1 + A = 3$

$\therefore A = 1$

$$\frac{3x^2 - 2x - 14}{(x-2)(x-3)(x+4)} = \frac{1}{x-2} + \frac{1}{x-3} + \frac{1}{x+4}$$

5. Resolve into Partial Fractions

$$\frac{lx+m}{(a-x)^2} = \frac{2x+3}{(1-x)^2} = \frac{A}{(1-x)^2} + \frac{B}{(1-x)}$$

Current Method

Put $p = 1 - x$

$\therefore x = 1 - P$

The given expression is $\frac{2(1-P)+3}{P^2} = \frac{2-2P+3}{P^2} = \frac{-2P+5}{P^2}$

$$= \frac{-2}{P} + \frac{5}{P^2} = \frac{-2}{1-x} + \frac{5}{(1-x)^2}$$

$$\frac{2x+3}{(1-x)^2} = \frac{-2}{1-x} + \frac{5}{(1-x)^2}$$

Vedic Method

(i) $\frac{2x+3}{(1-x)^2} = \frac{A}{(1-x)^2} + \frac{B}{1-x}$

$2x+3 = A + B(1-x)$

$2x+3 = A + B - Bx$

Comparing like terms on both sides

$A + B = 3,$

$B = -2, A = 5$

$$\frac{2x+3}{(1-x)^2} = \frac{5}{(1-x)^2} - \frac{2}{1-x}$$

(ii) $2x+3 = A + B(1-x)$

$\therefore$By Parvartya making

$1 - x = 0; x = 1$

$\therefore$ We have $A = 5$

Put $x = 0$

$A + B = 3 \Rightarrow B = -2$

$\therefore \frac{2x+3}{(1-x)^2} = \frac{5}{(1-x)^2} - \frac{2}{1-x}$

6. Resolve into Partial Fractions

$$\frac{lx^2+mx+n}{(a-bx)^3}=\frac{2x^2+5x+1}{(1-2x)^3}=\frac{A}{(1-2x)^3}+\frac{B}{(1-2x)^2}+\frac{C}{(1-2x)}$$

Current Method

Put $1-2x=P$

$x=(1-P)/2$

The given expression is $\dfrac{2(\frac{1-P}{2})^2+5(\frac{1-P}{2})+1}{P^3}$

$$=\frac{\frac{1+P^2-2P}{2}+5(\frac{1-P}{2})+1}{P^3}$$

$$=\frac{1+P^2-2P+5-5P+2}{2P^3}=\frac{P^2-7P+8}{2P^3}$$

$$=\frac{1}{2P}-\frac{7}{2P^2}+\frac{4}{P^3}=\frac{1}{2(1-2x)}-\frac{7}{2(1-2x)^2}+\frac{4}{(1-2x)^3}$$

$$\therefore\frac{2x^2+5x+1}{(1-2x)^3}=\frac{4}{(1-2x)^3}-\frac{7}{2(1-2x)^2}+\frac{1}{2(1-2x)}$$

Vedic Method

(i) $\dfrac{2x^2+5x+1}{(1-2x)^3}=\dfrac{A}{(1-2x)^3}+\dfrac{B}{(1-2x)^2}+\dfrac{C}{(1-2x)}$

$2x^2+5x+1=A+B(1-2x)+C(1-2x)^2$

$=A+B-2Bx+C+4Cx^2-4Cx$

$=4Cx^2-(2B+4C)x+A+B+C$

$4C=2\Rightarrow C=2/4=\frac{1}{2}$

$B=-7/2$

$A+B+C=1$

$\therefore A=4$

$$\therefore\frac{2x^2+5x+1}{(1-2x)^3}=\frac{4}{(1-2x)^3}-\frac{7}{2(1-2x)^2}+\frac{1}{2(1-2x)}$$

(ii) $2x^2 + 5x + 1 = A + B(1 - 2x) + C(1 - 2x)^2$

By Paravartya making $1 - 2x = 0$
i.e., $x = \frac{1}{2}$
$\therefore$ We have $A = 4$
Put $x = 0$ $\quad A + B + C = 1$ —— (1)
Put $x = 1$ $\quad A - B + C = 8$ —— (2)

Adding 1 and 2, we get
$2A + 2C = 9$
$C = \frac{1}{2}$

Substituting A and C in equation 1, we get
$4 + \frac{1}{2} + B = 1$
$\therefore B = -7/2$

$$\therefore \frac{2x^2 + 5x + 1}{(1-2x)^3} = \frac{4}{(1-2x)^3} - \frac{7}{2(1-2x)^2} + \frac{1}{2(1-2x)}$$

7. Resolve into Partial Fractions $\frac{3x^3+2x^2+x+4}{(x^2-4)(x-2)(x+1)}$

x^2-4 is equal to $(x-2)(x+2)$

$(x-2)$ is repeated fraction hence the partial fractions are to be represent as given below

Current Method

$$\frac{3x^3+2x^2+x+4}{(x^2-4)(x-2)(x+1)} = \frac{3x^3+2x^2+x+4}{(x-2)^2(x+2)(x+1)}$$

$$\frac{3x^3+2x^2+x+4}{(x-2)^2(x+2)(x+1)} = \frac{A}{(x-2)^2}+\frac{B}{x-2}+\frac{C}{x+1}+\frac{D}{x+2}$$

$3x^3+2x^2+x+4 = A(x^2+3x+2)+B(x^3+x^2-4x-4)+C(x^3-2x^2-4x+8)+D(x^3-3x^2+4)$

$B+C+D=3$ ——— I

$A+B-2C-3D=2$ ——— II

$3A-4B-4C=1$ ——— III

$2A-4B+8C+4D=4$ ——— IV

II X 4	$4A+4B-8C-12D$	$=8$
IV X 3	$6A-12B+24C+12D$	$=12$
+	$10A-8B+16C$	$=20$

$5A-4B+8C=10$ ——— V

Vedic Method

$$\frac{3x^3+2x^2+x+4}{(x-2)^2(x+2)(x+1)} = \frac{A}{(x-2)^2}+\frac{B}{x-2}+\frac{C}{x+1}+\frac{D}{x+2}$$

$3x^3+2x^2+x+4 = A(x^2+3x+2)+B(x^3+x^2-4x=4)+C(x^3-2x^2-4x+8)+D(x^3-3x^2+4)$

This is valid for any values of x

Let $x=2 \Rightarrow 38=12A \Rightarrow A=\frac{19}{6}$

Let $x=-2 \Rightarrow 14=16d \Rightarrow D=\frac{7}{8}$

Let $x=-1 \Rightarrow 2=9C \Rightarrow C=\frac{2}{9}$

By comparing the x^3 coefficient on both sides

$B+C+D=3 \Rightarrow B=3-\frac{2}{9}-\frac{7}{8}=\frac{216-16-63}{72}$; $B=\frac{137}{72}$

$$\begin{array}{lll} \text{I X 3} & 0 + 3B + 3C + 3D = 9 & \\ \text{II} & A + B - 2C - 3D = 2 & \\ + & \overline{A + 4B + C \quad = 11} & \text{---------- VI} \end{array}$$

$$\begin{array}{lll} \text{III} & 3A - 4B - 4C = 1 & \\ \text{VI} & A + 4B + C = 11 & \\ + & \overline{4A \quad - \quad 3C = 12} & \text{---------- VII} \end{array}$$

$$\begin{array}{lll} \text{III} & 3A - 4B - 4C = 1 & \\ \text{V} & 5A - 4B + 8C = 10 & \\ - & \overline{-2A \quad - 12C = -9} & \text{---------- VIII} \end{array}$$

$$\begin{array}{ll} \text{VII} & 4A - 3C = 12 \\ \text{VIII X 2} & -4A - 24C = -18 \\ + & \overline{-27C = -6 \Rightarrow C = \frac{2}{9}} \end{array}$$

Substituting in VII $\Rightarrow 4A - 3 \times \frac{2}{9} = 12$

$$\therefore \frac{3x^3 + 2x^2 + x + 4}{(x-2)^2(x+2)(x+1)} =$$

$$\frac{19}{6(x-2)^2} + \frac{137}{72(x-2)} + \frac{2}{9(x+1)} + \frac{7}{8(x+2)}$$

$$\Rightarrow A = \frac{19}{6}$$

Substituting the values of A and C in VI

$$\frac{19}{6} + 4B + \frac{2}{9} = 11$$

Continuation of Current method

$$\Rightarrow 4B = \frac{198 - 4 - 57}{18} = \frac{137}{18} \Rightarrow B = \frac{137}{72}$$

Substituting the values of B, C in I

$$\frac{137}{72} + \frac{2}{9} + D = 3$$

$$D = 3 - \frac{2}{9} - \frac{137}{72} = \frac{7}{8}$$

$$\therefore \frac{3x^3 + 2x^2 + x + 4}{(x-2)^2(x+2)(x+1)} =$$

$$\frac{19}{6(x-2)^2} + \frac{137}{72(x-2)} + \frac{2}{9(x+1)} + \frac{7}{8(x+2)}$$

Appendix

Current method of solving the partial fractions:

Usual and routine method i.e. adopted to get the values A, B is as follows

$\frac{lx+m}{(x-a)(x-b)}$ is defined for x = a and x ≠ a and x ≠ b

for x = a and x ≠ a and x ≠ b

let $\frac{lx+mb}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$

i .e

$$\frac{lx+mb}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$$

Since x ≠ a and x ≠ b, we have lx + mb = A (x-b) + B(x-a)

Now taking the limit

As $x \to a$ i.e.

$$\underset{x\to a}{\text{Lt}} (lx+m) = \underset{x\to a}{\text{lt}} (A(x-b)+B(x-a))$$

i.e la + m = A(a-b)

$$A = \frac{la+m}{a-b}$$

Similarly by taking lt $x \to b$

lb + m = B (b-a)

$$\Leftrightarrow B = \frac{lb+m}{b-a}$$

In the text Swamiji method is explained using the parvartya, elimination and retenction process the same result could be obtained.

Example:

$\frac{2x+3}{(1-x)^2} = \frac{A}{(1-x)^2} + \frac{B}{1-x}$ is defined for x ≠ 1

$$\frac{2x+3}{(1-x)^2} = \frac{A+B(1-x)}{(1-x)^2}$$

For x ≠ 1, 2x + 3 = A+B (1-x)

Taking the limit as $x \to 1$ i.e.

$$\underset{X\to 1}{\text{Lt}} (2x+3) = \underset{x\to 1}{\text{lt}} (A + B(1-x))$$

$$5 = A + \underset{x\to 1}{\text{lt}}\ B(1-x) = A$$

$$A = 5$$

Hence 2x + 3 = 5 + B (1-x) is defined for all x ≠ - 1 in particular

x = 0 , 3 = 5 + B => B =-2 or equating the constant of x on both sides ,we get B = -2.

8. $\dfrac{3x+7}{(x+5)(x-2)}$

Current Method

$$\frac{3x+7}{(x+5)(x-2)} = \frac{A}{(x+5)} + \frac{B}{(x-2)}$$

By taking LCM and comparing the like terms

A(x-2) + B (x+5) = 3x + 7

$A + B = 3$ -- I
$-2A + 5B = 7$ -- II
$2A + 2B = 6$ -- I X 2

$7B = 13 \Rightarrow B = \frac{13}{7}$

$A = 3 - \frac{13}{7} = \frac{8}{7}$

Vedic Method

$$\frac{3x+7}{(x+5)(x-2)} = \frac{A}{(x+5)} + \frac{B}{(x-2)}$$

$x = -5$ $\quad A = \frac{-15+7}{7} = \frac{8}{7}$

$x = 2$ $\quad B = \frac{6+7}{2+5} = \frac{13}{7}$

9. $\dfrac{5x+4}{(x+2)(x+3)}$

Current Method

$$\frac{5x+4}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$$

$$\frac{5x+4}{(x+2)(x+3)} = \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}$$

$A + B = 5$

$3A + 2B = 4$

$3A + 3B = 15$

$-B = -11 \Rightarrow B = 11;\ A = -6$

Vedic Method

$$\frac{5x+4}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$$

$x = -2 \Rightarrow A = \dfrac{-6}{1} = -6$

$x = -3 \Rightarrow B = \dfrac{-11}{-1} = 11$

10. $\dfrac{2x+7}{(x-1)(x-4)}$

Current Method

$$\frac{2x+7}{(x-1)(x-4)} = \frac{A}{(x-1)} + \frac{B}{(x-4)}$$

$A(x-4) + B(x-1) = 2x + 7$

$A + B = 2$

$\underline{-4A - B = 7}$

$-3A = 9$

$\Rightarrow A = -3;\quad B = 5$

Vedic Method

$$\frac{2x+7}{(x-1)(x-4)} = \frac{A}{(x-1)} + \frac{B}{(x-4)}$$

$x = 1\ A = \dfrac{9}{-3} = -3$

$x = 4\ \ B = \dfrac{15}{3} = 5$

11. $\dfrac{2x^2+11x-9}{(x+1)(x-2)(x-3)}$

Current Method

$$\frac{2x^2+11x-9}{(x+1)(x-2)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$2x^2+11x-9 = A\,(x-2)(x-3) + B\,(x+1)(x-3) + C(x+1)(x-2)$

x^2 coefficient $\Rightarrow$ $A + B + C = 2$

x coefficient$\Rightarrow$ $-5A - 2B - C = 11$

x^0 coefficient$\Rightarrow$ $6A - 3B - 2C = -9$

$-4A - B = 13$

$8A - B = -5$

$-12A = 18 \Rightarrow A = \dfrac{-3}{2}$

$-4\left(\dfrac{-3}{2}\right) - B = 13 \Rightarrow B = -7$

$\dfrac{-3}{2} - 7 + C = 2 \Rightarrow C = \dfrac{21}{2}$

Vedic Method

$$\frac{2x^2+11x-9}{(x+1)(x-2)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$x = -1$ $\quad A = \dfrac{-18}{(-3)(-4)} = \dfrac{-18}{12} = \dfrac{-3}{2}$

$x = 2$ $\quad B = \dfrac{8+22-9}{(3)(-1)} = \dfrac{-21}{3} = -7$

$x = 3$ $\quad C = \dfrac{18+33-9}{(4)(1)} = \dfrac{21}{2}$

12. $\dfrac{3x^2+5x-2}{(x+5)(x-1)(x-2)}$

Current Method

$$\frac{3x^2+5x-2}{(x+5)(x-1)(x-2)} = \frac{A}{(x+5)} + \frac{B}{(x-1)} + \frac{C}{(x-2)}$$

$3x^2 + 5x - 2 = A(x-1)(x-2) + B(x+5)(x-2) + C(x+5)(x-1)$

$A + B + C = 3$
$-3A + 3B + 4C = 5$
$2A - 10B - 5C = -2$

$-6A + 6B + 8C = 10$
$6A - 30B - 15C = -6$

$-24B - 7C = 4$

$3A + 3B + 3C = 9$
$-3A + 3B + 4C = 5$

$6B + 7C = 14$

$-24B - 7C = 4$
$6B + 7C = 14$

$-18B = 18 \Rightarrow B = -1$; $C = \dfrac{20}{7}$

$A + B + C = 3 \Rightarrow A = \dfrac{8}{7}$

Vedic Method

$$\frac{3x^2+5x-2}{(x+5)(x-1)(x-2)} = \frac{A}{(x+5)} + \frac{B}{(x-1)} + \frac{C}{(x-2)}$$

$x = -5$, $A = \dfrac{75-25-2}{-6X-7} = \dfrac{48}{42} = \dfrac{8}{7}$

$x = 1$, $B = \dfrac{3+5-2}{6X-1} = \dfrac{6}{-6} = -1$

$x = 2$, $C = \dfrac{12+10-2}{7X1} = \dfrac{20}{7}$

13. $\dfrac{7x^2+13x+8}{(x+1)(x+2)(x+3)}$

Current Method

$$\frac{7x^2+13x+8}{(x+1)(x+2)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)}$$

A (x+2)(x+3) + B(x+1) (x+3) + C (x+1)(x+2)

$A + B + C = 7$
$5A + 4B + 3C = 13$
$6A + 3B + 2C = 8$

$6A + 3B + 2C = 8$
$6A + 6B + 6C = 42$

$-3B - 4C = -34$

$5A + 5B + 5C = 35$
$5A + 4B + 3C = 13$

$B + 2C = 22$
$3B + 6C = 66$
$-3B - 4C = -34$

$2C = 32 \Rightarrow C = 16$
$B = -10\ ;\ A = 1$

Vedic Method

$$\frac{7x^2+13x+8}{(x+1)(x+2)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)}$$

$x = -1,\ A = \dfrac{7-13+8}{1X2} = \dfrac{2}{2} = 1$

$x = -2,\ B = \dfrac{28-26-+8}{-1X2} = \dfrac{10}{-2} = -5$

$x = -3,\ C = \dfrac{63-39+8}{-2(-1)} = \dfrac{32}{2} = 16$

14. $\dfrac{5x+9}{(x-2)^2}$

Current Method

$$\frac{5x+9}{(x-2)^2} = \frac{A}{(x-2)^2} + \frac{B}{(x-2)}$$

$5x+9 = A + B\,(x-2)$

$B = 5$
$A - 2B = 9 \Rightarrow A = 19$

$$\frac{5x+9}{(x-2)^2} = \frac{19}{(x-2)^2} + \frac{5}{(x-2)}$$

Vedic Method

$$\frac{5x+9}{(x-2)^2} = \frac{A}{(x-2)^2} + \frac{B}{(x-2)}$$

By paravartya $x = 2$

$A = 19$
This is valid for any value of x, Let $x = 0$
$9 = A + B(-2) \Rightarrow B = 5$

$$\frac{5x+9}{(x-2)^2} = \frac{19}{(x-2)^2} + \frac{5}{(x-2)}$$

15. $\dfrac{x^2+2x+3}{(1-2x)^3}$

Current Method

$$\frac{x^2+2x+3}{(1-2x)^3} = \frac{A}{(1-2x)^3} + \frac{B}{(1-2x)^2} + \frac{C}{(1-2x)}$$

$x^2+2x+3 = A+B(1-2x)+C(1-2x)^2 = (A+B+C)+(-2B-4C)x+4Cx^2$

$4C = 1 \Rightarrow C = \frac{1}{4}$; $-2B - 1 = 2 \Rightarrow B = \frac{-3}{2}$

$A + B + C = 3 \Rightarrow A = \frac{17}{4}$

Vedic Method

$$\frac{x^2+2x+3}{(1-2x)^3} = \frac{A}{(1-2x)^3} + \frac{B}{(1-2x)^2} + \frac{C}{(1-2x)}$$

By Paravartya $x = 1/2$

$A = \frac{1}{4} + \frac{2}{2} + 3 \Rightarrow A = \frac{17}{4}$

This is valid for any value of x
Let $x = 0 \therefore A + B + C = 3$
Let $x = 1 \therefore A - B + C = 6$

$\Rightarrow B = \frac{-3}{2}; C = \frac{1}{4}$

16. $\frac{4x^2+7x+3}{(x-1)^3}$

Current Method

$$\frac{4x^2+7x+3}{(x-1)^3} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)}$$

$4x^2+7x+3 = A + B(x-1) + C(x-1)^2 = (A+B+C)+x(B-2C)+Cx^2$

$C = 4$; $B - 2C = 7 \Rightarrow B = 7 + 8 = 15$

$A-B+C = 3$; $A = 3 + B - C = 3 + 15 - 4 = 14$

$$\frac{4x^2+7x+3}{(x-1)^3} = \frac{14}{(x-1)^3} + \frac{15}{(x-1)^2} + \frac{4}{(x-1)}$$

Vedic Method

$$\frac{4x^2+7x+3}{(x-1)^3} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)}$$

By Paravartya $x = 1$
$A = 4 + 7 + 3 = 14$
This is valid for any value of x
Let $x = 0$
$A-B+C = 3$
Let $x = -1$
$A - 2B + 4C = 0$
$\Rightarrow B = 15, C = 4$

$$\frac{4x^2+7x+3}{(x-1)^3} = \frac{14}{(x-1)^3} + \frac{15}{(x-1)^2} + \frac{4}{(x-1)}$$

17. $\dfrac{3x^2+7x+2}{(x-1)^2(x+3)}$

Current Method

$$\frac{3x^2+7x+2}{(x-1)^2(x+3)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+3)}$$

$3x^2 + 7x + 2 = A(x+3) + B(x-1)(x+3) + C(x-1)^2$

$= x^2(B+C) + x(A+2B-2C) + (3A - 3B+C)$

$B + C = 3$
$A + 2B - 2C = 7$
$3A - 3B + C = 2$

$3A + 6B - 6C = 21$
$3A - 3B + C = 2$

$9B - 7C = 19$
$7B + 7C = 21$

$16B = 40 \Rightarrow B = 5/2$

$B + C = 3 \Rightarrow C = \frac{1}{2}$

$A+2B-2C=7 \Rightarrow A = 3$

$$\frac{3x^2+7x+2}{(x-1)^2(x+3)} = \frac{3}{(x-1)^2} + \frac{5}{2(x-1)} + \frac{1}{2(x+3)}$$

Vedic Method

$$\frac{3x^2+7x+2}{(x-1)^2(x+3)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+3)}$$

By Paravartya $x = 1$

$4A = 12 \Rightarrow A = 3$

By parvartya $x = -3$

$27-21 + 2 = 16C \Rightarrow c = \frac{1}{2}$

This is valid for any value of x
Let $x = 0$

$3A - 3B + C = 2 \Rightarrow B = 5/2$

$$\frac{3x^2+7x+2}{(x-1)^2(x+3)} = \frac{3}{(x-1)^2} + \frac{5}{2(x-1)} + \frac{1}{2(x+3)}$$

Integration by Partial Fractions

Examples

1. Integrate $\frac{5x-3}{x^2-2x-3}$

Current Method

$$\int\frac{5x-3}{x^2-2x-3}dx = \int\frac{5x-3}{x^2-3x+x-3}dx = \int\frac{5x-3}{(x-3)(x+1)}dx$$

$$\frac{5x-3}{(x-3)(x+1)} = \frac{A}{x-3}+\frac{B}{x+1}$$

$5x-3 = A(x+1)+B(x-3)$
$5x-3 = Ax+A+Bx-3B$
$\quad = (A+B)x+A-3B$
$A+B=5$
$A-3B=-3$
Hence $A=3, B=2$

$$\int\frac{5x-3}{(x-3)(x+1)}dx = \int\frac{3}{x-3}dx+\int\frac{2}{x+1}dx$$

$\Rightarrow 3\log(x-3)+2\log(x+1)$

Vedic Method

$$\int\frac{5x-3}{x^2-2x-3}dx = \int\frac{5x-3}{(x-3)(x+1)}dx$$
$$= \int\frac{A}{x-3}dx+\int\frac{B}{x+1}dx$$

$a=3, b=-1$

$A = \frac{la+m}{(a+1)}, B = \frac{lb+m}{(b-3)}$

By using Vedic Method of Partial Fractions

$$= \int\left(\frac{3}{x-3}+\frac{2}{x+1}\right)dx = \int\frac{3}{x-3}dx+\int\frac{2}{x+1}dx$$

$= 3\log(x-3)+2\log(x+1)$

2. Integrate $\frac{2x+5}{20x^2+x-1}$

Current Method

$$\int\frac{2x+5}{20x^2+x-1}dx = \int\frac{2x+5}{(4x+1)(5x-1)}dx$$

$$\frac{2x+5}{(4x+1)(5x-1)} = \frac{A}{4x+1}+\frac{B}{5x-1}$$

$2x+5 = A(5x-1)+B(4x+1)$

Therefore,

$5A+4B=2$ —— (1)

$-A+B=5$ —— (2)

Multiplying equation 2 with 5 and adding with equation 1, we get

$5A+4B = 2$

$\underline{-5A+5B=25}$

$9B=3 \Rightarrow B=3, A=-2$

$$= \int\frac{3}{5x-1}dx - \int\frac{2}{4x+1}dx$$

$$= \frac{3}{5}\int\frac{5}{5x-1}dx - \frac{1}{2}\int\frac{4}{4x+1}dx$$

$$= \frac{3}{5}\log(5x-1) - \frac{1}{2}\log(4x+1)$$

$$= \frac{3}{5}\log(5x-1) - \frac{1}{2}\log(4x+1)$$

Vedic Method

$$\int\frac{2x+5}{20x^2+x-1}dx$$

By Anurupyena and Adyamadyenantyamantyena sub-sutras Denominator can be factorized as

$$= \int\frac{2x+5}{(4x+1)(5x-1)}dx$$

$$\int\frac{2x+5}{(4x+1)(5x-1)}dx = \int\frac{A}{(4x+1)}dx + \int\frac{B}{(5x-1)}dx$$

By Vedic Method of Partial Fractions

A and B can be derived as $A=-2$ $B=3$

$$\int\frac{2x+5}{(4x+1)(5x+1)}dx = \int\left(\frac{3}{5x-1}-\frac{2}{4x+1}\right)dx$$

$$\Rightarrow \int\frac{-2}{4x+1}dx + \int\frac{3}{5x-1}dx$$

$$= \frac{3}{5}\int\frac{5}{5x-1}dx - \frac{1}{2}\int\frac{4}{4x+1}dx$$

$$= \frac{3}{5}\log(5x-1) - \frac{1}{2}\log(4x+1)$$

3. Integrate $\int \frac{5x^2+10x+3}{(x^2-1)(x+2)}dx$

Current Method

$$\int\frac{5x^2+10x+3}{(x^2-1)(x+2)}dx = \int\frac{5x^2+10x+3}{(x-1)(x+1)(x+2)}dx$$

$$\frac{5x^2+10x+3}{(x-1)(x+1)(x+2)}dx = \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x+2}$$

$$5x^2+10x+3 = A(x^2+3x+2)+B(x^2+x-2)+C(x^2-1)$$

Therefore, $A+B+C=5$ ——— (1)

$3A+B=10$ ——— (2)

$2A-2B-C=3$ ——— (3)

$(1)+(3) \Rightarrow 3A-B=8$

$(2) \Rightarrow \underline{3A+B=10}$

$6A=18 \Rightarrow A=3, B=1, C=1$

$$\therefore \int\frac{5x^2+10x+3}{(x-1)(x+1)(x+2)}dx = \int\left(\frac{3}{x-1}+\frac{1}{x+1}+\frac{1}{x+2}\right)dx$$

$$= \int\frac{3}{x-1}dx+\int\frac{1}{x+1}dx+\int\frac{1}{x+2}dx$$

$$= 3\log(x-1)+\log(x+1)+\log(x+2)$$

Vedic Method

$$\int\frac{5x^2+10x+3}{(x^2-1)(x+2)}dx = \int\frac{5x^2+10x+3}{(x-1)(x+1)(x+2)}dx$$

By Vedic Method of Partial Fractions

$$\int\frac{5x^2+10x+3}{(x^2-1)(x+2)} = \int\frac{A}{(x-1)}dx+\frac{B}{(x+1)}dx+\int\frac{C}{(x+2)}dx$$

$$\frac{5x^2+10x+3}{(x-1)(x+1)(x+2)} = \frac{lx^2+mx+n}{(x-1)(x+1)(x+2)}$$

$A=3, B=1, C=1$

$$\int\frac{5x^2+10x+3}{(x^2-1)(x+2)}dx = \int\left(\frac{3}{x-1}+\frac{1}{x+1}+\frac{1}{x+2}\right)dx$$

$$= \int\frac{3}{x-1}dx+\int\frac{1}{x+1}dx+\int\frac{1}{x+2}dx$$

$$= 3\log(x-1)+\log(x+1)+\log(x+2)$$

Section-5
POWER SERIES (POWERS OF POLYNOMIALS)

Procedure to expand any power of the type $(a+bx+cx^2+----)^n$ by current and Vedic methods are as follows. In the current method the given expression can be condensed to $(a+px)^n$ where p can be either a number or polynomial. If p is a number, a single binomial expansion will give the result. If the p is a polynomial then successive condensation is to be carried out until we reach final expansion in the form of a + bx where b^1 is mere number. At every stage of condensation binomial expansion should be carried out and the results are to be incorporated into successive steps. An example is given when a single condensation is explained. The binomial expansion for +ve and –ve values of n using the given formula.

In Vedic method, the following principle is followed. The given problem $(a+bx+cx^2+\ldots)^n$ is written in the form of a power series taking into consideration, an identity between the two and also the maximum power in the power of the power series is determined by the multiplication of n with the highest +ve power in the problem.

If n is –ve, even then the same rule applies but it is to be expressed in the form of infinite series. The following points may be noted in equating the given problem in power series

(1) The problem in general form can be taken to be $(a+bx+cx^2+\ldots\ldots kx^m)^n$. It is a polynomial of definite order raised to the power n, n is a natural number.

If n is not a natural number then the expansion will result in infinite series.

In both the cases the method adopted by Nicholas group of workers is to express the given polynomial either in the finite series or an infinite series of the form as the case may be.

$A + Bx + Cx^2 + Dx^3 + \ldots\ldots\ldots\ldots + Kx^{mn}$ $n \in N$ (Natural number)

$A + Bx + Cx^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $n \notin N$

In case the polynomial consists of –ve powers of x, then also if n is +ve we get finite series, if n is –ve, infinite series. In addition, the expansion consists of the –ve powers of x the minimum –ve power is determined by the quantity n multiplied by the minimum power in the given polynomial. Taking these points into consideration the expansion of polynomial in the power series can be written as follows:

$(2 + 3x + 4x^2)^3 = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6$

$(2 + 3x^{-1} + 4x^2)^3 = D^1x^{-3} + C^1x^{-2} + B^1x^{-1} + A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6$

$(2+3x^{-1} + 4x^{-2})^3 = G^1x^{-6} + F^1x^{-5} + E^1x^{-4} + D^1x^{-3} + C^1x^{-2} + B^1x^{-1} + A$

$(2 + 3x + 4x^2)^{-3} = A + Bx + Cx^2 + Dx^3 + \ldots\ldots\ldots\ldots\ldots$ infinite series

This is only being way of example shown here. For any polynomial raised to any power, this procedure is applicable.

The method actually consists of

1) Considering derivative of the logarithm of the given polynomial.

For example:

$(2 + 3x + 4x^2)^3 = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6$

2) Taking derivative of logarithms on both sides we get

$(\log(2 + 3x + 4x^2)^3 = [\log(A + Bx + Cx^2 + Dx^3 + Ex^4+Fx^5+Gx^6)]$

$$\frac{3(3+8x)}{(2+3x+4x^2)} = \frac{B+2Cx+3Dx^2+4Ex^4+5Fx^4+6Gx^5}{A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6}$$

3) By cross multiplication and comparison the coefficients of each power of x on both sides, one can evaluate A, B, C ..... A few examples are worked out taking certain powers using both the current method and the method adopted by Nicholas group. Using directly the straight division method described by Vedic principles for a division, by polynomials, the expansion of a polynomial when n takes a –ve value also is demonstrated here.

For example when n takes a –ve value $(1 + x + x^2)^{-4}$

It can be identified by Paravartya as

$$\frac{1}{(1+x+x^2)^4}$$

Concentrating first on the denominator and proceeding in the same manner as described above, one can get

$$(1+x+x^2)^4 = A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6+Hx^7+Ix^8$$

Where A, B, C, D, E, F, G, H and I can be definitely worked out by the above method.

In order to get the coefficients in the expansion of $(1+x+x^2)^{-4}$ one has to actually divide 1 by the power series so obtained (Reciprocal of the power series). At this stage applying straight division principles as explained in the vedic system, one has to consider 1 as the dividend and the power series as the divisor, with the proper partition of dhwajanka and part divisor. This is explained in the example $(3x+2)^{-3}$.

This method in case of –ve powers of the polynomial expansion into power series can be taken to be general whatever may be the degree of the polynomial.

Example: Expansion of $(1 + x^{-1} + x^2)^3$

Given polynomial is expressed in terms of a power series as

$D^1x^{-3} + C^1x^{-2} + B^1x^{-1} + A + BX + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6$

On taking logs and the derivatives w. r .t. x on both sides one gets $\frac{3(2x - x^{-2})}{1 + x^{-1} + x^2} =$

$$\frac{-3D^1x^{-4} - 2c^1x^{-3} - B^1x^{-2} + B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + 6Gx^5}{D^1x^{-3} + C^1x^{-2} + B^1x^{-1} + A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6}$$

On cross multiplication and collecting the coefficient of equal power of x

$-3C + 6B^1 = B + 2C - B^1 \longrightarrow x^0$
$-B = +5C - 7B^1$

$2C + 3D = 6A - 3D \longrightarrow x^1$
$C = 3A - 3D$

$B + 4E + 3D = 6B - 3E \longrightarrow x^2$
$3D = 5B - 7E$

$4E + 5F + 2C = 6C - 3F \longrightarrow x^3$
$E = C - 2F$

$5F + 6G + 3D = 6D - 3G \longrightarrow x^4$
$5F = 3D - 9G$

$6E = 6G + 4E \longrightarrow x^5$
$E = 3G$

$-3B + 6C^1 = B - 2C^1 \longrightarrow x^{-1}$
$B = 2C^1$

$-B^1 - 3D^1 = 6D^1 - 3A \longrightarrow x^{-2}$
$-B^1 = 9D^1 - 3A$

$-3B^1 = -2C^1 - B^1 \longrightarrow x^{-3}$
$C^1 = B^1$

$-3D^1 - 2C^1 = -3C^1 \longrightarrow x^{-4}$
$3D^1 = C^1$

From these one can deduce the following relation

$(1 + x^{-1} + x^2)^3 = \{1 + x(x^{-2}+x)\}^3$

Let $(x^{-2} + x) = p$. Hence the given polynomial

$= (1+px)^3$

$1 + 3px + 3p^2x^2 + p^3x^3$

$p = (x^{-2} + x)$

$P^2 = (x^{-2}+x)^2 = x^2 + x^{-4} + 2x^{-}$

$p^3 = (x^{-2}+x)^3 = x^{-6} + 3 + 3x^{-3} + x^3$

On substitution of the values of p, p^2 and p^3 we get

$(1 + x^{-1} + x^2)^3 = 4 + 6x + 3x^2 + 3x^3 + 3x^4 + x^6 + x^{-3} + 3x^{-2} + 3x^{-1}$

$A = \frac{4}{3}C^1$

$B = 2C^1$

$C = C^1$

$D = C^1$

$E = C^1$

$G = \frac{1}{3}C^1$

$B^1 = C^1$

$C^1 = C^1$

$D^1 = \frac{1}{3}C^1$

$$[1 + x^{-1} + x^2]^3 = C^1\left[\frac{4}{3} + 2x + x^2 + x^3 + x^4 + \frac{1}{3}x^6 + x^{-1} + x^{-2} + \frac{1}{3}\ x^{-3}\right]$$

To evaluate C^1 we can give a value to x for example $x = 1$

$$(1+1+1)^3 = 27 = C^1\left[\frac{4}{3}+2+1+1+1+\frac{1}{3}+1+1+\frac{1}{3}\right] = C^1 \times 9 = 27$$

$C^1 = 3$

$27 = 9\,C^1$

$C^1 = 3$

Hence the given expansion is $x^{-3} + 3x^{-2} + 3x^{-1} + 4 + 6x + 3x^2 + 3x^3 + 3x^4 + x^6$

Example: $(1 + x^2 + 2x\,)^{0.5}$

Taking logarithm and differentiate w. r. t. x

$$\frac{0.5(2x+2)}{1+x^2+2x}$$

$$\frac{x+1}{1+x^2+2x} = \frac{B+2Cx+3Dx^2+4Ex^3+5Fx^4+6Gx^5}{A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6}$$

$A = B = 1$

$C = 0, D = 0$

x^0 $\quad A = B = 1$

x^1 $\quad A + B = 2B + 2C \qquad A - B = 2C \Rightarrow C = 0$

x^2 $\quad B + C = 4C + B + 3D) = 0 \rightarrow 3C + 3D = 0 \Rightarrow D = 0$

x^3 $\quad C + D = 4E + 2C + 6D \Rightarrow E = 0$

x^4 $\quad D + E = 5F + 8E + 3D \Rightarrow F = 0$

$\therefore (1 + x^2 + 2x)^{0.5} = 1 + x$

Example: $(8x^6 + 12x^5 + 42x^4 + 37x^3 + 63x^2 + 27x + 27)^{1/3}$

$$= A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots\ldots\ldots$$

$$\left(\frac{\begin{array}{l}(16x^5+20x^4+56x^3+37x^2+42x+9 \\ \frac{1}{3}(48x^5+60x^4+168x^3+111x^2+126x+27)\end{array}}{8x^6+12x^5+42x^4+37x^3+63x^2+27x+27} \right) = \frac{B+2Cx+3Dx^2+\ldots.}{A+Bx+Cx^2+Dx^3+\ldots..}$$

$A = 3$

x^0 $\quad 9A = 27B;\ A = 3B \qquad \Rightarrow \qquad B = 1$

x^1 $\quad 9B + 42A = 54C + 27B \qquad \Rightarrow \qquad C = 2$

$\therefore (8x^6 + 12x^5 + 42x^4 + 37x^3 + 63x^2 + 27x + 27)^{1/3} = (3 + x + 2x^2 \ldots)$

POWERS OF POLYNOMIALS

Examples:

(1) Expand $(1+x)^5$

Current Method

$(1+x)^5$

By using Binomial Expansion

$$(1+x)^5 = 5c_0 \times 1^5 + 5c_1 \times 1^4 \times x + 5c_2 \times 1^3 \times x^2$$

$$+ 5c_3 \times 1^2 \times x^3 + 5c_4 \times 1 \times x^4 + 5c_5 \times x^5$$

$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

Vedic Method

Let $(1+x)^5 = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5$

Take log and differentiate w.r.t x,

$$\frac{5}{1+x} = \frac{B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4}{A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5}$$

By cross Multiplication, we get

$5A + 5Bx + 5Cx^2 + 5Dx^3 + 5Ex^4 + 5Fx^5$

$= B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + 5Fx^5$

$$5A + 5Bx + 5Cx^2 + 5Dx^3 + 5Ex^4 + 5Fx^5 = B + (B + 2C)\,x + (2C + 3D)\,x^2 + (3D + 4E)\,x^3$$

$$+ (4E + 5F)\,x^4 + 5Fx^5.$$

Equating Coefficients of powers of x

Constant term $5A = B$

x term $5B = B + 2C \Rightarrow 2B = C$

x^2 term $5C = 2C + 3D \Rightarrow C = D$

x^3 term $5D = 3D + 4E \Rightarrow D = 2E$

x^4 term $5E = 4E + 5F \Rightarrow E = 5F$

The first Coefficient, $A = 1^5 = 1$

$B = 5A = 5$
$C = 2B = 10$
$D = C = 10$
$E = \frac{D}{2} = 5$

$F = \frac{E}{5} = 1$

$\therefore (1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

(2) Expand $(2x + 3)^4$

<u>Current Method</u>

$(2x + 3)^4$

By using Binomial Expansion

$$(2x+3)^4 = 4c_0\,(2x)^4 + 4c_1(2x)^3 \times 3 + 4c_2\,(2x)^2 \times 3^2 + 4c_3\,(2x) \times 3^3 + 4c_4 \times 3^4$$

$$= 16x^4 + 96x^3 + 216x^2 + 216x + 81$$

<u>Vedic Method</u>

Let $(2x+3)^4 = A + Bx + Cx^2 + Dx^3 + Ex^4$

Take log and differentiate w.r.t. x, we get

$$\frac{4 \times 2}{2x+3} = \frac{B + 2Cx + 3Dx^2 + 4Ex^3}{A + Bx + Cx^2 + Dx^3 + Ex^4}$$

$$8A + 8Bx + 8Cx^2 + 8Dx^3 + 8Ex^4$$

$$= 3B + 6Cx + 9Dx^2 + 12Ex^3 + 2Bx + 4Cx^2 + 6Dx^3 + 8Ex^4$$

$8A + 8Bx + 8Cx^2 + 8Dx^3 + 8Ex^4$

$$= 3B + (2B+6C)x + (4C + 9D)x^2 + (6D + 12E)x^3 + 8Ex^4$$

Equating Coefficient of power x

$8A = 3B$

$8B = 2B + 6C \Rightarrow B = C$

$8C = 4C + 9D \Rightarrow 4C = 9D$

$8D = 6D + 12E \Rightarrow D = 6E$

The First Coefficient, $A = 34 = 81$

$$B = \frac{8A}{3} = \frac{8 \times 81}{3} = 216$$

$$C = B = 216$$

$$D = \frac{4C}{9} = \frac{4 \times 216}{9} = 96$$

$$E = \frac{D}{6} = \frac{96}{6} = 16$$

$$\therefore (2x + 3)^4 = 81 + 216x + 216x^2 + 96x^3 + 16x^4$$

(3) Expand $(3 - 2x + x^2)^5$

<u>Current Method:</u>

$(3 - 2x + x^2)^5$

Let $(x - 2) = p$

$(3 - 2x + x^2)^5 = (3 + px)^5$

By using binomial Expansion

$$(3 + px)^5 = 5c_0 3^5 + 5c_1 3^4 (px) + 5c_2 3^3 (px)^2 + 5c_3 3^2 (px)^3 + 5c_4 3 (px)^4 + 5c_5(px)^5$$

$$= 243 + 405px + 270p^2x^2 + 90p^3x^3 + 15p^4x^4 + p^5x^5$$

$p^5 = (x - 2)^5$

$$= 5c_0 x^5 + 5c_1 x^4 (-2) + 5c_2 x^3 (-2)^2 + 5c_3 x^2(-2)^3 + 5c_4 x(-2)^4 + 5c_5(-2)^5$$

$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

$p^4 = (x-2)^4$

$= 4c_0 x^4 + 4c_1 x^3 (-2) + 4c_2 x^2 (-2)^2 + 4c_3 x (-2)^3 + 4c_4 (-2)^4$

$= x^4 - 8x^3 + 24x^2 - 32x + 16$

$p^3 = (x-2)^3 = x^3 - 6x^2 + 12x - 8$

$p^2 = (x-2)^2 = x^2 - 4x + 4$

$(3 - 2x + x^2)^5 = 243 + 405(x-2)x + 270(x^2 - 4x + 4)x^2 + 90(x^3 - 6x^2 + 12x - 8)x^3 + 15$

$(x^4 - 8x^3 + 24x^2 - 32x + 16)x^4 + (x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32)x^5$

$= 243 + 405x^2 - 810x + 270x^4 - 1080x^3 + 1080x^2 + 90x^6 - 540x^5 + 1080x^4 - 720x^3 + 15x^8$

$- 120x^7 + 360x^6 - 480x^5 + 240x^4 + x^{10} - 10x^9 + 40x^8 - 80x^7 + 80x^6 - 32x^5$

$= 243 - 810x + 1485x^2 - 1800x^3 + 1590x^4 - 1052x^5 + 530x^6 - 200x^7 + 55x^8 - 10x^9 + x^{10}$

Vedic Method

Let $(3 - 2x + x^2)^5 = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots\ldots$

Take log and differentiate w.r.t. x, we get

$$\frac{5(2x-2)}{3-2x+x^2} = \frac{B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots}$$

$10Ax + 10Bx^2 + 10Cx^3 + 10Dx^4 + 10Ex^5 - 10A - 10Bx - 10Cx^2 - 10Dx^3 - 10Ex^4 + \ldots$

$= 3B + 6Cx + 9Dx^2 + 12Ex^3 - 2Bx - 4Cx^2 - 6Dx^3 - 8Ex^4 + Bx^2$
$+ 2Cx^3 + 3Dx^4 + 4Ex^5 + \ldots\ldots$

$-10A + (10A - 10B)x + (10B - 10C)x^2 + (10C - 10D)x^3 + (10D - 10E)x^4 + 10Ex^5 + \ldots.$
$= 3B + (6C - 2B)x + (9D - 4C + B)x^2 + (12E - 6D + 2C)x^3 + \ldots..$

Equating coefficients of power of x

$-10A = 3B$

$10A - 10B = 6C - 2B \Rightarrow 5A - 4B = 3C$

$10B - 10C = 9D - 4C + B \Rightarrow 3B - 2C = 3D$

$10C - 10D = 12E - 6D + 2C \Rightarrow 2C - D = 3E$

The first Coefficient $A = 3^5 = 243$

$$B = \frac{-10A}{3} = \frac{-10 \times 243}{3} = -810$$

$$C = \frac{5A - 4B}{3} = \frac{5 \times 243 - 4(-810)}{3}$$

$$= (5 \times 81) + (4 \times 270)$$

$$= 1485$$

$$D = \frac{3B - 2C}{3} = \frac{3(-810) - 2(1485)}{3}$$

$$= 3(-270) - 2(495)$$

$$= -1800$$

$$E = \frac{2C - D}{3} = \frac{2(1485) - (-1800)}{3}$$

$$= 2(495) + 600$$

$$= 1590$$

$\therefore\ (3 - 2x + x^2)^5 = 243 - 810x + 1485x^2 - 1800x^3 + 1590x^4 + \ldots..$

Expansion of (a + b + c) _ _ _ _ _ to the power of any integer can be directly written down using the tables that are formed as the general expansion derived in Volume III (b) page 75 and the tables N, O, P, Q, R, S, T.

One can derive such tables for the expansion (a + b + c) _ _ _ raised to any integer n. For example let us consider the expansion $(3 - 2x + x^2)^5$.

The terms in the expansion can be simply written down easily from the table (2) page no.379 in Volume III (b) and details are as follows.

Consider x^2 as a, -2x as b and 3 as c in the expansion $(3-2x+x^2)^5$ the terms are derived using the symmetry relations.

1) $a^5 = x^{10} \times 1 = x^{10}$

$b^5 = -32x^5 \times 1$

$c^5 = 243 \times 1$

2) $x^9 \to a^4b = -2x^9 \times 5 \quad = -10x^9$

3) $x^8 \to a^4c = 3x^8 \times 5 \quad = +15x^8$

$a^3b^2 = x^6.4x^2 \times 10 = \underline{40x^8}$

$+\underline{55x^8}$

4) $x^7 \to a^2b^3 = x^4.(-8x^3) \times 10 \quad = -\ 80x^7$

$a^3bc = x^6.(-2x) \times 3 \quad = -6x^7 \times 20 = \underline{-120x^7}$

$\underline{-200x^7}$

5) $x^6 \to ab^4 = x^2.(16x^4) \times 5 \quad = \ 80x^6$

$a^3c^2 = x^6 \times 9 = 9x^6 \times 10 \quad = \ 90x^6$

$a^2b^2c = x^4.(4x^2) \times 3 = 12x^6 \times 30 = \underline{360x^6}$

$+\underline{530x^6}$

6) $x^5 \to b^5 = (-32)x^5 \times 1 \quad = -32x^5$

$b^3ca = (-2x)^3.3 \times x^2 = -8x^3 \times 3x^2 \times 20 = -480x^5$

$a^2c^2b = x^4.9.(-2x) \times 30 \quad = \underline{-540x^5}$

$\underline{-1052x^5}$

7) $x^4 \to b^4c = 16x^4 \times 3 \times 5 \quad = \ 240x^4$

$c^3a^2 = 27 \times x^4 \times 10 \quad = \ 270x^4$

$b^2c^2a = 4x^2.9.x^2 \times 30 = \underline{1080x^4}$

$\underline{1590x^4}$

8) $x^3 \to b^3c^2 = (-8x^3) \times 9 \times 10 \quad = -\ 720x^3$

$c^3ab = 27 \times x^2(-2x) \times 20 = \underline{-1080x^3}$

$\underline{-1800x^3}$

9) $x^2 \to ac^4 = x^2.81 \times 5 \quad = \quad 405x^2$

$c^3b^2 = 27 \times 4x^2 \times 10 = \underline{1080x^2}$

$\underline{1485x^2}$

10) $x \to c^4b = 81 \times (-2x) \times 5 = -810x$

$\therefore (3-2x+x^2)^5 = x^{10} - 10x^9 + 55x^8 - 200x^7 + 530x^6 - 1052x^5 + 1590x^4 - 1800x^3 +$

$1485x^2 - 810x + 243$

This is well compared with the current method and also with the method described wherein Logs and derivatives are used as given by the British authors.

(4) Expand $(2x^2 - 3x + 1)^{22}$

<u>Vedic Method</u>

Let $(2x^2 - 3x + 1)^{22} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \ldots\ldots$

Take log and differentiate w.r.t x, we get

$$\frac{22(4x-3)}{2x^2-3x+1} = \frac{B+2Cx+3Dx^2+4Ex^3+5Fx^4+\ldots}{A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\ldots}$$

$$\frac{88x-66}{2x^2-3x+1} = \frac{B+2Cx+3Dx^2+4Ex^3+5Fx^4+\ldots}{A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\ldots}$$

$88Ax + 88Bx^2 + 88Cx^3 + 88Dx^4 + 88Ex^5 + 88Fx^6 + \ldots$

$-66A - 66Bx - 66Cx^2 - 66Dx^3 - 66Ex^4 - 66Fx^5 + \ldots\ldots$

$= 2Bx^2 + 4Cx^3 + 6Dx^4 + 8Ex^5 + 10Fx^6 \ldots\ldots$

$-3Bx - 6Cx^2 - 9Dx^3 - 12Ex^4 - 15Fx^5 \ldots + B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \ldots\ldots$

$\Rightarrow -66A + (88A - 66B)x + (88B - 66C)x^2 + (88C - 66D)x^3 + (88D - 66E)x^4$

$+ (88E - 66F)x^5 + \ldots\ldots$

$$= B + (2C - 3B)\,x + (3D - 6C + 2B)\,x^2 + (4E - 9D + 4C)\,x^3 + (5F - 12E + 6D)\,x^4 + \ldots.$$

Equating Coefficients of powers of x,

$\Rightarrow \quad -66A = B$

$88A - 66B = 2C - 3B \;\Rightarrow 88A - 63B = 2C$

$88B - 66C = 3D - 6C + 2B \Rightarrow 86B - 60C = 3D$

$88C - 66D = 4E - 9D + 4C \Rightarrow 84C - 57D = 4E$

$88D - 66E = 5F - 12E + 6D \Rightarrow 82D - 54E = 5F$

First Coefficient $A = 1^{22} = 1$

$B = -66A = -66$

$$C = \frac{88A - 63B}{2} = \frac{88 - 63(-66)}{2} = 2123$$

$$D = \frac{86B - 60C}{3} = -44352$$

$$E = \frac{84C - 57D}{4} = 676599$$

$$F = \frac{82D - 54E}{5} = -8034642$$

$$\therefore\ (2x^2 - 3x + 1)^{22} = 1 - 66x + 2123x^2 - 44352x^3 + 676599x^4 - 8034642x^5 + \ldots.$$

(5) Expand $(2x + 1)^{2.5}$

<u>Vedic Method</u>

Let $(2x + 1)^{2.5} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots..$

Take log and differentiate w.r.t x,

$$\frac{2.5 \times 2}{2x+1} = \frac{B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots..}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots.}$$

$$\frac{5}{2x+1} = \frac{B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots..}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots.}$$

$5A + 5Bx + 5Cx^2 + 5Dx^3 + 5Ex^4 + \ldots..$

$= B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots\ldots. + 2Bx + 4Cx^2 + 6Dx^3 + 8Ex^4 + \ldots..$

Equating Coefficients of powers of x,

$5A = B$

$5B = 2B + 2C \Rightarrow 3B = 2C$

$5C = 3D + 4C \Rightarrow C = 3D$

$5D = 4E + 6D \Rightarrow -D = 4E$

First Coefficient $A = 1^{2.5} = 1$

$B = 5A = 5$

$C = \frac{3B}{2} = \frac{3 \times 5}{2} = \frac{15}{2} = 7.5$

$D = \frac{C}{3} = \frac{15}{2 \times 3} = 2.5$

$E = \frac{-D}{4} = \frac{-5}{2 \times 4} = -0.625$

$\therefore (2x + 1)^{2.5} = 1 + 5x + 7.5x^2 + 2.5x^3 - 0.625x^4 + \ldots\ldots$

(6) Expand $(1-5x+x^2)^{1.6}$

<u>Vedic Method</u>

Let $(1-5x+x^2)^{1.6} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots$

Take log and differentiate w.r.t. x,

$$\frac{1.6(2x-5)}{1-5x+x^2} = \frac{B+2Cx+3Dx^2+4Ex^3+\ldots}{A+Bx+Cx^2+Dx^3+Ex^4+\ldots}$$

$$\frac{3.2x-8}{1-5x+x^2} = \frac{B+2Cx+3Dx^2+4Ex^3+\ldots}{A+Bx+Cx^2+Dx^3+Ex^4+\ldots}$$

$$3.2Ax + 3.2Bx^2 + 3.2Cx^3 + 3.2Dx^4 + 3.2Ex^5 + \ldots - 8A - 8Bx - 8Cx^2 - 8Dx^3 - 8Ex^4 + \ldots$$
$$= B + 2Cx + 3Dx^2 + 4Ex^3 - 5Bx - 10Cx^2 - 15Dx^3 - 20Ex^4 + Bx^2 + 2Cx^3 + 3Dx^4 + 4Ex^5 + \ldots$$

$$\Rightarrow \quad -8A + (3.2A - 8B)\,x + (3.2B - 8C)\,x^2 + (3.2C - 8D)\,x^3 + (3.2D - 8E)\,x^4 + \ldots$$

$$= B + (2C - 5B)\,x + (3D - 10C + B)\,x^2 + (4E - 15D + 2C)\,x^3 + \ldots$$

Equating Coefficients of powers of x,

$B = -8A$

$3.2A - 8B = 2C - 5B \Rightarrow 3.2A - 3B = 2C$

$3.2B - 8C = 3D - 10C + B \Rightarrow 2.2B + 2C = 3D$

$3.2C - 8D = 4E - 15D + 2C \Rightarrow 1.2C + 7D = 4E$

First Coefficient A = 1

$B = -8A = -8$

$C = \dfrac{3.2A - 3B}{2} = 13.6$

$D = \dfrac{2.2B + 2C}{3} = 3.2$

$E = \dfrac{1.2C + 7D}{4} = 9.68$

$\therefore (1-5x+x^2)^{1.6} = 1 - 8x + 13.6x^2 + 3.2x^3 + 9.68x^4 + \ldots$

(7) $(3x+2)^{-3}$

<u>Vedic Method</u>

$$(3x+2)^{-3} = \frac{1}{(3x+2)^3}$$

Let $(3x+2)^3 = A + Bx + Cx^2 + Dx^3$
Take log and differentiate w.r.t x, we get

$$\frac{3\times 3}{3x+2} = \frac{B+2Cx+3Dx^2}{A+Bx+Cx^2+Dx^3}$$

$$\frac{9}{3x+2} = \frac{B+2Cx+3Dx^2}{A+Bx+Cx^2+Dx^3}$$

$$9A + 9Bx + 9Cx^2 + 9Dx^3$$
$$= 2B + 4Cx + 6Dx^2 + 3Bx + 6Cx^2 + 9Dx^3$$
$$9A + 9Bx + 9Cx^2 + 9Dx^3$$

$$= 2B + (4C+3B)x + (6D+6C)x^2 + 9Dx^3$$

Equating Coefficients of powers of x

$9A = 2B$
$9B = 4C + 3B \Rightarrow 3B = 2C$

$9C = 6D + 6C \Rightarrow C = 2D$

First Coefficient $A = 2^3 = 8$

$$B = \frac{9A}{2} = \frac{9\times 8}{2} = 36$$
$$C = \frac{3B}{2} = \frac{3\times 36}{2} = 54$$

$$D = \frac{C}{2} = \frac{54}{2} = 27$$

$$\therefore\ (3x+2)^3 = 8 + 36x + 54x^2 + 27x^3$$

$$\frac{1}{(3x+2)^3} = \frac{1}{8+36x+54x^2+27x^3}$$

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	D_1	D_2	D_3	
	$36x$	$54x^2$	$27x^3$	1 (Dividend)
8				$\underset{Q_1}{\frac{1}{8}} - \underset{Q_2}{\frac{9x}{16}} + \underset{Q_3}{\frac{27x^2}{16}} - \underset{Q_4}{\frac{135x^3}{32}} + \underset{Q_5}{\frac{1215x^4}{128}} + \ldots$

<u>Step1:</u> $\frac{1}{8}$ (Q_1)

<u>Step2:</u> $ND = 0 - \begin{array}{c} D_1 \\ \begin{pmatrix} 36x \\ \uparrow \\ \frac{1}{8} \end{pmatrix} \\ Q_1 \end{array} = \frac{-9x}{2}$

$\frac{-9x}{2} \times \frac{1}{8} = \frac{-9x}{16}$ (Q_2)

<u>Step3:</u> $ND = 0 - \begin{array}{cc} D_1 & D_2 \\ \multicolumn{2}{c}{\begin{pmatrix} 36x & 54x^2 \\ \frac{1}{8} & \frac{-9x}{16} \end{pmatrix}} \\ Q_1 & Q_2 \end{array}$

$= \frac{27x^2}{2}$

$$\frac{27x^2}{2}\times\frac{1}{8}=\frac{27x^2}{16}\ (Q_3)$$

Step4: ND = 0 –

$$\begin{array}{ccc} D_1 & D_2 & D_3 \\ \left(\begin{array}{ccc} 36x & 54x^2 & 27x^3 \\ \frac{1}{8} & \frac{-9x}{16} & \frac{27x^2}{16} \end{array}\right) \\ Q_1 & Q_2 & Q_3 \end{array}$$

$$=\frac{-135x^3}{4}\times\frac{1}{8}$$

$$\frac{-135x^3}{32}\ (Q_4)$$

Step5: ND = 0 –

$$\begin{array}{ccc} D_1 & D_2 & D_3 \\ \left(\begin{array}{ccc} 36x & 54x^2 & 27x^3 \\ \frac{-9x}{16} & \frac{27x^2}{16} & \frac{-135x^3}{32} \end{array}\right) \\ Q_2 & Q_3 & Q_4 \end{array}$$

$$=\frac{1215x^4}{16}$$

$$\frac{1215x^4}{16}\times\frac{1}{8}=\frac{1215x^4}{128}\ (Q^5)$$

$$(3x+2)^{-3}=\frac{1}{8}-\frac{9x}{16}+\frac{27x^2}{16}-\frac{135x^3}{32}+\frac{1215x^4}{128}+\ldots\ldots$$

(8) $(1+x+x^2)^{-4}$

Vedic Method

$$(1+x+x^2)^{-4} = \frac{1}{(1+x+x^2)^4}$$

Let $(1+x+x^2)^4 = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + Hx^7 + Ix^8$

Take log and differentiate w.r.t. x, we get

$$\frac{4(2x+1)}{1+x+x^2} = \frac{B+2Cx+3Dx^2+4Ex^3+5Fx^4+6Gx^5+7Hx^6+8Ix^7}{A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6+Hx^7+Ix^8}$$

$8Ax + 8Bx^2 + 8Cx^3 + 8Dx^4 + 8Ex^5 + 8Fx^6 + 8Gx^7 + 8Hx^8 + 8Ix^9$

$+ 4A + 4Bx + 4Cx^2 + 4Dx^3 + 4Ex^4 + 4Fx^5 + 4Gx^6 + 4Hx^7 + 4Ix^8$

$= B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + 6Gx^5 + 7Hx^6 + 8Ix^7$

$+ Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + 5Fx^5 + 6Gx^6 + 7Hx^7 + 8Ix^8$

$+ Bx^2 + 2Cx^3 + 3Dx^4 + 4Ex^5 + 5Fx^6 + 6Gx^7 + 7Hx^8 + 8Ix^9$

$4A + (8A + 4B)x + (8B + 4C)x^2 + (8C + 4D)x^3 + (8D + 4E)\,x^4 + (8E + 4F)x^5$

$+ (8F + 4G)\,x^6 + (8G + 4H)x^7 + (8H + 4I)x^8 + 8Ix^9$

$= B + (B + 2C)\,x + (B + 2C + 3D)x^2 + (2C + 3D + 4E)x^3$

$+ (3D + 4E + 5F)x^4 + (4E + 5F + 6G)\,x^5 + (5F + 6G + 7H)x^6$

$(6G + 7H + 8I)\,x^7 + (7H + 8I)x^8 + 8Ix^9$

Equating Coefficients of powers of x

$4A = B$
$8A + 4B = B + 2C \Rightarrow 8A + 3B = 2C$
$8B + 4C = B + 2C + 3D \Rightarrow 7B + 2C = 3D$
$8C + 4D = 2C + 3D + 4E \Rightarrow 6C + D = 4E$
$8D + 4E = 3D + 4E + 5F \Rightarrow D = F$
$8E + 4F = 4E + 5F + 6G \Rightarrow 4E - F = 6G$
$8F + 4G = 5F + 6G + 7H \Rightarrow 3F - 2G = 7H$
$8G + 4H = 6G + 7H + 8I \Rightarrow 2G - 3H = 8I$
$8H + 4I = 7H + 8I \quad \Rightarrow H = 4I$
First Coefficient $A = 1^4 = 1$

$B = 4A = 4$

$C = \frac{8A + 3B}{2} = \frac{8 + 12}{2} = 10$

$D = \frac{7B + 2C}{3} = \frac{28 + 20}{3} = 16$

$E = \frac{6C + D}{4} = \frac{60 + 16}{4} = 19$

$F = D = 16$

$G = \frac{4E - F}{6} = \frac{76 - 16}{6} = 10$

$H = \frac{3F - 2G}{7} = \frac{48 - 20}{7} = 4$

$I = \frac{2G - 3H}{8} = \frac{20 - 12}{8} = 1$

$\therefore (1 + x + x^2)^4 = 1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8$

$$(1 + x + x^2)^{-4} = \frac{1}{1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8}$$

$4x \quad 10x^2 \quad 16x^3 \quad 19x^4 \quad 16x^5 \quad 10x^6 \quad 4x^7 \quad x^8$ | 1

1

$1 - 4x + 6x^2 + 0x^3 - 15x^4$

$Q_1 \quad Q_2 \quad Q_3 \quad Q_4 \quad Q_5$

$+ 24x^5 - 6x^6 + 36x^7 + \ldots$

$(Q_6) \quad (Q_7) \quad (Q_8)$

Step: 1 $\quad \frac{1}{1} = 1\ (Q_1)$

$$\underline{\text{Step:2}} \quad ND = 0 - \begin{array}{c} D_1 \\ \begin{pmatrix} 4x \\ \uparrow \\ 1 \end{pmatrix} \\ Q_1 \end{array} = -4x$$

$$\frac{-4x}{1} = -4x \; (Q_2)$$

$$\underline{\text{Step3:}} \quad ND = 0 - \begin{array}{c} \begin{array}{cc} D_1 & D_2 \end{array} \\ \begin{pmatrix} 4x & 10x^2 \\ 1 & -4x \end{pmatrix} \\ \begin{array}{cc} Q_1 & Q_2 \end{array} \end{array}$$

$$= 6x^2$$

$$\frac{6x^2}{1} = 6x^2 \; (Q_3)$$

$$\underline{\text{Step:4}} \quad ND = 0 - \begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{pmatrix} 4x & 10x^2 & 16x^3 \\ & \uparrow & \\ 1 & -4x & 6x^2 \end{pmatrix} \\ \begin{array}{ccc} Q_1 & Q_2 & Q_3 \end{array} \end{array}$$

$$= 0$$

$$0 \; (Q_4)$$

$$\underline{\text{Step:5}} \quad ND = 0 - \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & & D_4 \end{array} \\ \begin{pmatrix} 4x & 10x^2 & & 19x^4 \\ 1 & -4x & 6x^2 & 0x^3 \end{pmatrix} \\ \begin{array}{cccc} Q_1 & Q_2 & Q_3 & Q_4 \end{array} \end{array}$$

$$= 15x^4$$

$$\frac{-15x^4}{1} = -15x^4\ (Q_5)$$

<u>Step:6</u> ND = 0 –

$$\begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \\ 4x & 10x^2 & 16x^3 & 19x^4 & 16x^5 \\ 1 & -4x & 6x^2 & 0 & -15x^4 \\ Q_1 & Q_2 & Q_3 & Q_4 & Q_5 \end{array}$$

$= 24x^5\ (Q_6)$

<u>Step:7</u> ND = 0 –

$$\begin{array}{cccccc} D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \\ 4x & 10x^2 & 16x^3 & 19x^4 & 16x^5 & 10x^6 \\ 1 & -4x & 6x^2 & 0 & -15x^4 & 24x^5 \\ Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \end{array}$$

$= -6x^6\ (Q_7)$

<u>Step:8</u> ND = 0 –

$$\begin{array}{ccccccc} D_1 & D_2 & D_3 & D_4 & & D_6 & D_7 \\ 4x & 10x^2 & 16x^3 & 19x^4 & 16x^5 & 10x^6 & 4x^7 \\ 1 & -4x & 6x^2 & & -15x^4 & 24x^5 & -6x^6 \\ Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 & Q_7 \end{array}$$

$= -36x^7\ (Q_8)$

$(1+x+x^2)^{-4} = 1 - 4x + 6x^2 + 0x^3 - 15x^4 + 24x^5 - 6x^6 + 36x^7 + \ldots..$

(9) $(1+x^2+2x)^{\sqrt{2}} = A + Bx + Cx^2 + Dx^3$

$$\frac{\sqrt{2}(2x+2)}{1+x^2+2x} = \frac{2\sqrt{2}(x+1)}{1+x^2+2x} = \frac{B+2Cx+3Dx^2}{A+Bx+Cx^2+Dx^3}$$

x^0 Coefficient : $2\sqrt{2}\,A = B$

x^1 Coefficient : $2\sqrt{2}\,A + 2\sqrt{2}\,B = 2B + 2C$

$$\Rightarrow \sqrt{2}\,(A+B) = B + C$$

x^2 Coefficient : $2\sqrt{2}\,C + 2\sqrt{2}\,B = B + 4C + 3D \Rightarrow B = 2\sqrt{2}\,A$

$A = 1$

$B = 2\sqrt{2}$

$$\sqrt{2}\,(1+2\sqrt{2}) = 2\sqrt{2} + C \Rightarrow \sqrt{2}\,(2\sqrt{2}-1) = C$$

$\therefore (1+x^2+2x)^{\sqrt{2}} = A + Bx + Cx^2 + Dx^3 + \ldots\ldots = 1 + (2\sqrt{2})x + \sqrt{2}\,(2\sqrt{2}-1)\,x^2 + \ldots.$

To show that the Expansion of power $\frac{1}{3}$ (Cube Root) obtained in either ascending order or descending order gives same result for a Perfect Cube:

$8x^6 + 12x^5 + 42x^4 + 37x^3 + 63x^2 + 27x + 27$

Ascending Order (Result)

$(8x^6 + 12x^5 + 42x^4 + 37x^3 + 63x^2 + 27x + 27)^{1/3} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6$

Taking logarithm and differentiate on both sides w.r.t. x

$$\frac{1}{3}\left(\frac{48x^5 + 60x^4 + 168x^3 + 111x^2 + 126x + 27}{8x^6 + 12x^5 + 42x^4 + 37x^3 + 63x^2 + 27x + 27}\right) = \frac{B + 2Cx \ldots 4Ex^3 + 5Fx^4 + 6Gx^5}{A \ldots Ex^4 + Fx^5 + Gx^6}$$

$$\left(\frac{16x^5 + 20x^4 + 56x^3 + 37x^2 + 42x + 9}{8x^6 + 12x^5 + 42x^4 + 37x^3 + 63x^2 + 27x + 27}\right) = \frac{B \ldots 4Ex^3 + 5Fx^4 + 6Gx^5}{A + Bx \ldots + Ex^4 + Fx^5 + Gx^6}$$

Comparing the like terms on both sides

$x^0 \Rightarrow \quad 9A = 27B \qquad\qquad A = (27)^{1/3} = 3$

$\qquad A = 3B \qquad\qquad B = 1$

$x^1 \Rightarrow \quad 27B + 54C = 9B + 42A$

$54C = 42A - 18B$

$54C = 126 - 18 = 108$

$\Rightarrow C = 2$

$x^2 \Rightarrow \quad 63B + 54C + 81D = 9C + 42B + 37A$

$21B + 45C - 37A + 81D = 0$

$21 + 90 - 111 + 81D = 0$

$\Rightarrow D = 0$

$x^3 \Rightarrow \quad$ [illegible]$B + 126C + 81D + 108E = 56A + 37B + 42C + 9D$

$37 + 252 + 108E = 168 + 37 + 84$

$289 + 108E = 289 \Rightarrow E = 0$

$x^4 \Rightarrow \quad 42B + 74C + 189D + 108E + 135F = 20A + 56B + 37C + 42D + 9E$

$42 + 148 + 135F = 60 + 56 + 74$

$190 + 135F = 190 \Rightarrow F = 0$

$x^5 \Rightarrow \quad 12B + 84C + 111D + 172E +$ [illegible]$5F + 162G = 16A + 20B + 56C + 37D + 42E + 9F$

$12 + 168 + 162G = 48 + 20 + 112$

$180 + 162G = 180$

$G = 0$

$(8x^6 + 12x^5 + 42x^4 + 37x^3 + 63x^2 + 27x + 27)^{1/3} = (3 + x$ [illegible]$x^2)$

Descending Order (incase of given expansion is [illegible]

$(27 + 27x + 63x^2 + 37x^3 + 42x^4 + 12x^5 + x6)^{1/3} =$ [illegible]x

Taking log and differentiate w. r. t. x

$$\frac{1}{3}\left[\frac{27 + 126x + 111x^2 + 168x^3 + 60x^4 + 48x^5}{27 + 27x + 63x^2 + 37x^3 + 42x^4 + 12x^5 + 8x^6}\right] = \frac{2C\ \ + B}{Cx^2 + Bx + A}$$

$$\left[\frac{9 + 42x + 37x^2 + 56x^3 + 20x^4 + 16x^5}{27 + 27x + 63x^2 + 37x^3 + 42x^4 + 12x^5 + 8x^6}\right] = \frac{2Cx + B}{Cx^2 + Bx + A}$$

Comparing like terms on both sides

x^0 Coefficient : $9A = 27B$ $A = (27)^{1/3} = 3$

x^1 Coefficient : $42A + 9B = 27B + 54C$ $B = 1$

$126 - 18B = 54C \Rightarrow C = 2$

Verification:

x^2 Coefficient : $37A + 42B + 9C = 63B + 54C$

$111 + 42 + 18 = 63 + 108 \Rightarrow 171 = 171$

Hence Given expansion is Perfect Cube:

Imperfect Cube:

To show that the roots obtained from two different methods i.e. Straight Division method and Power series method $(A + Bx + Cx^2 + \ldots\ldots)$ in both ascending and descending order.

$(x^3 + 6x^2 + 15x + 27)^{1/3}$

Descending Order (Starting with highest power of x)

$(x^3 + 6x^2 + 15x + 27)^{1/3} = Bx + A + B^1x^{-1} + C^1x^{-2} + D^1x^{-3} + E^1x^{-4} \ldots\ldots$

Take log and differentiate w.r.t. x

$$\frac{1}{3}\left[\frac{3x^2 + 12x + 15}{x^3 + 6x^2 + 15x + 27}\right] = \frac{B - B^1x^{-2} - 2C^1x^{-3} - 3D^1x^{-4} - 4E^1x^{-5}}{Bx + A + B^1x^{-1} + C^1x^{-2} + D^1x^{-3} + E^1x^{-4}}$$

$$\left[\frac{x^2 + 4x + 5}{x^3 + 6x^2 + 15x + 27}\right] = \frac{B - B^1x^{-2} - 2C^1x^{-3} - 3D^1x^{-4} - 4E^1 \ldots}{Bx + A + B^1x^{-1} + C^1x^{-2} + D^1x^{-3}}$$

x^0 Coefficient : $C^1 + 4B^1 + 5A = -2C^1 - 6B^1$

$27B - 10B^1 = 3C^1 + 5A$ ---------------1

x^1 Coefficient : $-B^1 + 15B = B^1 + 4A + 5B$

$-2B^1 + 10B = 4A$ --------------- 2

x^2 Coefficient : $A + 4B = 6B \Rightarrow A = 2B$ --------------- 3

X^3 Coefficient : $B = B$

x^{-1} Coefficient : $D^1+4C^1+5B^1 = -3D^1-12C^1-15B^1$

$$4D^1 = -16C^1-20B^1 \Rightarrow D^1 = -4C^1 - 5B^1 \text{------------} \quad 4$$

$A = 2$

From 3 $B = 1$

From 2 $B^1=1$

From 1 $C^1 = \frac{7}{3}$

From 4 $D^1 = \frac{-43}{3}$

$\therefore (x^3 + 6x^2 + 15x + 27)^{1/3} = x + 2 + x^{-1} + \frac{7}{3}x^{-2} + \frac{-43}{3}x^{-3} + \ldots..$

Straight Division Method

CD= $3x^2$	X^3	$6x^2$	$15x$	27	0
			$-12x$	-20	$\frac{-43}{x}$
			$-3ab^2$	$-(b^3+6abc)$	$3b^2c+3ac^2 + 6abd$
	x	2	$\frac{1}{x}$	$\frac{7}{3x^2}$	$\frac{-43}{3x^3}$
	a	b	c	d	e

Ascending Order

$(x^3 + 6x^2 + 15x + 27)^{1/3} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots\ldots$

Taking logarithm and differentiate on both sides w.r.t. x

$$\frac{1}{3}\left[\frac{3x^2+12x+15}{x^3+6x^2+15x+27}\right] = \frac{B+2Cx+3Dx^2+4Ex^3\ldots}{A+Bx+Cx^2+Dx^3+Ex^4\ldots}$$

$$\left[\frac{x^2+4x+5}{x^3+6x^2+15x+27}\right] = \frac{B+2Cx+3Dx^2+4Ex^3\ldots}{A+Bx+Cx^2+Dx^3+Ex^4\ldots}$$

Comparing like terms on both sides

x^0 Coefficient : $5A = 27B$ $\qquad A = (27)^{1/3} = 3$

$$B = \frac{5A}{27} = \frac{15}{27} = \frac{5}{9}$$

x^1 Coefficient: $15B + 54C = 5B + 4A$

$54C = -10B + 4A$

$$= \frac{-50}{9} + 12 = \frac{58}{9} \Rightarrow C = \frac{58}{9X54} = \frac{29}{243}$$

x^2 Coefficient : $6B + 30C + 81D = A + 4B + 5C$

$81D = -2B - 25C + A$

$$= \frac{-10}{9} - \frac{25X29}{243} +$$

$$81D = \frac{-270-725+729}{243} = -$$

$$D = \frac{-266}{19683}$$

x^3 Coefficient : $B + 12C + 45D + 108E = B + 4C + 5D$

$108E = -8C - 40D$

$$27E = -2C - 10D$$

$$= \frac{-58}{243} + \frac{2660}{19683} = \frac{-2038}{19683}$$

$$E = \frac{-2038}{27X19683} = \frac{-2038}{531441}$$

Straight Division

	27	15x	$6x^2$	x^3	$0x^4$
$CD = 3a^2 = 27$					
			$\frac{-25x^2}{9}$	$\frac{995x^3}{729}$	$\frac{-2038x^4}{19683}$
			$-3ab^2$	(b^3+6abc)	$3b^2c+3ac^2$ + 6abd
	3	$\frac{5x}{9}$	$\frac{29x^2}{243}$	$\frac{-266x^3}{19683}$	$\frac{-2038x^4}{531441}$
	a	b	c	d	e

$$(27 + 15x + 6x^2 + x^3)^{1/3} = 3 + \frac{5}{9}x + \frac{29}{243} \quad x^2 + \frac{-266}{19683}x^3 + \frac{-2038}{531441}x^4 \ldots\ldots$$

(1)

Ex:

$$\begin{pmatrix} 3+ & 3x \\ +4y+ & 9xy \end{pmatrix}^2$$

$$\begin{pmatrix} 3+ & 3x \\ +4y+ & 9xy \end{pmatrix}^2 = \begin{pmatrix} a_0 + & b_0 x + & c_0 x^2 \\ +a_1 y + & b_1 xy + & c_1 x^2 y \\ +a_2 y^2 + & b_2 xy^2 + & c_2 x^2 y^2 \end{pmatrix} \quad \rightarrow (1)$$

Taking log and differentiating with respect to x

$$\frac{2(\ . \ +9y)}{\begin{pmatrix} 3+ & 3\ \\ +4y+ & 9xy \end{pmatrix}} = \frac{\begin{pmatrix} b_0 + & 2c_0 x \\ +b_1 y & + 2c_1 xy \\ +b_2 y^2 + & 2c_2 xy^2 \end{pmatrix}}{\begin{pmatrix} 9 & 18 & 9 \\ a_0 + & b_0 x + & c_0 x^2 \\ 24 & 78 & 54 \\ +a_1 y + & b_1 xy + & c_1 x^2 y \\ 16 & 72 & 81 \\ +a_2 y^2 + & b_2 xy^2 + & c_2 x^2 y^2 \end{pmatrix}} \quad \rightarrow (2)$$

Let x = y = 0 in equation (1), we ge.

1) $a_0 = 9$, by cross multiplication method,

2) Unity in x, $6a_0 = 3b_0 \Rightarrow b_0 = \frac{54}{3} = 18$

3) X Coefficient: $6b_0 = 3b_0 + 6c_0 \Rightarrow 6c_0 = 3b_0$

$\Rightarrow c_0 = \ = 9$

4) x^2 coefficient : $6\ c_0 = 6c_0$

5) y Coefficient: $6a_1 + 18a_0 = 4b_0 + 3b_1$

$$144 + 162 = 72 + 3b_1 \Rightarrow b_1 = \frac{23.}{3} = \ ,$$

6) Xy Coefficient

$$6b_1 + 18b_0 = 6c_1 + 3b_1 + 8c_0 + 9b_0$$

$$468 + 324 = 6c_1 + 234 + 72 + 162$$

$$6c_1 = 324$$

$$c_1 = 54$$

$$6a_2 + 18a_1 = 3b_2 + 4b_1$$

$96 + 432 = 3b_2 + 312$

$3b_2 = 216 \Rightarrow b_2 = 72$

(3) $\begin{pmatrix} 3+ & 3x \\ +4y+ & 9xy \end{pmatrix}^2 = \begin{bmatrix} a_0 + b_0x + c_0x^2 \\ + a_1y + b_1xy + c_1x^2y \\ + a_2y^2 + b_2xy^2 + c_2x^2y^2 \end{bmatrix}$

Taking log and differentiate w. r . t y

$$\frac{2(4+9x)}{\begin{bmatrix} 3+3x \\ 4y+9xy \end{bmatrix}} = \frac{\begin{bmatrix} a_1 + b_1x + c_1x^2 \\ + 2a_2y + 2b_2y + 2c_2x^2y \end{bmatrix}}{\begin{bmatrix} a_0 + b_0x + c_0x^2 \\ + a_1y + b_1xy + c_1x^2y \\ + a_2y^2 + b_2xy^2 + c_2x^2y^2 \end{bmatrix}}$$

Unity: $8a_0 = 3a_1 \Rightarrow 72 = 3a_1 \Rightarrow a_1 = 24$

Y coefficient: $8a_1 = 4a_1 + 6a_2$

$6a_2 = 4a_1$

$a_2 = \frac{96}{6} = 16$

$8c_2 + 18b_2 = 8c_2 + 18b_2$

xy^2 Coefficient: $8b_2 + 18a_2 = 18a_2 + 8b_2$

x^2y Coefficient: $8c_1 + 18b_1 = 6c_2 + 6b_2 + 4c_1 + 9b_1$

$\Rightarrow 6c_2 = 4c_1 + 9b_1 - 6b_2 = 216 + 702 - 432 = 486$

$\Rightarrow c_2 = 81$

(4)

From the comparisons of the quotients obtained fr multiplications of the Equations (2) & (3) we get

$$\begin{pmatrix} 3+ & 3x \\ +4y+ & 9xy \end{pmatrix}^2 = \begin{pmatrix} 9+ & 18x+ & 9x^2 \\ +24y+ & 78xy+ & 54x^2y \\ +16y^2+ & 72xy^2+ & 81x^2y^2 \end{pmatrix}$$

E.g.

$$\begin{pmatrix} 3+ & 2x+ & 4x^2 \\ +4y+ & 3xy+ & 3x^2y \\ +y^2+ & 2xy^2+ & 3x^2y^2 \end{pmatrix}^2 = \begin{pmatrix} 9 & 12 & 28 & 16 & 16 \\ a_0+ & b_0x+ & c_0x^2+ & d_0x^3+ & e_0x^4 \\ 24 & 34 & 62 & 36 & 24 \\ +a_1y+ & b_1xy+ & c_1x^2y+ & d_1x^3y+ & e_1x^4y \\ 22 & 40 & 67 & 46 & 33 \\ +a_2y^2+ & b_2xy^2+ & c_2x^2y^2+ & d_2x^3y^2+ & e_2x^4y^2 \\ 8 & 22 & 42 & 30 & 18 \\ +a_3y^3+ & b_3xy^3+ & c_3x^2y^3+ & d_3x^3y^3+ & e_3x^4y^3 \\ 1 & 4 & 10 & 12 & 9 \\ +a_4y^4+ & b_4xy^4+ & c_4x^2y^4+ & d_4x^3y^4+ & e_4x^4y^4 \end{pmatrix} \rightarrow (1)$$

Taking log and differentiating w.r.t x

$$\frac{2\begin{pmatrix} 2+ & 8x \\ +3y+ & 6xy \\ +2y^2+ & 6xy^2 \end{pmatrix}}{\begin{pmatrix} 3+ & 2x+ & 4x^2 \\ +4y+ & 3xy+ & 3x^2y \\ +y^2+ & 2xy^2+ & 3x^2y^2 \end{pmatrix}} = \frac{\begin{pmatrix} b_0+ & 2c_0x+ & 3d_0x^2+ & 4e_0x^3 \\ +b_1y+ & 2c_1xy+ & 3d_1x^2y+ & 4e_1x^3y \\ +b_2y^2+ & 2c_2xy^2+ & 3d_2x^2y^2+ & 4e_2x^3y^2 \\ +b_3y^3+ & 2c_3xy^3+ & 3d_3x^2y^3+ & 4e_3x^3y^3 \\ +b_4y^4+ & 2c_4xy^4+ & 3d_4x^2y^4+ & 4e_4x^3y^4 \end{pmatrix}}{\begin{pmatrix} 9 & 12 & 28 & 16 & 16 \\ a_0+ & b_0x+ & c_0x^2+ & d_0x^3+ & e_0x^4 \\ 24 & 34 & 62 & 36 & 24 \\ +a_1y+ & b_1xy+ & c_1x^2y+ & d_1x^3y+ & e_1x^4y \\ 22 & 40 & 67 & 46 & 33 \\ +a_2y^2+ & b_2xy^2+ & c_2x^2y^2+ & \ldots y^2+ & e_2x^4y^2 \\ 8 & 22 & 42 & \ldots 0 & 18 \\ +a_3y^3+ & b_3xy^3+ & \ldots x^2y^3+ & d_3x^3y^3+ & e_3x^4y^3 \\ 1 & 4 & & 12 & 9 \\ +a_4y^4+ & b \ldots & & d_4x^3y^4+ & e_4x^4y^4 \end{pmatrix}} \rightarrow (2)$$

Let substitute x = y = 0 in equation (1) we get a_0

By equating the like terms in the equation (2) by c... ...ultiplication, we get the following values:

(1) $a_0 = 9$

(2) $4a_0 = 3b_0 \Rightarrow 36 = 3b_0 \Rightarrow b_0 = 12$

(3) $16a_0 + 4b_0 = 2b_0 + 6c_0$

$6c_0 = 144 + 24 = 168 \Rightarrow c_0 = 28$

(4) $16b_0 + 4c_0 = 9d_0 + 4c_0 + 4b_0$

$$9d_0 = 12b_0 \Rightarrow 9d_0 = 144 \Rightarrow d_0 = 16$$

(5) $4d_0 + 16c_0 = 12e_0 + 6d_0 + 8c_0$

$$12e_0 = 8c_0 - 2d_0 \Rightarrow 12e_0 = 192 \Rightarrow e_0 = 16$$

(6) $4a_1 + 6a_0 = 3b_1 + 4b_0$

$$96 + 54 = 3b_1 + 48 \Rightarrow 3b_1 = 102 \Rightarrow b_1 = 34$$

(7) $4b_1 + 16a_1 + 6b_0 + 12a_0 = 6c_1 + 2b_1 + 8c_0 + 3b_0$

$$136 + 384 + 72 + 108 = 6c_1 + 68 + 224 + 36$$

$$6c_1 = 372 \Rightarrow c_1 = 62$$

(8) $9d_1 = 12b_1 - 12d_0 + 9b_0$

$$9d_1 = 408 - 192 + 108 = 324$$

$$d_1 = 36$$

(9) $4d_1 + 16c_1 + 6d_0 + 12c_0 = 12e_1 + 6d_1 + 8c_1 + 16e_0 + 9d_0 + 6c_0$

$$12e_1 = -2d_1 + 8c_1 - 3d_0 + 6c_0 - 16e_0$$

$$12e_1 = -72 + 496 - 48 + 168 - 256$$

$$12e_1 = 288 \Rightarrow e_1 = 24$$

(10) $4a_2 + 6a_1 + 4a_0 = 3b_2 + 4b_1 + b_0$

$$88 + 144 + 36 = 3b_2 + 136 + 12$$

$$3b_2 = 120 \Rightarrow b_2 = 40$$

(11) $4b_2 + 16a_2 + 6b_1 + 12a_1 + 4b_0 + 12a_0 = 6c_2 + 2b_2 + 8c_1 + 3b_1 + 2c_0 + 2b_0$

$$160 + 352 + 204 + 288 + 48 + 108 = 6c_2 + 80 + 496 + 102 + 56 + 24$$

$$6c_2 = 402 \Rightarrow c_2 = 67$$

(12) $4c_2 + 16b_2 + 6c_1 + 12b_1 + 4c_0 +$ [illegible] $= 9d_2 + 4c_2 + 4b_2 + 12d_1 + 6c_1 + 3b_1 + 3d_0 + 4c_0 + 3b_0$

$$9d_2 = 12b_2 + 9b_1 + 9b_0 - 12d_1 - 3d_0$$

$$= 480 + 306 + 108 - 432 - 48 = 414 \Rightarrow d_2 = 46$$

(13) $4d_2 + 16c_2 + 6d_1 + 12c_1 + 4d_0 + 12c_0 = 12e_2 + 6d_2 + 8c_2 + 16e_1 + 9d_1 + 6c_1 + 4e_0 + 6d_0 + 6c_0$

$12e_2 = 8c_2 - 3d_1 + 6c_1 - 2d_0 + 6c_0 -$ [illegible] $- 16e_1 - 4e_0$

$12e_2 = 536 - 108 + 372 - 32 + 168$ [illegible] $- 384 - 64$

$$12e_2 = 396 \Rightarrow e_2 = 33$$

(14) $4a_3 + 6a_2 + 4a_1 = 3b_3 + 4b_2 + b_1$

$$32 + 132 + 96 = 3b + 160 + 34 \Rightarrow b_3 = 22$$

(15) $4b_3 + 16a_3 + 6b_2 + 12a_2 + 4b_1 + 12a_1 = 6c_3 +$ [illegible] $2c_1 + 2b_1$

$6c_3 = 2b_3 +$ [illegible] $12a_2 + 12a_1 - 8c_2 - 2c_1$

$= 44 + 68 + 12$[illegible] $64 + 288 - 536 - 124 = 252$

$$c_3 = 42$$

(16) $4c_3 + 16b_3 + 6c_2 + 12b_2 + 4c_1 + 12b_1 = 9d_3 + 4c_3$ [illegible] $12d_2 + 6c_2 + 3b_2 + 3d_1 + 4c_1 + 3b_1$

$$9d_3 = 12b_3 + 9b_2 + 9b_1 - 12d_2 - 3d_1$$

$$9d_3 = 264 + 360 + 306 - 552 - 108 \Rightarrow d_3 = 30$$

(17) $4d_3 + 16c_3 + 6d_2 + 12c_2 + 4d_1 + 12c_1 = 12e_3 + 6d_3 + 8c_3 + 16e_2 + 9d_2 + 6c_2 + 4e_1 + 6d_1 + 6c_1$

$$12e_3 = -2d_3 + 8c_3 - 3d_2 + 6c_2 - 2d_1 + 6c_1 - 4e_1 - 16e_2$$

$$= -60 + 336 - 138 + 402 - 72 + 372 - 96 - 528$$

$$12e_3 = 216 \Rightarrow e_3 = 18$$

(18) $4a_4 + 6a_3 + 4a_2 = 3b_4 + 4b_3 + b_2$

$$3b_4 = 4 + 48 + 88 - 88 - 40$$

$$= 12 \Rightarrow b_4 = 4$$

(19) $4b_4 + 16a_4 + 6b_3 + 12a_3 + 4b_2 + 12a_2 = 6c_4 + 2b_4 + 8c_3 + 3b_3 + 2c_2 + 2b_2$

$$16 + 16 + 132 + 96 + 160 + 264 = 6c_4 + 8 + 336 + 66 + 134 + 80$$

$$684 = 6c_4 + 624$$

$$6c_4 = 60 \Rightarrow c_4 = 10$$

(20) $4c_4 + 16b_4 + 6c_3 + 12b_3 + 4c_2 + 12b_2 = 9d_4 + 4c_4 + 4b_4 + 12d_3 + 6c_3 + 3b_3 + 3d_2 + 4c_2 + 3b_2$

$$40 + 64 + 252 + 264 + 268 + 480 = 9d_4 + 40 + 16 + 360 + 252 + 66 + 138 + 268 + 120$$

$$1368 = 9d_4 + 1260 \Rightarrow d_4 = 12$$

(21)

$4d_4 + 16c_4 + 6d_3 + 12c_3 + 4d_2 + 12c_2 = 12e_4 + 6d_4 + 8c_4 + 16e_3 + 9d_3 + 6c_3 + 4e_2 + 6d_2 + 6c_2$

$$48 + 160 + 180 + 504 + 184 + 804 = 12e_4 + 72 + 80 + 288 + 270 + 252 + 132 + 276 + 402$$

$$1880 = 12e_4 + 1772 \Rightarrow 12e_4 = 108 \Rightarrow e_4 = 9$$

$$\begin{pmatrix} 3+ & 2x+ & 4x^2 \\ +4y+ & 3xy+ & 3x^2y \\ +y^2+ & 2xy^2+ & 3x^2y^2 \end{pmatrix}^2 = \begin{pmatrix} \overset{9}{a_0+} & \overset{12}{b_0x+} & \overset{28}{c_0x^2+} & \overset{16}{d_0x^3+} & \overset{16}{e_0x^4} \\ \overset{24}{+a_1y+} & \overset{34}{b_1xy+} & \overset{62}{c_1x^2y+} & \overset{36}{d_1x^3y+} & \overset{24}{e_1x^4y} \\ \overset{22}{+a_2\;{}^2+} & \overset{40}{b_2xy^2+} & \overset{67}{c_2x^2y^2+} & \overset{46}{d_2x^3y^2+} & \overset{33}{e_2x^4y^2} \\ \overset{8}{+a_3y^3+} & \overset{22}{b_3xy^3+} & \overset{42}{c_3x^2y^3+} & \overset{30}{d_3x^3y^3+} & \overset{18}{e_3x^4y^3} \\ +a_4y^4+ & b_4xy^4+ & c_4x^2y^4+ & d_4x^3y^4+ & e_4x^4y^4 \end{pmatrix} \rightarrow (2)$$

Taking log and differentiating w.r.t y

$$\frac{2\begin{pmatrix} 4+ & 3x+ & 3x^2 \\ +2y+ & 4xy+ & 6x^2y \end{pmatrix}}{\begin{pmatrix} 3+ & 2x+ & 4x^2 \\ +4y+ & 3xy+ & 3x^2y \\ +y^2+ & 2xy^2+ & 3x^2y^2 \end{pmatrix}} = \frac{\begin{pmatrix} a_1+ & b_1x+ & c_1x^2+ & d_1x^3+ & e_1x^4 \\ +2a_2y+ & 2b_2xy+ & 2c_2x^2y+ & 2d_2x^3y+ & 2e_2x^4y \\ +3a_3y^2+ & 3b_3xy^2+ & 3c_3x^2y^2+ & 3d_3x^3y^2+ & 3e_3x^4y^2 \\ +4a_4y^3+ & 4b_4xy^3+ & 4c_4x^2y^3+ & 4d_4x^3y^3+ & 4e_4x^4y^3 \end{pmatrix}}{\begin{pmatrix} 9 & 12 & 28 & 16 & 16 \\ a_0+ & b_0x+ & c_0x^2+ & d_0x^3+ & e_0x^4 \\ 24 & 34 & 62 & 36 & 24 \\ +a_1y+ & b_1xy+ & c_1x^2y+ & d_1x^3y+ & e_1x^4y \\ 22 & 40 & 67 & 46 & 33 \\ +a_2y^2+ & b_2xy^2+ & c_2x^2y^2+ & d_2x^3y^2+ & e_2x^4y^2 \\ 8 & 22 & 42 & 30 & 18 \\ +a_3y^3+ & b_3xy^3+ & c_3x^2y^3+ & d_3x^3y^3+ & e_3x^4y^3 \\ +a_4y^4+ & b_4xy^4+ & c_4x^2y^4+ & d_4x^3y^4+ & e_4x^4y^4 \end{pmatrix}}$$

(1) $8a_0 = 3a_1 \Rightarrow a_1 = 24$

(2) $8a_1 + 4a_0 = 6a_2 + 4a_1$

$$192 + 36 = 6a_2 + 96 \Rightarrow a_2 = \frac{132}{6} = 22$$

(3) $8a_2 + 4a_1 = 9a_3 + 8a_2 + a_1$

$176 + 96 = 9a_3 + 176 + 24$

$9a_3 = 72 \Rightarrow a_3 = 8$

(4) $8a_3 + 4a_2 = 12a_4 + 12a_3 + 2a_2$

$12a_4 = -4a_3 + 2a_2$

$= -32 + 44 = 12 \Rightarrow a_4 = 1$

From all the comparisons of like terms, we get

$$\begin{pmatrix} 3+ & 2x+ & 4x^2 \\ +4y+ & 3xy+ & 3x^2y \\ +y^2+ & 2xy^2+ & 3x^2y^2 \end{pmatrix}^2 = \begin{pmatrix} 9+ & 12x+ & & 16x^3+ & 16x^4 \\ +24y+ & 34xy & & 36x^3y+ & 24x^4y \\ +22y^2+ & 40x & & 46x^3y^2+ & 33x^4y^2 \\ +8y^3+ & 22xy^3+ & + & 30x^3y^3+ & 18x^4y^3 \\ +y^4+ & 4xy^4+ & + & 12x^3y^4+ & 9x^4y^4 \end{pmatrix}$$

Section-6
EVALUATION OF LOGARITHMS & EXPONENTIALS

The method of evaluation is described by British authors making use of Taylor's theorem, differentiation and argumentation principles.

For example ln (3.8)
It is written as ln (3+8x)
This can be written in power series of x as
Step1: ln (3+8x) = $a+bx+cx^2+dx^3+ex^4+fx^5+$ _ _ _ _ _ _ — (1)
When x = 0, log 3 = a ⇒ a = 1.098612289

Step2: Differentiating both sides w. r. t. x

$$\frac{8}{3+8x} = b+2cx+3dx^2+4ex^3+5fx^4+ \_\_\_\_\_\_\_ \quad — (2)$$

In comparison of the quotients, we get

$3b=8 \Rightarrow b = \frac{8}{3} = 2.666666667$

Examples:

1. ln (3.8)
ln(3.8) = ln(3+8x) with x = 1/10
ln (3+8x) = $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + \ldots$
a = ln3 = 1.098612289
Differentiate w.r.t. x

$$\frac{8}{3+8x} = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4 + 6gx^5 + \ldots$$

By Cross multiplication and comparing the like terms, we get

$3b = 8 \Rightarrow b = \frac{8}{3} = 2.666666667$

$8b + 6c = 0$

$\Rightarrow 6c = -8b \Rightarrow c = \frac{-8b}{6} = -3.555555555$

$16c + 9d = 0$

$\Rightarrow d = \frac{-16c}{9} = 6.320987654$

$24d + 12e = 0$
$\Rightarrow e = -2d = 0.001264197531$

ln (3.8) = 1.098612289 + 0.266666666 - 0.03555555 + 0.006320987654 – 0.001264197531 = 1.33478019

Explanation:

Referring to equation (1).

ln (3.8) = ln (3+8x) where x = 0.1

Can be written in the form of Taylor's theorem as a.bcdef.......in decimal notation.

When x = 0 log 3 = a.,

Considering the differentiation equation (2)

In order to obtain the quotients, one has to multiply the cross wise of the equation (2).And comparing the quotients on both sides (Argumentation) one gets the values of b, c, d, e.......

The final result

$$Ln (3.8) = 1.33478019.$$

The verification can be obtained by evaluating the exponentials.

Refer example ln (3.13520794)

2. ln (49)

$\ln (7^2) = 2 \ln 7 = 2 \times 1.945910149 = 3.891820298$

3. $\ln (27) = \ln(3^3) = 3\ln 3 = 3 \times 1.098612289 = 3.295836867$

4. ln (183)

$183 = 3 \times 61$

$\ln (183) = \ln(3) + \ln(61)$

$\frac{61}{6} = 10.1666666.... = 10 \times 1.01666666....$

$$\ln (183) = \ln(3) + \ln(6 \times \frac{61}{6})$$

$$= \ln (3) + \ln (6) + \ln (10) + \ln (1.016666)$$

Evaluation of $\ln (1.016666) = a + bx + cx^2 + dx^3 + ex^4 + ...$

$$Ln (1 + 0x + x^2 + 6x^3 + 6x^4 + 6x^5 + ...) = a + bx + cx^2 \cdot \quad ^3 + ...$$
0 (1

Differentiating w.r.t. x

$$\frac{2x + 18x^2 + 24x^3 + 30x^4 + ...}{1 + 0x + x^2 + 6x^3 + 6x^4 + 6x^5 + ...} = b + 2cx \qquad x^4 + 6gx^5 + ...$$

By cross multiplication and comparing the , we get

$2c = 2 \Rightarrow c = 1$

$3d + b = 18 \Rightarrow d = 6$

$4e + 2c = 24 \Rightarrow 4e = 22 \Rightarrow e = \frac{11}{2}$

$5f + 3d + 12e = 30 \Rightarrow f = 0$

ln (1.016666) = 0 + 0.000 + 0.01 + 0.006 + 0.00055 = 0.016555

∴ ln (183) = ln(3) + ln(6) + ln(10) + ln(1.016666)

ln (3)	=	1.098 612289
ln (6)	=	1.791759469
ln (10)	=	2.302585509
ln (1.016666)	=	0.01655
ln (183)	=	5.209507267

5. ln (2.4612)

ln (2.4612) = $\ln(2 + 4x+6x^2+x^3 + 2x^4) = a + bx + cx^2 + dx^3 + ex^4 + \ldots$

ln (2) = a = 0.693147180

Differentiating w.r.t. x

$$\frac{4 + 12x + 3x^2 + 8x^3}{2 + 4x + 6x^2 + x^3 + 2x^4} = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4$$

By Cross multiplication and comparing the like terms, we get

$2b = 4 \Rightarrow b = 2$

$4c + 4b = 12 \Rightarrow c = 1$

$6d + 8c + 6b = 3 \Rightarrow 6d = 3\text{-}8\text{-}12 = -17 \Rightarrow d = \frac{-17}{6}$

$8e + 12d + 12c + b = 8 \Rightarrow e = \frac{186}{24} = 7.75$

ln (2.4612) = 0.693147180 + 0.2..00000+0.010000000-0.00283333 + 0.000280000
= 0.900593847

Evaluation of exponentials:

Example:

Exp 0.3$\overline{2}$4

$\mathrm{Ln}\,(a + bx + cx^2 + dx^3 + - - - -) = 0 + 3x - 2x^2 + 4x^3$

$$\frac{b + 2cx + 3dx^2 + - - -}{a + bx + cx^2 + dx^3 + - - -} = 3 - 4x + 12x^2$$

$1 \quad 3 \quad \frac{5}{2} \quad \frac{5}{2}$

$3a = b \Rightarrow b = 3$

$-4a + 3b = 2c$

$-4 + 9 = 2c \Rightarrow c = 2.5$

$12a - 4b + 3c = 3d$

$3d = 12 - 12 + \frac{15}{2} \Rightarrow d = \frac{15}{3x2} = \frac{5}{2}$

exp 0.3$\overline{2}$4 = 1.00000 + 0.30000 + 0.02500 + 0.0025 = 1.3275

The given exponential is rewritten if necessary in smaller figures for example
exp (0.284) can be written as
exp $(0.3\bar{2}4)$

The working as follows Log power series is equated to the given exponential to be determined. For example:

Step2: The given exponential $0.3\bar{2}4$

It is written in the form of
$\ln(a+bx+cx^2+dx^3+.......) = 0+3x-2x^2+4x^3+......$ — (1)
Where $x = 0.1$
On differentiation on both sides, we get to obtain

$$\frac{b+2cx+......}{a+bx+.......} = 3-4x+12x^2+......... \quad — (2)$$

On cross multiplication and applying argumentation, the coefficients can be obtained.
When $x = 0$ one can get the value of a from equation (1)

exp 0.284
B = exp 0.284
exp $0.3\bar{2}4$
$\text{Ln}(a+bx+cx^2+dx^3+......) = 0+3x-2x^2+4x^3$

$$\frac{b+2cx+3dx^2+....}{a+bx+cx^2+dx^3.....} = 0+3-4x+12x^2$$

$$1 \quad 3 \quad \frac{5}{2} \quad \frac{5}{2}$$

$3a = b \Rightarrow b = 3$
$-4a+3b = 2c$
$-4+9 = 2c \Rightarrow c = 2.5$

$12a-4b+3c = 3d$

$$3d = 12-12+\frac{15}{2}$$

$$d = \frac{15}{3\times 2} = \frac{5}{2}$$

exp $0.3\bar{2}4 = 1.00000+0.30000+0.02$ ... $= 1.3275$

Note on describing the interpolation
1. exp (0.284)

ln 8	=	2.079441542
ln 6	=	1.791759469
$\ln(\frac{8}{6})$	=	0.287682073
ln (B)	=	0.284

$\ln(B)-\ln(\frac{8}{6}) = -0.003682073 \; = -(0.004\bar{3}\bar{2}21\bar{3}3)$

$\ln(\frac{6B}{8}) = \ln(a+bx+cx^2+dx^3+ex^4+fx^5+..) = 0+0x+0x^2-4x^3+3x^4+2x^5-2x^6-x^7+3x^8-3x^9$

Differentiate w.r.t x

$\frac{b+2cx+3dx^2+4ex^3+5fx^4+6gx^5+7hx^6+...}{a+bx+cx^2+dx^3+ex^4+fx^5+gx^6+hx^7+...} = 0+0x-12x^2+12x^3+10x^4-12x^5-7x^6+24x^7-27x^8$

By cross multiplication and comparing like terms, we get

$a = 1$

$a.x0 = b \Rightarrow b = 0$

$0+0 = 2c \Rightarrow c = 0$

$-12a = 3d \Rightarrow d = -4$

$12a = 4e \Rightarrow e = 3$

$10a = 5f \Rightarrow f = 2$

$-12a-12d = 6g \Rightarrow g = 6$

$\frac{6B}{8} = 1+0+0-0.004+0.0003+0.00002+0.000006 = 0.99632600$

$\Rightarrow B = \exp(0.284) = 1.328434667$

2. exp (4.2184)

$B = 4.2184$

ln (8)	=	2.07944154
ln (9)	=	2.197224577
ln (72)	=	4.276666119
expB		=4.2184

$\ln(\frac{B}{72}) \quad = \quad -0.058266119 = -(0.1\bar{4}\bar{2}3$

$\ln(\frac{B}{72}) = \ln(a+bx+cx^2+dx^3+ex^4+fx^5+..) = 0-x- \qquad 3x^4+3x^5+4x^6-x^7-2x^8+x^9$

Differentiate w.r.t x

$\frac{b+2cx+3dx^2+4ex^3+5fx^4+6gx^5+7hx^6+...}{a+bx+cx^2+dx^3+ex^4+fx^5+gx^6+hx^7+...} = -1+8x+6x^2-12x^3+15x^4+24x^5-7x^6-16x^7+9x^8$

By cross multiplication and comparing like terms, we get

$a = 1$

$-a = b \Rightarrow b = -1$

$8a-b = 2c \Rightarrow c = \frac{9}{2}$

$6a+8b-c = 3d \Rightarrow d = \frac{-13}{6}$

$-12a+6b+8c-d = 4e \Rightarrow e = \frac{121}{24}$

$15a-12b+6c+8d-e = 5f \Rightarrow f = \frac{253}{40}$

$24a+15b-12c+6d+8e-f = 6g \Rightarrow g = \frac{-2879}{720}$

$\frac{B}{72} = 1.0-0.1+0.045-0.00216667+0.00050416+0.00006325-0.00000399+ 0.000001672$
$= 0.943398422$

$B = 72 \times 0.943398422 = 67.92468638$

$\exp(4.2184) = 67.92468638$

3. exp (32)

ln (3)	=	1.098612289
ln (8)	=	2.079441542
ln (24)	=	3.178053831
ln (24^{10})	=	31.78053831
exp (B)	=	32.00000000
		0.21946169

$\frac{B}{24^{10}} = 0.22\bar{1}5\bar{4}2\bar{3}\bar{1}$

$\frac{B}{24^{10}} = \ln(a+bx+cx^2+dx^3+ex^4+fx^5+\ldots) = 0+2x-2x^2-x^3+5x^4-4x^5+2x^6-3x^7-x^8$

Differentiate w.r.t x

$$\frac{b+2cx+3dx^2+4ex^3+5fx^4+6gx^5+7hx^6+\dots}{a+bx+cx^2+dx^3+ex^4+fx^5+gx^6+hx^7+\dots} = 2+4x-3x^2+20x^3-20x^4+12x^5-21x^6-8x^7$$

By Cross multiplication and comparing the like terms, we get

$a = 1$

$2a = b \Rightarrow b = 2$

$4a + 2b = 2c \Rightarrow c = 4$

$-3a+4b+2c = 3d \Rightarrow d = \frac{13}{3}$

$20a-3b+4c+2d = 4e \Rightarrow e = \frac{29}{3}$

$-20a+20b-3c+4d+2e = 5f \Rightarrow f = \frac{134}{15}$

$12a-20b+20c-3d+4e+2f = 6g \Rightarrow g = \frac{1433}{90}$

$\frac{B}{24^{10}} = 1+0.2+0.0404+0.00433333+0.00096666+0.00008933+0.00001592$

$= 1.24540524$

$B = 7.896290289 \times 10^{13}$

4. exp (11.08)

ln (8)	=	2.079441542
5ln (8)	=	10.39720771
ln (2)	=	0.693147180
ln (2 X 8^5)	=	11.09035489
ln (B)	=	11.08000000
ln (B) – ln(2X8^5)	=	-0.01035489

$\frac{B}{2X8^5} = -0.01035489 = -(0.010355\dot{1}\dot{1})$

$= \ln(a+bx+cx^2+dx^3+ex^4+fx^5+\dots) = 0+0x-1x^2+0x^3-3x^4-5x^5-5x^6+x^7+x^8$

Differentiate w.r.t x

$$\frac{B}{2X8^5} = \frac{b+2cx+3dx^2+4ex^3+5fx^4+6gx^5+7hx^6+\ldots}{a+bx+cx^2+dx^3+ex^4+fx^5+gx^6+hx^7+\ldots} = 0-2x+0x^2-12x^3-25x^4-30x^5+7x^6+8x^7$$

By cross multiplication and comparing like terms, we get

$a = 1$

$a \times 0 = b \Rightarrow b = 0$

$-2a = 2c \Rightarrow c = -1$

$-2b = 3d \Rightarrow d = 0$

$-12a - 2c = 4e \Rightarrow e = \dfrac{-5}{2}$

$-25a+12b-2d = 5f \Rightarrow f = -5$

$-30a-25b-12c-2e = 6g \Rightarrow g = \dfrac{-13}{6}$

$7a-30b-25c-12d-2f = 7h \Rightarrow h = 6$

$8a+7b-30c-25d-12e-2g = 8i \Rightarrow i = \dfrac{217}{24}$

$$\frac{B}{2X8^5} = 1 + 0.0 - 0.01 + 0.0 - 0.00025 - 0.00005 - 0.00000216 + 0.0000006 + 0.00000009$$

$$= 0.98969853$$

$B = (2 \text{ X } 8^5)(0.98969853) = 64860.88286 \cong 64860.883$

5. exp (-0.672)

$A = -0.672$

$\ln(3)$	=	1.098612289
$\ln(6)$	=	1.791759469
$\ln(\frac{3}{6})$	=	-0.69314718
$\ln(A) - \ln(\frac{3}{6}) =$		0.02114718
$\ln(\frac{6A}{3})$	=	$0.02114718 = 0.02115\dot{3}2\dot{2}$

$\ln(\frac{6A}{3}) = \ln(a+bx+cx^2+dx^3+ex^4+fx^5+..) = 0+0x+2x^2+x^3+x^4+5x^5-3x^6+2x^7-2x^8$

Differentiate w.r.t x

$\frac{B}{2X8^5} = \frac{b+2cx+3dx^2+4ex^3+5fx^4+6gx^5+7hx^6+8ix^7...}{a+bx+cx^2+dx^3+ex^4+fx^5+gx^6+hx^7+ix^8...} = 0+4x+3x^2+4x^3+25x^4-$
$18x^5+14x^6-16x^7$

By cross multiplication and comparing like terms, we get

$a = 1$

$a \times 0 = b \Rightarrow b = 0$

$4a = 2c \Rightarrow c = 2$

$3a+4b=3d \Rightarrow d = 1$

$4a+3b+4c = 4e \Rightarrow e = 3$

$25a + 4b + 3c + 4d = 5f \Rightarrow f = 7$

$-18a+25b+4c+3d+4e = 6g \Rightarrow g = \frac{5}{6}$

$14a-18b+25c+4d+3e+4f = 7h \Rightarrow h = 15$

$-16a+14b-18c+25d+4e+3f+4g = 8i \Rightarrow i = \frac{28}{24}$

$\frac{6A}{3} = 1 + 0 + 0.02+0.001+0.0003+0.00007+0.0000083+0.00000150+0.00000001$
$= 1.02137231$
$A = 0.510686155$

Section-7
TRIGONOMETRIC, HYPERBOLIC AND INVERSE FUNCTIONS

The evaluation of these functions are from the formulae in terms of a series functions of x which is based on the series expansion in power series. (x in radians)

$\text{Cos}\,(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - - - - -$

$\text{Sin}\,(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - - - -$

$\text{Tan}^{-1}\,(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - - - -$

$\text{Sinh}x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + - - - - - - -$

$\text{Cosh}\,(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + - - -$

$\text{Tan h}^{-1}\,(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + - - -$

Evaluation has been worked out by British Authors*. We have worked out in a slightly different way (direct method) and a comparison of the results obtained by both the methods shows that in the forme method the accuracy to a particular decimal is being built up step by step starting from the first decimal, where as in the method suggested by the author, one starts with the required decimal accuracy, which is being modified systematically.The evaluation in the first step itself can be aimed at by choosing the expression of x in the power series with the choice of requ... decimal in the beginning itself. This later method is a direct substitution. A few examples are illustrated below.

The results are verified with the standard t... also those obtained from computer / scientific calculator.

Cos 0.8235
Cos 0.15
Tan^{-1} 0.323
Tan^{-1} 0.35
Sin 0.15
Sin 0.323

(1) Example:

Cos (0.8235) (up to nine decimal)

x = 0.8235 radian

$$Cos = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + - - - - - -$$

Step (1):

$x^2 = (0.8235)^2$ By using Duplex Method

0.	0	64	32	52	92	29	30	25

= 0.67815225

and $\frac{x^2}{2!} = \frac{0.67815225}{2} = 0.339076125$

Step (2):

$(x^2)^2 = (0.67815225)^2$ By using Duplex Method

0.	0	36	84	145	124	138	110	133	130	131	104	34	58	24	20	25

$= 0.459890474$ and $\frac{x^4}{4!} = 0.019162103$

step (3):

$x^6 = (x^4, x^2)$ By using Duplex Method

$x^4 \rightarrow 0.459890474$, $x^2 \rightarrow 0.678152250$

0.	0	24	58	121	155	207	169	167	167	209	167	117	92	42	42	43	20	[illegible]

$x^6 = 0.311875759696665 = 0.31187576$ and $\frac{x^6}{6!} = \frac{0.31187576}{720} =$ 00043316

Similarly $\frac{x^8}{8!} = 0.000005245$ $\frac{x^{10}}{10!} = 0.000000039$

$$\therefore Cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} - - - -$$

$\Rightarrow$ 1 − 0.339076125 + 0.019162103 − 0.00[illegible] [illegible]5245 − 0.000000039

= 0.679657424 = 0.679658023

Cos (0.8235) = 0.679652818

* Evaluated in British Authors Method

Cos (0.8235)

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$$

$$A = \frac{x^2}{2!}, B = \frac{A^2}{6}, C = \frac{AB}{15}, D = \frac{AC}{28}, E = \frac{AD}{45}$$

$1-\frac{x^2}{2!}$	1	.	$\bar{3}$	$\bar{3}$	$\bar{9}$	0	$\bar{7}$	$\bar{6}$	$\bar{1}$	$\bar{2}$	$\bar{5}$
$\frac{x^4}{24}$	0	.	0	1	9	1	6	2	1	0	3
$-\frac{x^6}{720}$	0	.	0	0	0	$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{6}$	0
$\frac{x^8}{40320}$	0	.	0	0	0	0	0	5	2	4	5
$\frac{-x^{10}}{3628800}$	0	.	0	0	0	0	0	0	0	$\bar{3}$	$\bar{9}$
	1	.	$\bar{3}$	$\bar{2}$	0	$\bar{3}$	$\bar{4}$	$\bar{2}$	1	$\bar{7}$	$\bar{6}$

$$= 0.6\ 7\ 9\ 6\ 5\ 8\ 0\ 2\ 4$$

<u>Duplex method:</u>

<u>Row1</u>

1) $-\frac{0.8235^2}{2} = 0.\ \bar{3}\,\bar{4}\,1\,\bar{1}\,2\,4\,\bar{1}\,\bar{2}\,\bar{5}$

<u>Row2</u>

2) $B = \frac{A^2}{6}$

$D(\bar{3}) = 9 + 6 = 1\ r_3$

$D(\bar{3}\,\bar{4}) = 24 + 30 = 54 + 6 = 9\ r_0$

$D(\bar{3}\,\bar{4}\,1) = \bar{6} + 16 = 10 + 6 = 1\ r_4$

$D(\bar{3}\,\bar{4}\,1\,\bar{1}) = 6 + \bar{8} + 40 = 38 + 6 = 6\ r_2$

$D(\bar{3}\,\bar{4}\,1\,\bar{1}\,2) = \overline{12} + 8 + 1 + 20 = 17 + 6 = 2\ r_5$

$D(\bar{3}\,\bar{4}\,1\,\bar{1}\,2\,4) = \overline{24} + \overline{16} + \bar{2} + 50 = 8 + 6 = 1\ r_2$

$D(\bar{3}\,\bar{4}\,1\,\bar{1}\,2\,4\,\bar{1}) = 6 + \overline{32} + 4 + 1 + 20 = \bar{1} + 6 = 0\ r_1$

$D(\bar{3}\,\bar{4}\,1\,\bar{1}\,2\,4\,\bar{1}\,\bar{2}) = 12 + 8 + 8 + \bar{4} - 10 = 28 - 14 = 14 + 6 = 2\ r_2$

$D(\bar{3}\,\bar{4}\,1\,\bar{1}\,2\,4\,\bar{1}\,\bar{2}\,\bar{5}) = 30 + 16 + \bar{2} + \bar{8} + 4 + 20 = 60 + 6 = 10\ r_0$

3) $C = \frac{AB}{15}$

$(\bar{3}) = \bar{6} + 15 = \bar{4}\ 0\ r_{\bar{6}}$

$1\ r_{\bar{7}}$

$\begin{pmatrix} \bar{3} & \bar{4} \\ 1 & 9 \end{pmatrix} = \overline{54} + \bar{8} + \overline{60}$

Row3

$0 + 15 = 0\ r_0$

$\bar{3} + 15 = 0\ r_{\bar{3}}$

$\overline{31} + \overline{30} = \overline{61} + 15 = \bar{4}\ r_{\bar{1}}$

$\overline{38} + \overline{10} = \overline{48} + 15 = \bar{3}\ r_{\bar{3}}$

$\overline{14} + \overline{30} = \overline{44} + 15 = \bar{3}\ r_1$

$\overline{36} + 10 = \overline{26} + 15 = \bar{1}\ r_{\overline{11}}$

$16 + \overline{110} = \overline{94} + 15 = \bar{6}\ r_{\bar{4}}$

$29 + \overline{40} = \overline{11} + 15 = 0\ r_{\overline{11}}$

Row4

$D = \frac{AC}{28}$ By UT Multiplication

$0.\ \bar{3}\ \bar{4}\ 1\ \bar{1}\ 2\ 4\ \bar{1}\ \bar{2}\ \bar{5}$

$0\ 0\ 0\ \bar{4}\ \bar{3}\ \bar{3}\ \bar{1}\ \bar{6}\ 0$

0.	0	0	12	25	25	16	12

AC = 0.000012 25 25 16

Row4

$\frac{12}{28} = 0\ r$

$12 + 28 = 0\ r_{12}$

$25 + 120 = 145 + 28 = 5\ r_5$

$17 + 50 = 67 + 28 = 2\ r_{11}$

$16 + 110 = 126 + 28 = 4\ r_{14}$

$12 + 140 = 152 + 28 = 5\ r_{12}$

Row5

$D = \dfrac{AD}{45}$

By UT Multiplication

$0.\ \overline{3}\ \overline{4}\ 1\ \overline{1}\ 2\ 4\ \overline{1}\ \overline{2}\ \overline{5}$

$0.\ 0\ 0\ 0\ 0\ 0\ 5\ 2\ 4\ 5$

$0.\ \big|\ 0\ 0\ 0\ 0\ 0\ \overline{15}\ \overline{26}\ \overline{15}\ \overline{34}$

$AD = 0.00000\overline{15}\ \overline{26}\ \overline{15}\ \overline{34}$

B is obtained by $\left(\dfrac{A^2}{6}\right)$ using Duplex

Row5

$\dfrac{AD}{45} = \overline{15} \div 45 = 0\, r_{\overline{15}}$

$\overline{26} + \overline{150} = \overline{176} \div 45 = \overline{3}\, r_{\overline{41}}$

$\overline{15} + \overline{410} = \overline{425} \div 45 = \overline{9}\, r_{\overline{20}}$

$AC = A = \overline{3}\ \overline{4}\ 1\ \overline{1}\ 2\ 4\ \overline{1}\ \overline{2}\ \overline{5}$

$C = \overline{4}\ \overline{3}\ \overline{3}\ \overline{1}\ \overline{6}\ 0$

I) $CD\begin{pmatrix} \overline{3} \\ \overline{4} \end{pmatrix} = 12$

$CD\begin{pmatrix} \overline{3} & \overline{4} \\ \overline{4} & \overline{3} \end{pmatrix} = 25$

$CD\begin{pmatrix} \overline{3} & \overline{4} & 1 \\ \overline{4} & \overline{3} & \overline{3} \end{pmatrix}$

(2) Example:

Cos (0.15) **(Substituting the value 0.15 as x value in radians.)**

$\text{Cos}(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} + \text{-------}$

$x = 0.15$

0.	0	0	0	0	0	10	10	37	16	44	20	25	0	0	0	0

Step (1):

$x^2 = (0.15)^2$ By using Duplex Method

0.	0	1	10	25

$= 0.0225$

and $\frac{x^2}{2!} = 0.01125$

Step (2):

$x^4 = (0.0225)^2$ By Duplex Method

0.	0	0	0	4	8	24	20	25

$= 0.00050625$

and $\frac{x^4}{4!} = \frac{(x^2)^2}{24} = \frac{0.00050625}{24} = 0.00002109375$

Step (3):

$x^6 = (x^4, x^2)$ By using Duplex Method.

$x^4 \rightarrow 0.00050625$

$x^2 \rightarrow 0.02250000$

$x^6 =$

$= 0.000011390625$ and $\frac{x^6}{6!} = 0.0000000158203125$

$\therefore \text{Cos}(0.15) = 0.988771077$ (up to 9 decimals)

(3) Example:

Sin (0.15) (Up to nine decimal) $x = 0.15$

$\text{Sin}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \text{-------}$

Step(1):

$x^2 =$

0.	0	1	10	25

$= 0.0225$

Step (2):

$x^3 = (x)(x^2) \cong 0.00337500$ and $\frac{x^3}{3!} = 0.0005625$

Step (3):

$x^5 = (x^3)(x^2)$

0.	0	0	0	0	6	12	35	39	45	25	0	0

= 0.0000759375

and $\frac{x^5}{5!} = 0.0000006328125$

Step (4):

$x^7 = (x^5)(x^2)$

0.	0	0	0	0	0	0	14	24	63	49	65	39	45	25	0	0	0

= 0.00000170859375

and $\frac{x^7}{7!} = 0.0000000003390066964$

$\therefore$ Sin (0.15) = 0.149438133

Verification: -

Cos (0.15) = 0.988771077 from Eg. (1)

$\sin^2(0.15) + \cos^2(0.15)$

$\sin^2(x) + \cos^2(x) = 1$

0.022331755 + 0.977668242 = 0.999999997 ~ 1

(4) **Example:**

Sin (0.323) $\qquad x = (0.323)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \quad -$$

Step (1):

$x^2 =$

0.	0	9	12	22	12	9

= 0.104329

Step (2):

$x^3 = (x^2)(x)$

0.	0	3	2	15	17	24	4

= 0.033698267 and $\frac{x^3}{3!} = 0.005616377833$

Step (3):

$x^5 = (x^3)(x^2)$

0.	0	0	3	3	18	30	47	89	104	111	127	0	122	51	68	63	0	0	0

$x^5 =$ 0.00351570649784300 and $\frac{x^5}{5!} = 0.000029298$

$\therefore$ Sin (0.323) = 0.31741292.

(5) Example:

$Tan^{-1}(0.35)$ $x = (0.35)$

$$Tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{------}$$

Step (1):

$x^2 = (0.35)^2$

0.	0	9	30	25

$x^2 = \ = 0.1225$

Step (2):

$x^3 = (x^2)(x) =$

0.	0	3	11	16	25	25	0	0

$= 0.0428750$

and $\frac{x^3}{3} = 0.014291666$

Step (3):

$x^5 = (x^3)(x^2) =$

0.	0	0	4	10	20	47	45	64	45	25	0

$= 0.0052521875$

and $\frac{x^5}{5} = 0.0010504375$

Step (4):

$x^7 = (x^5)(x^2)$

0.	0	0	0	5	12	19	41	25	39	35	40	64	45	25

$= 0.00064339296875$ and $\frac{x^7}{7} = 0.00009191328125$

$$\therefore Tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{----}$$

$Tan^{-1}(0.35) = 0.336666858$

(6) Example:

$Tan^{-1}(0.323)$ $x = 0.323$

$$Tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{----}$$

Step (1):

$x^2 =$

0.	0	9	12	22	12	9

$= 0.104329$

Step (2):

$x^3 = (x^2)(x) =$

0.	0	3	2	15	17	24	40	24	27	0

$= 0.033698267$

and $\frac{x^3}{3} = 0.011232755$

Step (3):

$x^5 = (x^3)(x^2)$

0.	0	0	3	3	18	30	47	89	104	111	127	122	51	68	63

$= 0.003515706497843$ and $\frac{x^5}{5} = 0.0007031412$

Step (4):

$x^7 = (x^5)(x^2)$

0.	0	0	0	3	5	13	34	32	60	96	40	83	81	12	54	0	0	0

$= 0.000366790091274000$ and $\frac{x^7}{7} = \frac{0.000366790}{7} = 0.000052399$

$\therefore \mathrm{Tan}^{-1}(0.323) = 0.312417987$

Hyper Bolic Functions:

Example:

Sinh (0.323) $\quad x = 0.323$

$\mathrm{Sinh}x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \text{-----}$

Steps 1 to 5 are same terms as sin (0.323)) working

Sinh (0.323) = 0.323 + 0.005616378 + 0.000 45676

Cosh (0.8235)

$\mathrm{Cos\,h}(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \text{---}$

Steps 1 to 6 are same terms as Cos (0.8235) Refer Cos (0.8235) working

Cosh (0.8235) = 1 + 0.339076125 + 0.019162103 + 0.00043316 + ---------

= 1.358671388

Example:

Tan h^{-1} (0.35)

$$\text{Tan } h^{-1}(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + - - -$$

Steps 1 to 7 are same terms as $\tan^{-1}(0.35)$ Refer $\text{Tan}^{-1}(0.35)$ working

Tan h^{-1} (0.35) = 0.35 + 0.014291666 + 0.001050438 + 0.00091913 = 0.365434017

Inverse Sin, Cosine and Tangent

Inverse Sin

$$\text{Sin}x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots\ldots \qquad \text{Sin } \sin^{-1}(x) = x$$

$$x = \sin x + \frac{x^3}{3!} + \frac{x^5}{5!} + - - - - - -$$

Example:

$\text{Sin}^{-1}(0.432) \Rightarrow \text{Sin } x = (0.432)$

$$x = 0.432 + \frac{x^3}{3!} - \frac{x^5}{5!} + - - - - - -$$

For lower angles Sinx = x

By considering x = 0.432

<u>Step 1</u>:

$x^2 = D(x) = D(0.432)$

By urdhva tiryak multiplication.

```
    0.432
    0.432
  ________________
    0.0162425124
    0.186624
```

<u>Step 2</u>:-

$x^3 = (x^2)(x)$ By urdhva tiryak multiplication

```
    0.186624
    0.432000
```

0	0	4	35	50	5' [illegible]	16	8	0 0	0

<u>Step 3</u>:-

$$\frac{x^3}{3!} = \frac{0.080621568}{6} = 0.013436928$$

<u>Step 4</u>:-

$x^5 = (x^3)(x^2)$

By urdhva tiryak multiplication

$x^3 = 0.080621568$

$x^2 = 0.186624000$

0	0	0	8	64	54	98	69	93	76	120		
140	98	80	40	32	0	0	0					

0.015045919506432000

$\cong 0.015045920$

Step 5:-

$$\frac{x^5}{5!} = \frac{0.015045920}{120}$$

$= 0.000125382667$

$= 0.000125383$

$$x = 0.432 + \frac{x^3}{3!} + \frac{x^5}{5!} + \text{------}$$

$= 0.432 - 0.013436928 + 0.000125383 = 0.418688455$

Method explained by British Author

$\sin^{-1}(0.432)$

$$x = 0.432 + \frac{x^3}{3!} + \frac{x^5}{5!} + \text{------}$$

Inverse Trigonometric Function: (British Authors method)

In these calculations left to right multiplication is very conspicuously brought out. This is followed by working out digit by digit as they appear. This principle is used to obtain the next digits. As an example let us consider inverse Sin Function.

To find $Sin^{-1}(0.3)$

$Sin^{-1}(\sin(x)) = x$

$$\text{Sin } x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \quad \text{------------} \quad (1)$$

$$x = \sin(x) + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots \quad \text{------------} \quad (2)$$

For small angles x, sin (x) = x starting with such sinx value to obtain x. Applying the formula (2)

$$x = 0.3 + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots \quad \text{------------} \quad (2)$$

In order to convert into a standard formulae to find out the value of x using formulae (2). From formulae (I) Sinx is 0.3. For small angles, Sinx can be considered as x. Hence in the formulae (2), one can substitute for Sinx, a value 0.3. The value is obtained, to the required accuracy, by considering the various power series terms.

<u>Step</u> : (1)

The working details are shown in the table.

Sin x = 0. 3 0 0 0 0

<u>Step</u> : (2)

The value of x in relation to the 1st decimal is noted as 0.3 i.e. without considering $\frac{x^3}{3!}, \frac{x^5}{5!}$ etc.

<u>Step</u> : (3)

In order to arrive at the $\frac{x^3}{3!}$, one can x^2 considering the value of x upto the point shown originally i.e. 0.3

$\therefore x^2 = 0.09 = 0.1\bar{1}$ shown as $0.1_{\bar{1}}$

This $\bar{1}$ has to be carried to second decimal position as $\bar{1}0$ ($\bar{1}$ X 10) again shown as $\bar{1}_0$ again "0" has to be carried to the 3rd decimal position. (refer table).

Step: (4)

The contribution of $\frac{x^3}{3!}$ is obtained by multiplying x and x^2 up to the decimal points already obtained.

i.e. $x = 0.3$ $\qquad x^2 = 0.1$

x^3 is this obtained by multiplying 0.3 with 0.1 = 0.03 when it is divided by "6", the value is shown as 0.00_3 against the row representing $\frac{x^3}{3!}$. This 3 has to be carried to the 3rd Decimal Point as 30 (3 x 10)

Step (5):

To arrive at the 1st decimal contribution from $x^5 = (x^3 \text{ X } x^2)$ it is clearly seen as '0' = (0.0 x 0.1) one has to multiply x^3 and x^2 concerned with the 1st decimal points, it is 0.0 and 0.1 this gives 0.00. To arrive at the contribution to the 2nd decimal of x^5 term $(x^3 \text{ X } x^2) = 0.00 \times 0.1\bar{1}$ gives value 0.

Step (6):

Thus the x value is 0.30. One has to workout x^2 with x = 0.30 which is $(0.30)^2 = 0.0900$. This can be again written as $.01\bar{1} = 0.1\bar{1}00$. The second decimal with respect to x^2 is $\bar{1}$ carried from the 1st decimal point

Sinx	0.3	0	0	0	0
$\frac{x^3}{3!}$	0.0	0_3	4_3	$7_{\bar{2}}$	1_1
$\frac{x^5}{5!}$	0.0	0	0	0	$\bar{2}$
X	0.3	0	--	7	$\bar{1}$
x^2	$0.1_{\bar{1}}$	$\bar{1}_0$	2	8_2	20

Step: (7)

In order to get the 3rd decimal point contribution of $\frac{x^3}{3!}$, one has to multiply x as 0.30 and x^2 as $0.1\bar{1} = 0.03\bar{3}0$ when it is divided by "6" we get $\frac{1}{6}(0.03\bar{3}0)$ which is to be understood as $0.00_3\bar{3}$ that means

$1/6\ [0.00(30+\bar{3})] = 0.004_3$

This 3 has to be reckoned as 30 under 4th decimal contribution.

<u>Step</u>: (8)

To workout the 3rd DP contribution against $\frac{x^5}{5!}$. One has to consider the multiplication of

$0.1\bar{1}\ (x^2)$ and $0.00(x^3) = 0.0000$

Third decimal point contribution against $\frac{x^5}{5!}$ is zero. Thus the value of x now is 0.304.

<u>Step</u>: (9)

To evaluate x^2 with $x = 0.304$ the square of $x = 0.09\ 0\ 24\ 0\ 16$. The 0.090 is already dealt with under the value of x^2 when $x = 0.30$. Now it is left that one has to work with 24 can be written as 2_4 with the carrying of zero from the 2nd decimal point naturally $0 + 2 = 2$. This 4 has to carried to the 4th decimal point as $4 \times 10 = 40$

<u>Step</u>: (10)

Now to workout the 4th decimal point contribution of $\frac{x^3}{3!}$ one has to workout the

multiplication of x as 0.304 with x^2 as $0.11\bar{2}$ which is equal to $0.\ 0\ 3\ \bar{3}\ 10\ \bar{4}\ 8$

$0.\ 0\ 3\ \bar{3}$ is already dealt with under the contribution of 3 decimals against $\frac{x^3}{3!}$ as such one has to consider 10. This on addition from arrived by carrying out 30 from the previous 3rd decimal point result becomes 40. This is to be divided by 6 which result can be written as $7_{\bar{2}}$.

<u>Step</u>: (11)

To workout the 4 decimal point contributic   ıs x One has to consider the

multiplication of $0.11\bar{2}(x^2)$ and $0.004\ (\frac{x^3}{3!})$ we get $u4\bar{4}8$

This on division by 20 is equal to 0.00002..... Thus !   .g to the contribution of 4th Decimal

Point against $\frac{x^5}{5!}$ is zero and be 5th decimal point contribution is $\bar{2}$ as one has to consider - $\frac{x^5}{5!}$. The 'x' value now is 0.3047.

Step: (12)

To determine the value of $\bar{x}$ one has to square the 0.3047, one gets the value 0.0 9 0 24 42 16 49. We have already considered up to 24. Leaving 42 which on consideration of previous decimal point contribution, as 40 becomes 40 + 42 = 82 as shown as 8_2 against X^2.

Step: (13)

In order to get $\frac{x^3}{3!}$ one has to multiply 0.3047 (x) with $0.1\bar{1}28$ (x^2) we get $0.0\ 3\bar{3}\ 10\ 27\ 14$ 46 56. We have already dealt with up to 4[th] DP for the 5[th] DP contribution against $\frac{x^3}{3!}$, one has to take 27 and one has to consider $\overline{20}$ as the contribution from the previous point on addition of this one gets 7. Which is to be divided by 6, the result is 1_1. Thus 5[th] in x is $\bar{1}$

finally the value is $0.3047\bar{1}$ = 0.30469.

Hence $\sin^{-1}(0.3) = 0.30469$

$\therefore$ x = 0.30469 in radians

(1). We can interpolate the sin value from the standard tables

0.3 radians = 17.1887°

1 radian = 57.2958

$\therefore$ 0.30469 radian = 57.2958 X 0.30469

= 17.4574573.

To convert the decimal value into minutes = $17°27447438^1$

= $17°27^1$ (app)

To read the Sin value of $17°27^1$ = 0.29987

(2). Verification from the standard tables using the x value as 0.30469.

$$\operatorname{Sin} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots\ldots\ldots\ldots$$

$$\frac{x^3}{6} = 0.004714366609$$

$$\frac{x^5}{120} = 0.000021883146$$

Sin x = 0.30469 - 0.004714366609 + 0.000021883146 = 0.299997517 ≅ 3

$1^{\circ} = 60^{1}$
$0.45246^{\circ} X 60^{1}$
27.44760

$17^{\circ} 27^{1} 27^{11}$

$17^{\circ}\ 27^{1} = 0.29987$
$17^{\circ}\ 28^{1} = 0.30015$

1 minute 0.00028

60 seconds → 0.00028
27 seconds → 0.00028 * 27

= 0.00012660

$17^{\circ} 27^{1} 27^{11} = 0.29987$
+ 0.000126
0.299996

(1) Eg. $\text{Sin}^{-1}(0.263)$

$\text{Sin x} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots$

$x = \sin x + \frac{x^3}{3!} - \frac{x^5}{5!} - \ldots\ldots\ldots$

Here sin x = 0.263

Sin x	0	2	6	3	0	0
$\frac{x^3}{3!} = \frac{x^3}{6}$	0	0	0	$\frac{12}{6} = 2_0$	$\frac{54}{6} = 9_0$	$\frac{114}{6} = 19_0$
$\frac{x^5}{5!} = \frac{x^5}{120}$	0	0	0	0	0	0
x	0	2	6	5	9	19
x^2	0	0_4	6_4	9_6	$\frac{60}{150}$	

1) $x^3 = 0.2$
$\underline{\quad 0.0 \quad}$
0.00

$x^5 = 0.0$
$\underline{\quad 0.0 \quad}$
0.00

2) $x^2 = 0.26$
$\underline{\quad 0.26 \quad}$
0.0 4 24 36

3) $x^3 : 0.06$
$\underline{\quad 0.26 \quad}$
0.0012 36

$x^5 = 0.06$
$\underline{\quad 0.00 \quad}$
0.0000

4) $x^2 : 0.265$
$\underline{\quad 0.265 \quad}$
0.04 24 56 60 25

$x^3 : 0.069$
$\underline{\quad 0.265 \quad}$
0.0012 54 84 45

$x^5 = 0.069$
$\underline{\quad 0.002 \quad}$
0.000012 18

5) $x^2 = 0.2659$
$\underline{\quad 0.2659 \quad}$
0.04 24 56 96 133 90 81

$x^3 = 0.06915$
$\underline{\quad 0.2659 \quad}$
0.00 12 54 114 189

x = 0.2659
= 0.26609

<u>**Verification**</u>:

$$\text{Sin } x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots\ldots$$

x	=	0.26609
$x^3/6$	=	(-) 0.00314
$x^5/120$	=	(+)0..00001
		0.26297 ≅ 0.263

(2) Eg. $Cos^{-1}(0.012)$

$Cos^{-1}(Cos\ x) \Rightarrow Cos\ x = 0.012$

Since the angle is close to 90°, we can find Sin^{-1} (0.012) and subtract the result from $\frac{\Pi}{2}$

Sin x	0.	0	1	2	0	0
$\frac{x^3}{3!} = \frac{x^3}{6}$	0.	0	0	0	0	0
x	0.	0	1	2	0	0
x^2	0 .	0	0	1		

$x^2 = 0.0$
0.0
0.00

0.01
0.01
0.0001

$x^3 = 0.012$
0.001
0.0000 12

$Cos^{-1}\ 0.012 = \frac{\Pi}{2} - 0.012$
$= 1.5588$

0.0120
0.0001
0.000000

(3) Eg. $Cos^{-1}\ 0.92$

$$Cos\ x = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots$$

$$x^2 = 2(1 - \cos x) + \frac{x^4}{12} - \frac{x^6}{360} + \ldots\ldots\ldots (1)$$

Here Cos x = 0.92

Let $y = x^2$

$$\Rightarrow (1) \Rightarrow y = 2(1 - \cos x) + \frac{y^2}{12} - \frac{y^3}{360} + - - - - - - -$$

$2(1-\cos x)$	0	1	6	0	0	0
$\frac{y^2}{12}$	0	0	0	10 12 $\overline{22} = 1_{10}$	100 38 $\overline{138} = 11_6$	60 34 $\overline{94} = 7_{10}$
$-\frac{y^3}{360}$	0	0	0	0	0	0
$y = x^2$	0	1	6	1	11	7

y = 0.1
y^2 = 0.1
0.1
0.01

y = 0.16
y^2 = 0.16
0.16
0.01 12 36

y = 0.161
y^2 = 0.161
0.161
0.01 12 38 12 1

y = 0.161
y^2 = 0. 1 61 11
0. 1 61 11
0. 1 61 11

0.01 12 38 34 133 22 121

y^3 = 0.161
0.001
0.000161

$y = x^2 = 0.1\ 6\ 1\ 11\ 7$
0. 1 6 2 1 7

$x = \sqrt{y} = 0.402703$

Square root of 0.16217

	$.\overline{16}\ \overline{21}\ \overline{70}\ 0_2\ 0$
	251 $\frac{4}{14}$
0.8	00
	0.402703

Tan (0.23)

$\text{Tans}^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots\ldots\ldots$

$x = \tan^{-1}x + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots\ldots\ldots\ldots$

1) $x = 0.2 \Rightarrow x^2 = 0.04$

$x^3 = 0.2$
0.0
0.00

2) x^2: 0.23
0.23
0.04 12 9

x^3: 0.23
0.05
0.00 10 15

x^5: 0.05
0.00
0.0000

3) $x = 0.233$

$x^2 = 0.233$
0.233
0.0 4 12 21 18 9

x^3: 0.233
0.054
0.0 0 10 23 27 12

x^5: 0.003
0.054
0.0 0 0 0 0 15 12

$\frac{15*3}{5} = 9$

4) $x = 0.23311$

$x^2 = 0.23311$
0.23311
0.0 4 12 21 62 75 121

x^3: 0.23311
0.0547
0.00 10 23 41 88 65 77

$x = 0.233114 = 0.23414$

$\tan^{-1}x$	0	2	3	0	.0	0
$\frac{x^3}{3}$	0	0	0	3_1	10 23 11_0 33	[illegible] 3 [illegible]
$-\frac{x^5}{5}$	0	0	0	0	0	9
X	0	2	3	3	11	4
x^2	0	0_4	5_2	4_1	7_2	

Tan (1.143)

1.143 radians is too large an angle for us to handle easily, so we go for the complementary angle, find its tangent and take the reciprocal.

Tan $(\frac{\Pi}{2} - 1.143) = \tan(0.4278)$

$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \_\_\_\_\_\_\_\_$

$x = \tan^{-1}(x) + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \_\_\_\_\_\_\_\_\_$

$\tan^{-1}(x)$	0.	4	2	7	8
$\frac{x^3}{3}$	0.	0	2_2	$\bar{1}_{\bar{1}}$	$\bar{\bar{6}}_0$
$-\frac{x^5}{5}$	0_1	0	0	$\bar{2}_2$	$\bar{\bar{7}}_{\bar{1}}$
$\frac{x^7}{7}$	0.	0	0	0	$3_{\bar{1}}$
X	0.	4	4	4	12
X^2	0.	$2_{\bar{4}}$	$\bar{\bar{8}}_0$	4_8	

1\. $x^2 = x \times x$ | $x^3 = x \times x^2$ | $x^5 = x^3 \times x^2$

$x = 0.4$
$x = 0.4$
$0.16 = 0.2\bar{4}$

$x = 0.4$
$x^2 = 0.2$
0.08

$x^3 = 0.02$
$x^2 = 0.20$
0.0040

$$\Rightarrow 0.004 \times 3 = \frac{0.012}{5} = 0.002_2$$

2\. $x = 0.44$ | $x^3 = x \times x^2$ | $x^7 = x^5 \times x^2$

$x^2 \Rightarrow x = 0.44$
$x = 0.44$
0.0163216

$x = 0.44$
$x^2 = 0.2\bar{8}$
$0.08\,\overline{24}\,\overline{32}$

$x^5 = 0.002$
$x^2 = 0.2\bar{8}0$
$0.0004\,\overline{16}\,0$

$$\Rightarrow \frac{0.0004}{7} \times 5 = \frac{0.0020}{7} = 0.0003_{\bar{1}}$$

3. $x = 0.4\,4\,4$

$x^2 \Rightarrow x = 0.4\,4\,4$
$x = 0.4\,4\,4$

$0.0\,16\,32\,48\,32\,16$

$x^3 = x^2 \times x$

$x = 0.4\,4\,4$
$x^2 = 0.2\,\bar{8}\,4$

$0.0\,8\,\overline{24}\,\bar{8}\,\overline{16}\,16$

$x^5 = x^3 \times x^2$

$x^3 = 0.0\,2\,\bar{1}\,\bar{6}$
$x^2 = 0.2\,\bar{8}\,4\,0$

$0.0\,0\,4\,\overline{18}\,4\,44\,\overline{24}$

$$\Rightarrow \frac{0.000\overline{18} \times 3}{5} = \overline{54} + 20 = \frac{\overline{34}}{5} = \bar{\bar{7}}_{\bar{1}}$$

Tan (1.143) = 1 + 0.4452 = 2.246181

Tan (1.982)

Here we observe that the tangent has a negative value, and as can be seen from the diagram we need to evaluate tan $(1.982 - \frac{\pi}{2})$, take the reciprocal and change the sign.

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \_\_\_\_\_\_\_$$

$$x = \tan^{-1}(x) + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \_\_\_\_\_\_\_\_$$

$\tan^{-1}(x)$	0	4	1	1	2
$\frac{x^3}{3}$	0	0	2_2	7_1	6_1
$-\frac{x^5}{5}$	0	0	0	2_2	10_0
$\frac{x^7}{7}$	0	0	0	0	$3_{\bar{1}}$
X	0	4	3	6	1
x^2	0	$2_{\bar{4}}$	$\bar{1}_{\bar{5}}$	$\bar{3}_0$	

1. $x = 0.4$

$x^2 \Rightarrow x = 0.4$
$x = 0.4$
$0.16 = 2_{\bar{4}}$

$x^3 = x^2 \times x$

$x^2 = 0.2$
$x = 0.4$
$0.0\,8$

2. $x = 0.43$ $\qquad$ $x^3 = x^2 \times x$ $\qquad$ $x^5 = x^2 \times x^3$

$$
\begin{array}{r}
x^2 \Rightarrow x = 0.4\,3 \\
x = \underline{0.4\,3} \\
\underline{0.0\,16\,24\,9}
\end{array}
\qquad
\begin{array}{r}
x = 0.4\,3 \\
x^2 = \underline{0.2\,\bar{1}} \\
\underline{0.08\,\bar{2}\,\bar{3}}
\end{array}
\qquad
\begin{array}{r}
x^3 = 0.0\,2 \\
x^2 = \underline{0.2\,\bar{1}} \\
\underline{0.0\,0\,4\,\bar{2}}
\end{array}
$$

$$\Rightarrow \frac{0.004 \times 3}{5} = \frac{0.012}{5} = 0.002_2$$

3. $x = 0.436$ $\qquad$ $x^3 = x^2 \times x$ $\qquad$ $x^5 = x^3 \times x^2$

$$
\begin{array}{r}
x^2 \Rightarrow x = 0.4\,3\,6 \\
x = \underline{0.4\,3\,6} \\
\underline{0.0\,16\,24\,57\,36\,36}
\end{array}
\qquad
\begin{array}{r}
x = 0.4\,3\,6 \\
x^2 = \underline{0.2\,\bar{1}\,\bar{3}} \\
\underline{0.0\,8\,2\,\bar{3}\,\overline{15}\,\overline{18}}
\end{array}
\qquad
\begin{array}{r}
x^3 = 0.0\,2\,6 \\
x^2 = \underline{0.2\,\bar{1}\,\bar{3}} \\
\underline{0.0\,0\,4\,10\,\overline{12}\,\overline{18}}
\end{array}
$$

$$0.00010 \times 3 = \frac{0.0030}{5} = 0.0006$$

$x^7 = x^5 \times x^2$

$$
\begin{array}{r}
x^5 = 0.0\,0\,2 \\
x^2 = \underline{0.2\,\bar{1}\,\bar{3}} \\
\underline{0.0\,0\,0\,4\,\bar{2}\,\bar{6}}
\end{array}
\Rightarrow 0.0004 \times 5 = \frac{0.0020}{7} = 0.0003_1
$$

$$\text{Tan}\left(1.982 - \frac{\pi}{2}\right) = 0.4361$$

$$\text{Tan}(1.982) = -1 + 0.4361 = -2.29305$$

Hyperbolic Functions:

$\text{Sinh}^{-1}(0.5324)$

$$\text{Sinh}x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \text{\_\_\_\_\_\_\_}$$

$$x = 0.5432 - \frac{x^3}{3!} - \frac{x^5}{5!}$$

Example (2)

$\text{Sin}^{-1}(0.263) = 0.266129 \rightarrow \text{Sin}(0.266129) \approx 0.263$

1 radian = 57.2958
= 570 17'44"

1 radian = 57.2958

0.266129 = 57.2958 × 0.266129
= 15.24800
= 150 14' 52"

$15^0\ 14' = 0.26275$
$15^0\ 15' = \underline{0.26303}$
$60'' \Rightarrow \underline{0.00028}$

$52 \text{ Seconds} \Rightarrow \frac{0.00028 \times 52}{60} = 0.00024266$

$15^0\ 14' \quad = 0.26275$
$15^0\ 14'\ 52'' = \underline{0.00024266}$

$\underline{0.26299266} \cong 0.263$

Example (3)

$\text{Cos}^{-1}(0.012) = 1.5588$

1 radian = 57.2958

1.5588 radian = 1.5588 × 57.2958
= 89.3127
= $89^0\ 18'\ 45''$

$\text{Cos}(89^0\ 18'\ 45'') = \text{Sin}(0^0\ 41'\ 15'')$

$41' = 0.01193$
$42' = \underline{0.01222}$
$60'' = \underline{0.00029}$

$15'' = 0.0000725$
$= \underline{0.012002}$
$\cong \underline{0.012}$

Example (4)

$\cos^{-1}(0.92) = 0.402703$

57.2958×0.402703

$23.07319055 \cong 23^0\ 04'\ 23''$

$\cos(23^0\ 04'\ 23'')$

$23^0\ 04' \rightarrow 0.92005$
$23^0\ 05' \rightarrow 0.91994$

$60' \rightarrow 0.00011$

$23'' \rightarrow 0.000042166$

$23^0\ 04'\ 23'' \rightarrow 0.92005$
$(-)\ 0.00004$

$0.92001 \cong 0.92$

Example (5)

$\tan(0.23) = 0.23414$

$$\tan^{-1}(0.23414) = \frac{13.17784665^0}{57.2958} = 0.22999673 \cong 0.23$$

$\tan(45) = 1 \rightarrow \tan^{-1}(1) = 45$

Example (6)

$\tan(1.143) = 2.24618$

$$\tan^{-1}(2.246181) = \frac{66.001367}{57.2958} = 1.1519$$

APPENDIX
Inverse Function

Example:

$Sin^{-1}(0.3)$ (Only for very small angles)

A Simpler method is attempted using direct evaluation of powers of x in the formulae in determination of$Sin^{-1}x$. The working details are as follows for $Sin^{-1}(0.3)$.

$Sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ - - - - - - - - - $Sin^{-1}(\sin x) = x$

$\Rightarrow x = Sinx + \frac{x^3}{3!} - \frac{x^5}{5!} +$ - - - - - - -

For small angles one can approximate the value of Sinx as x.
Sinx is considered as 0.3.

In the formulae, x = Sinx different powers of x which had to be evaluated. The method adopted is direct evaluation of x^3, x^5 _ _ _ _ with the successive computed values of x.

For the purpose of accuracy up to a particular decimal, the procedure is as follows.

Step 1): Considered 0.3 (Sinx) is the starting value of x.

Step 2): To the above value of x, one has to add $\frac{x^3}{3!}$ and subtract $\frac{x^5}{5!}$.

Thus the computed Value is

$x + \frac{x^3}{3!} - \frac{x^5}{5!}$ computed value

$\Rightarrow 0.3 + 0.004 - 0.000 = 0.304$ (up to 3 decimals)

Step 3): To workout the successive computed value with 0.304 as , and computed value is

$\Rightarrow 0.3 + 0.00468 - 0.00002 = 0.30466$ (up to 5 decimals)

Step 4): The next computed value with x = 0.3046

$x + \frac{x^3}{3!} - \frac{x^5}{5!}$

$\Rightarrow 0.3 + 0.00471 - 0.00002 = 0.30$ up to 5 decimals)

This procedure is adoptable with the value x, on, nally given to any decimal.

Note:

$Sin^{-1}(0.3)$

$Sin^{-1}x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$ - - - - - - - - -

$Sin^{-1}(0.3) = 0.3+0.0045+0.00002025+0.000000043$
$= 0.304520293$
$= 0.299835573$

The value obtained by the substitution in the direct formula $\sin^{-1}(0.3)$=0.304520293

For verification sin (0.304520293)=0.299835573. However for value evaluated from the authors adopted method is more accurate resulting sin (0.30464) = 0.29994977.

Example:

Let us consider evaluation of Sin^{-1} (0.263)
$\text{Sin}^{-1}(0.263) \Rightarrow \text{Sinx} = 0.263$, in case of small angles, $\text{Sinx} \equiv x$.
$\therefore$ In the formulae, $x = \text{Sinx} + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!}$ - - - - - - -
x is evaluated as per the RHS expression. The working details are as follows.

1) Let us consider 0.263 as the initial value of x.
2) $\frac{x^3}{3!}$ With x as 0.263 = 0.0030319.
3) $\frac{x^5}{5!}$ With x as 0.263 = 0.000010485.
4) Computed value of x is = 0.266021415.
   $\left(\text{Sin}(x) + \frac{x^3}{3!} - \frac{x^5}{5!}\right)$ Similarly
5) With a as the x value, one can again get the next computed value for x.
   $\frac{x^3}{3!} = 0.003137607.$
   $\frac{x^5}{5!} = 0.000011102.$
   The computed value of x
   $= (\text{Sinx} + \frac{x^3}{3!} - \frac{x^5}{5!}) = 0.263 + 0.003137607 - 0.000011102$
   $= 0.266126505$
   up to 5 decimal points x = 0.266126.

6) A further successive computation with x as '
   $\frac{x^3}{3!} = 0.0031413$
   $\frac{x^5}{5!} = 0.000011123$
   Computed value of $x = \text{Sinx} + \frac{x^3}{3!} - \frac{x^5}{5!}$
   $= 0.263 + 0.0031413 - 0.00011123$
   $= 0.266130177 \cong 0.266130$

The method adopted here is considered to be a direct one in the sense that one starts with the entire given value for Sinx and then computes the contribution in a successive manner.

A comparison of this result with the one's obtained by British authors is a manner where digit by digit calculations is considered.

All the working details shown under this chapter are considered to be general and can be adopted for angles of small values.

1) Example:

$\cos^{-1}(0.012)$

$\cos^{-1}(\text{Cosx}) = x \Rightarrow \text{Cos}(x) = 0.012$

Since the angle is close to 90^0, we consider $\sin^{-1}(0.012)$ and then subtract the result from $\frac{\pi}{2}$

$\sin(\sin^{-1}x) = x \Rightarrow \sin x = 0.012$

$\text{Sinx} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - - - - - - - - -$

$x = \sin x + \frac{x^3}{3!} - \frac{x^5}{5!} + - - - - - - - - -$

Let x = 0.012

$\frac{x^3}{3!} \quad = \quad 0.000000288$

$\frac{x^5}{5!} \quad = \quad \underline{0.000000000}$

0.000000288

Sinx (+) $\underline{0.012}$

$x_1 \rightarrow \underline{0.012000288}$

With $x \cong x_1$

$\frac{x^3}{3!} \quad = \quad 0.000000288$

$\frac{x^5}{5!} \quad = \quad \underline{0.000000000}$

0.000000288

Sinx (+) $\underline{0.012000000}$

$X_2 \rightarrow \underline{0.012000288}$

$\cos^{-1}(0.012) = \frac{\pi}{2} - 0.012$

$= 1.5708 - 0.012$

$= 1.5588$

<u>Verification:</u>

Cos (1.5588) = 0.011996 ~ 0.012

2) Example:

$\cos^{-1}(0.92)$

$$\text{Cosx} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{---------}$$

$$x^2 = 2(1-\cos x) + \frac{x^4}{12} - \frac{x^6}{360} + \text{---------} \qquad \rightarrow (1)$$

Here $\cos x = 0.92$

Let $y = x^2$ $\qquad 2(1-\text{Cosx}) = a$

$$y = 2(1-\cos x) + \frac{y^2}{12} - \frac{y^3}{360} + \text{---------}$$

$$= 2(1-\cos x) = 0.16$$

Let $y = 0.16$

$$\frac{y^2}{12} = 0.00213333$$

$$-\frac{y^3}{360} = 0.00001138$$

$$0.00212195$$

$$2(1-\cos x)\ (+)\ 0.16$$

$$0.16212195$$

$y_1 = 0.16212195$

$$\frac{y^2}{12} = 0.002190293$$

$$\frac{y^3}{360} = (-)\ 0.000011836$$

$$0.002178456$$

$$2(1-\cos x)\ (+)\ 0.16$$

$$0.162178456$$

$y_2 = 0.162178456$

$$\frac{y^2}{12} = 0.00219182$$

$$\frac{y^3}{360} = (-)\ 0.000011848$$

$$0.002179971$$

$$2(1-\cos x)\ \ 0.16$$

$$0.162179971$$

$y_3 = 0.162179971$

$y = 0.162179971$

$y = x^2 \Rightarrow x = \sqrt{y} = \sqrt{0.162179971} = 0.402715744$

Verification:

$\cos(0.402715744) = 0.920000038 \sim 0.92$

(3) Example:

Tan (0.23)

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - - - - - -$$

$$x = \tan^{-1}x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - - - - - -$$

Let $x = 0.23$

$\tan^{-1}x = 0.23$

$$\begin{aligned}
\frac{x^3}{3} &= 0.004055666 \\
\frac{x^5}{5} &= (-)\ 0.000128726 \\
\frac{x^7}{7} &= \underline{0.000004864} \\
& \quad 0.003931804 \\
\tan^{-1}x &\quad (+)\ \underline{0.23} \\
& \quad \underline{0.233931804}
\end{aligned}$$

$x_1 = 0.233931804$

$$\begin{aligned}
\frac{x^3}{3} &= 0.004267234 \\
\frac{x^5}{5} &= (-)\ 0.000140112 \\
\frac{x^7}{7} &= \underline{0.000004864} \\
& \quad 0.004131986 \\
\tan^{-1}x &\quad (+)\ \underline{0.23} \\
& \quad \underline{0.234131986}
\end{aligned}$$

$x_2 = 0.234131986$

$$\frac{x^3}{3} = 0.004278199$$
$$\frac{x^5}{5} = (-)\ 0.000140712$$
$$\frac{x^7}{7} = \underline{0.000005509}$$
$$0.004142996$$
$$\tan^{-1} x \quad (+)\ \underline{0.23}$$
$$\underline{0.234142996}$$

$\tan^{-1}(0.234142996) = 0.229999652 \cong 0.23$

$x_3 = 0.234142996$

$$\frac{x^3}{3} = 0.004278802$$
$$\frac{x^5}{5} = (-)\ 0.000140745$$
$$\frac{x^7}{7} = \underline{0.000005511}$$
$$0.004143568$$
$$\tan^{-1} x \quad (+)\ \underline{0.23}$$
$$\underline{0.234143568}$$

$\tan^{-1}(0.234143568) = 0.230000195 \cong 0.23$

(4) Example:

Tan (1.143)

1.143 radians is too large an angle for us to handle easily, so we go for the complementary angle, finds its tangent and take the reciprocal.

$\text{Tan}\left(\frac{\pi}{2} - 1.143\right) = \tan(0.42780)$

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots\cdots$

$$x = \tan^{-1}x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} ------$$

Let x = 0.42780

$$\frac{x^3}{3} = 0.02609763$$
$$\frac{x^5}{5} = (-)\ 0.00286572$$
$$\frac{x^7}{7} = \underline{0.00037462}$$
$$0.02360653$$
$$\tan^{-1}x \quad (+)\ 0.4278$$
$$\underline{0.45140653}$$

$x_1 = 0.45140653$

$$\frac{x^3}{3} = 0.030660713$$
$$\frac{x^5}{5} = (-)\ 0.003748600$$
$$\frac{x^7}{7} = \underline{0.000545603}$$
$$0.027457716$$
$$\tan^{-1}x \quad (+)\ \underline{0.4278}$$
$$\underline{0.455257716}$$

$x_2 = 0.455257716$

$$\frac{x^3}{3} = 0.031452175$$
$$\frac{x^5}{5} = (-)\ 0.003911258$$
$$\frac{x^7}{7} = \underline{0.000579032}$$
$$0.028119949$$
$$\tan^{-1}x \quad (+)\ \underline{0.4278}$$
$$\underline{0.455919949}$$

$x_3 = 0.455919949$

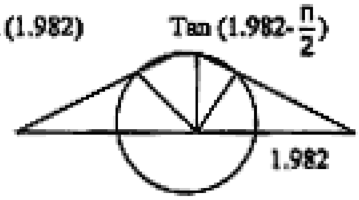
$\tan^{-1}(x_3) = 0.4277660 \cong 0.4278$

Tan (1.143) = 1 + 0.455919949 = 2.193367503

(5) Example:

Tan (1.982)
Here we observe that the tangent has a negative value, and as can be seen from the diagram.
We need to evaluate Tan $(1.982 - \frac{\pi}{2})$, take the reciprocal and change the sign.

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\cdots$$



$$x = \tan^{-1}(x) + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} \cdots\cdots$$

$(1.982 - \frac{\pi}{2}) = 1.982 - 1.5708 = 0.4112$

Let $\tan^{-1}(x) = 0.4112$
Let $x = 0.4112$

$\frac{x^3}{3}$ = 0.023175977

$\frac{x^5}{5}$ = (-) 0.002351232

$\frac{x^7}{7}$ = 0.000283970

0.021108715

$\tan^{-1} x$ 0.4112

0.432308715

$x_1 = 0.432308715$

$\frac{x^3}{3}$ = 0.026931510

$\frac{x^5}{5}$ = (-) 0.003019951

$\frac{x^7}{7}$ = 0.000403143

0.024314702

$\tan^{-1} x$ 0.4112

0.435514702 $\rightarrow$ 0.41074

$x_2 = 0.435514702$

$\frac{x^3}{3}$ = 0.027535134

$\frac{x^5}{5}$ = (-) 0.003133603

$$\frac{x^7}{7} = \underline{0.000424543}$$

0.024826074

tan^{-1} x $\quad$ $\underline{0.4112}$

0.436026074 $\quad \rightarrow$ 0.411172604

$x_3 = 0.436026074$

Tan (1.982) = -1 ÷ 0.436026074 = -2.293440828

(6) Example:

Sinh^{-1} **(0.5324)**

$$\text{Sinhx} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \text{---------}$$

$$x = \text{Sinhx} - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} \text{------}$$

3! = 6, 5! = 120, 7! = 5040

Sinhx = 0.5324

Let x = 0.5324

$$\frac{x^3}{3!} = 0.025151442$$

$$\frac{x^5}{5!} = 0.000356458$$

$$\frac{x^7}{7!} = \underline{0.000002405}$$

(-) 0.025510305

Sinhx $\quad$ (+) 0.5324

$\underline{0.506889695}$

$x_1 = 0.506889695$

let $x_1 = 0.506889695$

$$\frac{x^3}{3!} = (-)\ 0.021706466$$

$$\frac{x^5}{5!} = (-)\ 0.000278859$$

$$\frac{x^7}{7!} = (-)\ \underline{0.000001705}$$

(-) 0.02198703

Sinhx $\quad$ (+) 0.5324

$x_2 \rightarrow$ $\quad$ $\underline{0.51041297}$

$$\frac{x^3}{3!} = (-)\quad 0.02216225$$

$$\frac{x^5}{5!} = (-)\ 0.000288687$$

$$\frac{x^7}{7!} = (-)\ 0.00000179$$

(-) 0.022452727

Sinhx (+) 0.5324

0.509947273

$x_3 = 0.509947273$

Verification:

Sinh (0.509947273) = 0.03233807

$x_3 = 0.509947273$

$$\frac{x^3}{3!} = (-)\ 0.022101643$$

$$\frac{x^5}{5!} = (-)\ 0.000287372$$

$$\frac{x^7}{7!} = (-)\ 0.000001779$$

(-) 0.022390794

Sinhx (+) 0.5324

0.510009206

Sinh (0.510009206) = 0.532408 ~ 0.5324

Other wise $Sinh^{-1}(0.5324) = 0.51000193$

Section-8
DIFFERENTIAL AND INTEGRAL CALCULUS

An attempt is made to solve a few differential equations. However it is noticed that solving for the particular integral; the method adopted using the argumentation suggested in the Vedic method shows to be elegant and easier.

It is noticed in certain cases the solving of the particular integral in

1) $y'' + y' - y = -10x^4 + 38x^3 + 126x^2 + 8x + 4$

2) $y'' - 2y' + y = 3x^4 - 23x^3 + 30x^2 + 6x - 5$

by western method appears to be very laborious when compared to the Vedic method.

Certain problems are quoted from British authors (These are meant as problems to be worked) and some are constructed by authors. In this chapter an attempt is made to solve integro differential equations using Vedic method. The work on differential equations is not exhaustive. It is proposed by authors the more comprehensive and exhaustive and different types of equations will be taken up shortly as a separate work.

1) $y = \dfrac{ax+b}{cx+d}$

<u>Current method</u>:

$y = \dfrac{ax+b}{cx+d} = \dfrac{u}{v}$ Where u & v are functions of x

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(cx+d)a - (ax+b)c}{(cx+d)^2} = \frac{acx + ad - acx - bc}{(cx+d)^2}$$

i) $\dfrac{dy}{dx} = \dfrac{ad-bc}{(cx+d)^2}$

<u>Vedic method</u>:

$y = \dfrac{ax+b}{cx+d} = \dfrac{b+ax}{d+cx}$

<u>Table</u>: $\begin{array}{cc} x^0 & x^1 \\ \hline b & a \\ & \times \\ d & c \end{array} \quad = \quad \dfrac{(ad-bc)(1 - [illegible])}{(cx+d)^2}$

$$y^1 = \frac{ad-bc}{(cx+d)^2}$$

2) $y = \dfrac{(x+2)(x+3)}{7x-8}$

<u>Current method</u>:

G.P is $Y = \dfrac{x^2+5x+6}{(7x-8)} = \dfrac{U}{V}$, U, V are functions of x.

Using quotient rule (Q.R)

$$\frac{dy}{dx} = \frac{(7x-8)(2x+5)-(x^2+5x+6)7}{(7x-8)^2}$$

$$= \frac{(14x^2+35x-16x-40)-(7x^2+35x+42)}{(7x+8)^2}$$

$$= \frac{(14x^2+19x-40-7x^2-35x-42)}{(7x+8)^2} = \frac{7x^2-16x-82}{(7x+8)^2}$$

<u>Vedic method</u>:

$$y = \frac{x^2+5x+6}{7x-8} = \frac{6+5x+x^2}{-8+7x}$$

x^0	x^1	x^2
6	5	1
-8	7	0

$$y^1 = \frac{(-8.5-6.7)(1-0)+(-8.1-0.6)(2-0)x+(7.1-0.5)x^2}{(7x-8)^2}$$

$$= \frac{(-40-42)+(-8)(2x)+7x^2}{(7x-8)^2}$$

$$= \frac{-82-16x+7x^2}{(7x-8)^2}$$

3) $x^2y = 1$

<u>Current method</u>:

$$y = \frac{1}{x^2} = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{x^2(0)-1(2x)}{(x^2)^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}$$

<u>Vedic method</u>:

x^2	x	x^0
0	0	1
1	0	0

$$y^1 = \frac{[(1)(0)-(0)(0)](1-2)x^2+[(1)(1)-(0)(0)](0-2)x+[(0)(0)-(0)(1)](0-1)}{(x^2)^2}$$

$$= \frac{-2x}{x^4} = \frac{-2}{x^3}$$

4) $y = \dfrac{x^2 + 6x + 3}{x^2 + 2x - 15}$

<u>Current method</u>:

Using quotient rule

$$\frac{dy}{dx} = \frac{(x^2+2x-15)(2x+6)-(x^2+6x+3)(2x+2)}{(x^2+2x-15)^2}$$

$$= \frac{\{(2x^3+6x^2+4x^2+12x-30x-90)-(2x^3+2x^2+12x^2+12x+6x+6)\}}{(x^2+2x-15)^2}$$

$$= \frac{(2x^3+10x^2-18x-90)-(2x^3+14x^2+18x+6)}{(x^2+2x-15)^2}$$

$$= \frac{-4x^2-36x-96}{(x^2+2x-15)^2}$$

<u>Vedic method</u>:

$$y = \frac{x^2+6x+3}{x^2+2x-15} = \frac{3+6x+x^2}{-15+2x+x^2}$$

x^0	x^1	x^2
3	6	1
-15	2	1

$$y^1 = \frac{\{(-15.6-2.3)(1-0)+(-15.1-1.3)(2-0)x+(2.1-1.6)(2-1)x^2\}}{(x^2+2x-15)^2}$$

$$= \frac{\{(-90-6)+(-18)(2x)+(-4)x^2\}}{(x^2+2x-15)^2}$$

$$= \frac{-96-36x-4x^2}{(x^2+2x-15)^2}$$

5) $y = \dfrac{x^3+2}{x^2-x-2} = \dfrac{u}{v}$ Where u & v are functions of x

<u>Current method</u>:

Using quotient rule

$$\frac{dy}{dx} = \frac{[(x^2-x-2)3x^2-(x^3+2)(2x-1)]}{v^2}$$

$$= \frac{(3x^4-3x^3-6x^2)-(2x^4+4x-x^3-2)}{v^2}$$

$$= \frac{x^4-2x^3-6x^2-4x+2}{(x^2-x-2)^2}$$

<u>Vedic method:</u>

x^0	x^1	x^2	x^3
2	0	0	1
-2	-1	1	0

$$y' = \frac{\{-2.0-(-1.2)\}(1-0)+\{-2.0-(1.2)\}(2-0)x+\{-2.1-(0.2)\}(3-0)x^2+\{-1.1-(0.0)\}(3-1)x}{(x^2-x-2)^2}$$

$$(\,A = +\{1.1-(0.0)\}(3-2)x^4\,)$$

$$= \frac{(2-4x-6x^2-2x^3+x^4)}{(x^2-x-2)^2} \quad = \frac{(x^4-2x^3-6x^2-4x+2)}{(x^2-x-2)^2}$$

6) $y = \dfrac{x^2+3x}{2x-5} = \dfrac{u}{v}$

<u>Current method:</u>

$$\frac{dy}{dx} = \frac{(2x-5)(2x+3)-(x^2+3x)2}{(2x-5)^2}$$

$$= \frac{(4x^2-10x+6x-15)-(2x^2+6x)}{(2x-5)^2}$$

$$= \frac{2x^2-10x-15}{(2x-5)^2}$$

<u>Vedic method:</u>

x^0	x^1	x^2
0	3	1
-5	2	0

$$y' = \frac{(-5.3-2.0)1+(-5.1-0.0)(2-0)x+(2.1-0.3)(2-1)x^2}{(2x-5)^2}$$

$$= \frac{-15-10x+2x^2}{(2x-5)^2}$$

$$= \frac{2x^2-10x-15}{(2x-5)^2}$$

7) $y = \dfrac{x^2+x-6}{x^2-x-6}$

<u>Current method:</u>

$$\frac{dy}{dx} = \frac{(x^2-x-6)(2x+1)-(x^2+x-6)(2x-1)}{(x^2-x-6)^2}$$

$$= \frac{(2x^3+x^2-2x^2-x-12x-6)-(2x^3-x^2+2x^2-x-12x+6)}{(x^2-x-6)^2}$$

$$= \frac{-2x^2 - 12}{(x^2 - x - 6)^2}$$

<u>Vedic method</u>:

x^0	x^1	x^2
-6	1	1
-6	-1	1

$$y^{I} = \frac{(-6-6)(1-0) + (-6+6)(2-0)x + (-1-1)(2-1)x^2}{(x^2 - x - 6)^2} = \frac{-12 - 2x^2}{(x^2 - x - 6)^2}$$

8) $y = \dfrac{x^4 - 6 + x^3}{2x - 1 + x^{-1}} = \dfrac{u}{v}$

<u>Current method</u>:

$$\frac{dy}{dx} = \frac{[(2x - 1 + x^{-1})(4x^3 - 3x^{-4}) - (x^4 - 6 + x^{-3})(2 - x^{-2})]}{v^2}$$

$$= \frac{(8x^4 - 6x^{-3} - 4x^3 + 3x^{-4} + 4x^2 - 3x^{-5}) - (2x^4 - x^2 - 12 + 6x^{-2} + 2x^{-3} - x^{-5})}{v^2}$$

$$= \frac{6x^4 - 4x^3 + 5x^2 - \frac{2}{x^5} + \frac{3}{x^4} - \frac{8}{x^3} - \frac{6}{x^2} + 12}{v^2}$$

$$= \frac{6x^9 - 4x^8 + 3x^7 - 2 + x - 8x^2 - 6x^3 + 12x^5}{v^2}$$

$$= \frac{x^5(2x - 1 + x^{-1})^2}{v^2}$$

G.P can be written as

$$y = \frac{x^4 - 6 + \frac{1}{x^3}}{2x - 1 + \frac{1}{x}} = \frac{\frac{(x^7 - 6x^3 + 1)}{x^3}}{\frac{2x^2 - x + 1}{x}}$$

$$y = \frac{x^7 - 6x^3 + 1}{x^2(2x^2 - x + 1)} = \frac{(x^7 - 6x^3 + 1)}{2x^4 - x^3 + x^2} = \frac{u}{v}$$

$$y^{I} = \frac{[(2x^4 - x^3 + x^2)(7x^6 - 18x^2) - (x^7 - 6 \quad\quad + 2x)]}{v^2}$$

$$y^{I} = \frac{[(14x^{10} - 36x^6 - 7x^9 + 54x^5 + 7x^8 - 18x^4) - (8x^{10} - 3x^9 + 2x^8 - 48x^6 + 18x^5 - 12x^4 + 8x^3 - 3x^2 + }{v^2}$$

$$y^{I} = \frac{(6x^{10} - 4x^9 + 5x^8 + 12x^6 + 36x^5 - 6x^4 - 8x^3 + 3x^2 - 2x)}{v^2}$$

Vedic method:

$$y = \frac{x^4 - 6 + x^{-3}}{2x - 1 + x^{-1}}$$

x^{-3}	x^{-2}	x^{-1}	x^0	x^1	x^2	x^3	x^4
1	0	0	-6	0	0	0	1
0	0	1	-1	2	0	0	0

$$= \frac{[(0-0.1)(-2+3)+(0-1)(-1+3)x+(0+1)(0+3)x^2+(0-2)(1+3)x^3+(0-0)(2+3)x^4+(0-0)(3+3)x^5+(0-0)(4\cdot}{v^2}$$

$$= \frac{(0-2x+3x^2-8x^3)}{v^2}$$

9) $\quad z = \dfrac{4x+2}{2x^2+2x} = \dfrac{2+4x+0}{0+2x+2x^2}$

$$\frac{dz}{dx} = \frac{(0.4-2.2)(1-0)+(0.0-2.2)(2-0)x+(0.2-4.2)(2-1)x^2}{(2x+2x^2)^2}$$

$$= \frac{-4-8x-8x^2}{4x^2(1+x^2+2x)}$$

$$= \frac{-(1+2x+2x^2)}{x^2(1+x^2+2x)} = \frac{-1-2x-2x^2}{x^2+x^4+2x^3}$$

10. $\quad z = \dfrac{3+6x+9x^2+18x^3}{1+2x+4x^2+8x^3}$

$$\frac{dz}{dx} = \frac{(1.6-3.2)(1-0)+(1.9-3.4)(2-0)x+[(1.18-3.8)(3-0)+(2\,9-6.4)(2-1)]x^2+(2.18-6.8)(3-1)x^3+(4.18-9.}{(1+2\cdot}$$

$$\frac{dz}{dx} = \frac{-6x-24x^2-12x^3}{(1+2x+4x^2+8x^3)^2}$$

11. $z = \dfrac{2 + 7x + 3x^2}{4 + 9x + 2x^2}$

$$\frac{dz}{dx} = \frac{(4.7 - 2.9)(1 - 0) + (4.3 - 2.2)(2 - 0)x + (9.3 - 7.2)(2 - 1)x^2}{(4 + 9x + 2x^2)^2}$$

$$= \frac{10 + 16x + 13x^2}{(4 + 9x + 2x^2)^2}$$

12. $z = \dfrac{1 + 4x + 9x^2 + 3x^3}{2 + 3x + 4x^2 + 9x^3}$

$$\frac{dz}{dx} = \frac{(2.4 - 1.3)(1 - 0) + (2.9 - 1.4)(2 - 0)x + [(2.3 - 1.9)(3 - 0) + (3.9 - 4.4)(2 - 1)]x^2 + A}{(2 + 3x + 4x^2 + 9x^3)^2}$$

$$A = (3.3 - 4.9)(3 - 1)x^3 + (4.3 - 9.9)x^4$$

$$= \frac{5 + 28x + 2x^2 - 54x^3 - 69x^4}{(2 + 3x + 4x^2 + 9x^3)^2}$$

13. $z = \dfrac{2 + 3x + 4x^2 + x^3}{2 + 4x + 9x^2 + 2x^3}$

$$\frac{dz}{dx} = \frac{(2.3 - 2.4)(1 - 0) + (2.4 - 2.9)(2 - 0)x + [(2.1 - 2.2)(3 - 0) + (4.4 - 3.9)(2 - 1)]x^2 + A}{(2 + 4x + 9x^2 + \ldots}$$

$$A = (4.1 - 3.2)(3 - 1)x^3 + (9.1 - 4.2)(3 - 2)x^4$$

$$\frac{dz}{dx} = \frac{-2 - 20x - 17x^2 - 4x^3 + x^4}{(2 + 4x + 9x^2 + 2x^3)}$$

14. $z = \dfrac{1+4x+9x^2+3x^3}{2+3x+4x^2+9x^3}$

$$\frac{dz}{dx} = \frac{\left(2+3x+4x^2+9x^3\right)\left(4+18x+9x^2\right)-\left(1+4x+9x^2+3x^3\right)\left(3+8x+27x^2\right)}{\left(2+3x+4x^2+9x^3\right)^2}$$

$$\frac{dz}{dx} = \frac{\begin{array}{c}8+12x+16x^2+36x^2+36x+554x^2+72x^3+162x^4+18x^2+27x^3+36x^4+81x\\ -3-12x-27x^2-9x^3-8x-32x^2-72x^3-24x^4-27x^2-108x^3-243x^4-81x\end{array}}{\left(2+3x+4x^2+9x^3\right)^2}$$

$$= \frac{5+28x+2x^2-54x^3-69x^4}{\left(2+3x+4x^2+9x^3\right)^2}$$

Differential Equations

1. $(D^2 + 2D + 1)\, y = 2x + x^2$

Western Method:

Auxiliary equation is $m^2+2m+1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1,-1$

$\therefore\ y_c = (C_1+C_2x)e^{-x}$

$y_p = \dfrac{1}{(D+1)^2}(2x+x^2) = (1+D)^{-2}\,(2x+x^2)$

Since $(1+D)^{-2} = 1-2D+3D^2 - 4D^3 + \ldots + (-1)^r(r+1)D^2$

P.I $= (1-2D+3D^2-\ldots)\,(2x+x^2)$

$= 1(2x+x^2) - 2\,(2+x) + 3\,(2)$

$= x^2-2x+2$

$\therefore$ The general solution of the given equation is $y = y_c + y_p$

i.e. $y = (C_1+C_2x)e^{-x} + x^2-2x+2$

Vedic Method: $(D^2 + 2D + 1)\, y = 2x + x^2$

$\Rightarrow y^{11} + 2y^1 + y = 2x + x^2$

$y = a + bx + cx^2 + dx^3 + ex^4 + fx^5$

$y^1 = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4$

$2y^1 = 2b + 4cx + 6dx^2 + 8ex^3 + 10fx^4$

$y^{11} = 2c + 6dx + 12ex^2 + 20fx^3$

G. Equation is $y^{11} + 2y^1 + y = 2x + x^2$

Equating the coefficients of like terms on both sides
x^5 Coeff: $f = 0$
x^4 Coeff: $10f + e = 0 \Rightarrow e = 0$
x^3 Coeff: $20f + 8e + d = 0 \Rightarrow d = 0$
x^2 Coeff: $12e + 6d + c = 1 \Rightarrow c = 1$
x Coeff: $6d + 4c + b = 2 \Rightarrow 4 + b = 2 \Rightarrow b = -2$
Constant: $2c + 2b + a = 0$
$\Rightarrow 2 - 4 = a$ or $a = 2$
$\therefore$ Solution is $y = 2 - 2x + x^2$

2. $(D^2 - 6D + 9)\,y = 54x + 18$

Western Method:

$(D-3)^2\, y = 54x + 18$

$y_c = (C_1 + C_2x)e^{3x}$

$$y_p = \frac{1}{(D-3)^2}(54x+18)$$
$$= (3 - D)^{-2}\,(54x+18)$$
$$= \frac{1}{9}\left(1 - \frac{D}{3}\right)^{-2}(54x+18)$$
$$= \left(1 - \frac{D}{3}\right)^{-2}(6x+2)$$

since $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (r+$

$$\text{P.I is } \left[1 + \frac{2D}{3} + \frac{3D^2}{9} + \frac{4D^3}{27} + \ldots\right][6x+2]$$
$$= 1(6x+2) + \frac{2D}{3}(6x+2)$$
$$= 6x+2+4 = 6x+6$$

$\therefore$ The general solution of the given equation is $y = y_c + y_p$

i.e. $y = (C_1 + C_2x)e^{3x} + 6x + 6$

Vedic Method:

$(D^2 - 6D + 9)\,y = 54x + 18$

$y^{11} - 6y^{1} + 9y = 54x + 18$

$y = a + bx + cx^2 + dx^3 + ex^4$

$y^{1} = b + 2cx + 3dx^2 + 4ex^3$

$-6y^{1} = -6b - 12cx - 18dx^2 - 24ex^3$

$9y = 9a + 9bx + 9cx^2 + 9dx^3 + 9ex^4$

$y^{11} = 2c + 6dx + 12ex^2$

Equating the coefficients of like terms

x^4 Coeff: $9e = 0 \;\Rightarrow e = 0$

x^3 Coeff: $9d - 24e = 0 \Rightarrow d = 0$

x^2 Coeff: $12e + 9c - 18d = 0 \;\Rightarrow c = 0$

x Coeff: $6d + 9b - 12c = 54$

$9b = 54,\ b = 6$

Constant: $2c + 9a - 6b = 18 \Rightarrow 9a - 36 = 18$ or $9a = 54,\ a = 6$

$\therefore$ Solution: $6 + 6x = y$

3. $(D^3 - D^2 - D + 1)\,y = 1 + x^2$

Western method:

Auxiliary equation $= m^3 - m^2 - m + 1 = 0$

$F(1) = 1-1-1+1=0$

$\therefore (m-1)$ is a factor

$$
\begin{array}{l}
m-1\,)\; m^3 - m^2 - m + 1\,(\,m^2 - 1 \\
\qquad\quad \underline{m^3 - m^2} \\
\qquad\qquad -m + 1 \\
\qquad\qquad -m + 1
\end{array}
$$

$\therefore$ A.E $= (m-1)(m^2-1) = (m-1)(m-1)(m+1) = (m-1)^2(m+1)$

$\therefore m = 1, 1, -1$

$\therefore y_c = C_1 e^{-x} + (C_2 + C_3 x)e^x$

$$y_p = \frac{1}{(D-1)^2(D+1)}(1+x^2) = (1-D)^{-2}(1+x^2)$$

$= (1+2D+3D^2+\ldots)(1-D+D^2+\ldots)(1+x^2)$

$= (1 - D + D^2 + 2D - 2D^2 + 3D^2)(1+x^2)$

$= (1 + D + 2D^2)(1+x^2)$

$= 1 + x^2 + 2x + 2(2)$

$= x^2 + 2x + 5$

$\therefore$ P.I is $y_p = x^2 + 2x + 5$

$\therefore$ The general solution of the given equation is $y = y_c + y_p$

$\therefore$ i.e. $y = C_1 e^{-x} + (C_2 + C_3 x)e^x + x^2 + 2x + 5$

Vedic method:

$(D^3 - D^2 - D + 1)\, y = 1 + x^2$

$y^{III} - y^{II} - y^{I} + y = 1 + x^2$

$y = a + bx + cx^2 + dx^3 + ex^4$

$- y^{I} = -b - 2cx - 3dx^2 - 4ex^3$

$- y^{II} = -2c - 6dx - 12ex^2$

$y^{III} = 6d + 24ex$

x^4 Coeff: $e = 0$

x^3 Coeff: $d - 4e = 0 \Rightarrow d = 0$

x^2 Coeff: $c - 3d - 12e = 1 \Rightarrow c = 1$

x Coeff: $b - 2c - 6d + 24e = 0$

$\Rightarrow b - 2 = 0 \Rightarrow b = 2$

Constant: $a - b - 2c + 6d = 1$

$\Rightarrow a - 2 - 2 = 1, a = 5$

$\therefore$ Solution: $y = 5 + 2x + x^2$

4. $(D^4 - 2D^3 + D^2)\, y = x^3$

Western method:

$(D^4 - 2D^3 + D^2)\, y = x^3$

$\therefore$ Auxiliary equation is $m^4 - 2m^3 + m^2 = 0$ or $m^2(m^2 - 2m + 1) = 0$

$\Rightarrow m^2 (m - 1)^2 = 0 \therefore m = 0, 1$ or $m = 0,0,1,1$

$\therefore$ C F $= y_c = c_1 + c_2 x + (c_3 + c_4 x)e^x$

$$y_p = \frac{1}{D^2(D-1)^2} x^3 = \frac{1}{D^2}(D-1)^{-2} x^3$$

$(1 - x)^2 = 1 + 2c_1 x + 3c_2 x^2 + 4c_3 x^3 + ...$

$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + ...$ k (k is arbitrary constant)

$(D - 1)^2 = [- (1 - D)]^2 = (1 - D)^2$

$$\therefore y_p = \frac{1}{D^2}[1 + 2D + 3D^2 + 4D^3 + 5D^4 + 6D^5 + ...]\, x$$

$$= [\frac{1}{D^2} + \frac{2}{D} + 3 + 4D + 5D^2 + 6D^3 + ...]\, x^3$$

$$= \frac{1}{D}\frac{x^3}{D} + 2\,\frac{1}{D} x^3 + 3x^3 + 4Dx^3 + 5D^2x^3 + 6D^3x^3$$

$$= \frac{1}{D}\int x^3 dx + 2\int x^3 dx + 3x^3 + 4.\, 3x^2 + \frac{5d^2x^3}{dx^2} + \frac{6d^3x^3}{dx^3}$$

$$= \int \frac{x^4}{4}dx + \frac{2x^4}{4} + 3x^3 + 12x^2 + 30x + 36$$

$$y_p = \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2 + 30x + 36$$

$\therefore$ General solution is $y = y_c + y_p$

$$\Rightarrow y = (c_1 + c_2x)e^0 + (c_3 + c_4x)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2 + 30x + 36$$

$$\therefore y = c_1 + c_2x + (c_3 + c_4x)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2 + 30x + 36$$

Vedic Method:

$y^{IIII} - 2y^{III} + y^{II} = x^3$
$y = a + bx + cx^2 + dx^3 + ex^4 + fx^5$
$y^I = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4$
$y^{II} = 2c + 6dx + 12ex^2 + 20fx^3$
$-2y^{III} = -2(6d + 24ex + 60fx^2)$
$y^{IIII} = 24e + 120fx$

x^3 Coeff: $20f = 1 \Rightarrow f = \frac{1}{20}$

x^2 Coeff: $-120f + 12e = 0$, $12e = 6$, $e = \frac{1}{2}$

x Coeff: $120f - 48e + 6d = 0$

$$120 \times \frac{1}{20} - 48 \times \frac{1}{2} + 6d = 0$$

$6 - 24 + 6d = 0$ or $6d = 18$, $d = 3$

Constant: $24e - 12d + 2c = 0$

$$\frac{24}{2} - 36 + 2c = 0,\ 24 - 72 + 4c = 0,\ 4c = 48,\ c = 12$$

Solution $y = \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2 + 30x + k$ (k [illegible] arbitrary constant)

5. $(D^4 + D^3 + D^2)\,y = ax^2 + bx^3$

Western method:

$(D^4+D^3+D^2)\,y = x^2(a + bx)$
A.E is $m^4 + m^3 + m^2 = 0$
$\Rightarrow m^2(m^2+m+1) = 0$
$\Rightarrow m^2 = 0 \Rightarrow m = 0, 0$

or $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$$\therefore m = 0, 0, \frac{-1 \pm \sqrt{3}i}{2}$$

$$y_c = c_1 + c_2 x + e^{\frac{-x}{2}}\left[c_3 \cos\frac{x\sqrt{3}}{2} + c_4 \sin\frac{x\sqrt{3}}{2}\right]$$

$$y_p = \frac{1}{D^2(D^2+D+1)} x^2(a+bx)$$

$$= \frac{1}{D^2}\left[1+(D+D^2)\right]^{-1} x^2(a+bx)$$

$$= \frac{1}{D^2}\left[1-(D^2+D)+(D^2+D)^2-(D^2+D)^3+\ldots\right](ax^2+bx^3)$$

$$= \frac{1}{D^2}\left[1-D^2-D+D^4+D^2+2D^3-(D^6+D^3+3D^5+3D^4)+\ldots\right](ax^2+bx^3)$$

Since highest power of x is 3 we delete D^4, D^5 etc, since higher order derivatives other than 3rd order vanish.

$$= \frac{1}{D^2}\left[1-D+D^3\right](ax^2+bx^3)$$

$$= \frac{1}{D^2}\left[ax^2+bx^3-D(ax^2+bx^3)+D^3(ax^2+bx^3)\right]$$

$$= \frac{1}{D^2}\left[ax^2+bx^3-2ax-3bx^2+6b\right]$$

$$= \frac{1}{D}\int\left[bx^3+(a-3b)x^2-2ax+6b\right]$$

$$= \frac{1}{D}\left[b\int x^3 dx+(a-3b)\int x^2 dx-2a\int x dx+6b\int dx\right]$$

$$= \frac{1}{D}\left[\frac{bx^4}{4}+(a-3b)\frac{x^3}{3}-2a\frac{x^2}{2}+6bx\right]$$

$$= \frac{b}{4}\int x^4 dx+\frac{(a-3b)}{3}\int x^3 dx - a\int x^2 dx + 6b$$

$$= y = 3bx^2 - \frac{ax^3}{3} + \frac{(a-3b)}{12}x^4 + \frac{bx^5}{20}$$

General solution $y = y_c + y_p$

$$Y = c_1 + c_2 x + e^{\frac{-x}{2}}\left[c_3 \cos\frac{x\sqrt{3}}{2} + c_4 \sin\frac{x\sqrt{3}}{2}\right] + 3bx^2 - \frac{ax^3}{3} + \frac{(a-3b)}{12}x^4 + \frac{bx^5}{20}$$

Vedic method:

$y^{IIII} + y^{III} + y^{II} = ax^2 + bx^3$

$y = l + mx + nx^2 + px^3 + qx^4 + rx^5 + tx^6$

$y^{I} = m + 2nx + 3px^2 + 4qx^3 + 5rx^4 + 6tx^5$

$y^{II} = 2n + 6px + 12qx^2 + 20rx^3 + 30tx^4$

$y^{III} = 6p + 24qx + 60rx^2 + 120tx^3$

$y^{IIII} = 24q + 120rx + 360tx^2$

Coeff x^4: $30t = 0 \Rightarrow t = 0$

Coeff x^3: $20r + 120t = b \Rightarrow r = \dfrac{b}{20}$

Coeff x^2: $12q + 60r + 360t = a$

$\Rightarrow 12q + 60.\dfrac{b}{20} + 360t = a$

$\Rightarrow 12q + 3b = a - 360t \quad \therefore q = \dfrac{a - 3b - 360t}{12}$

$\therefore q = \dfrac{a - 3b}{12}$

$\Rightarrow 6p + 24q + 120r = 0$

$\Rightarrow 6p = -24\dfrac{(a - 3b)}{12} - 120\,\dfrac{b}{20}$

$\Rightarrow 6p = -2(a - 3b) - 6b = -2a + 6b - 6b = -2a$

$\therefore p = \dfrac{-a}{3}$

Constant: $2n + 6p + 24q = 0$

$\Rightarrow 2n + 6.\dfrac{-a}{3} + 24.\dfrac{(a - 3b)}{12} = 0$

$\Rightarrow 2n - 2a + 2a - 6b = 0$

$\Rightarrow 2n = 6b$ or $n = 3b$

$\therefore$ Particular solution (P I) is $y = 3bx^2 - \dfrac{ax^3}{3} + \dfrac{(a - 3b)}{12}x^4 + \dfrac{bx^5}{20}$

6) $(D^2 - 5D + 6)y = xe^{4x}$

Western method:

P.I of (1) is $\dfrac{1}{D^2 - 5D + 6}xe^{4x} = \dfrac{1}{(D-3)(D-2)}$

$= \left[\dfrac{1}{(D-3)} - \dfrac{1}{(D-2)}\right]xe^{4x} = \dfrac{xe^{4x}}{(D-3)} - \dfrac{xe^{4x}}{(D-2)}$

$= e^{3x}.\int xe^{4x}.e^{-3x}dx - e^{2x}\int xe^{4x}.e^{-2x}dx$

$= e^{3x}\int x.e^{x}dx - e^{2x}\int xe^{2x}dx = e^{3x}\left[xe^{x} - e^{x}\right] - e^{2x}\left[\dfrac{xe^{2x}}{2} - \dfrac{e^{2x}}{4}\right]$

$y_p = \dfrac{e^{4x}(2x - 3)}{4}$

Vedic method:

P.I: $y^{II} - 5y^{I} + 6y = xe^{4x}$ $\rightarrow$ (1)

Let $y = ae^{4x} + bxe^{4x}$ $\rightarrow$ (2)

$y^{I} = 4ae^{4x} + b(x.4e^{4x} + e^{4x})$

$\therefore y^{I} = 4ae^{4x} + 4bxe^{4x} + be^{4x}$

$y^{II} = 16ae^{4x} + 4b(x.4e^{4x} + e^{4x}) + 4be^{4x}$

$= 16ae^{4x} + 16bxe^{4x} + 4be^{4x} + 4be^{4x}$

$\therefore y^{II} = 16ae^{4x} + 16bxe^{4x} + 8be^{4x}$

$-5y^{I} = -20ae^{4x} - 20bxe^{4x} - 5be^{4x}$

$6y = 6ae^{4x} + 6bxe^{4x}$

$y^{II} - 5y^{I} + 6y = (16a - 20a + 6a + 8b - 20b + 6b)e^{4x} + (16b - 20b + 6b)xe^{4x}$

$= (2a + 3b)e^{4x} + 2bxe^{4x}$

$\therefore$ From (1) $(2a + 3b)e^{4x} + 2bxe^{4x} = xe^{4x}$

Equating the coefficients of like terms on both sides

$2a + 3b = 0 \Rightarrow a = -\frac{3}{2}b$

and $2b = 1 \Rightarrow b = \frac{1}{2}$

$\therefore a = \frac{3}{2}b \Rightarrow a = -\frac{3}{2}.\frac{1}{2} = -\frac{3}{4}$

$\therefore$ Solution is $y = ae^{4x} + bxe^{4x}$

$\Rightarrow y = -\frac{3}{4}e^{4x} + \frac{xe^{4x}}{2} = \frac{2xe^{4x} - 3e^{4x}}{4} = \frac{e^{4x}(2x-3)}{4}$

7) $(D^2 - 5D + 6)y = e^{4x}$

Western method:

Auxiliary equation A. E $= m^2 - 5m + 6 = 0$

$\Rightarrow m^2 - 3m - 2m + 6 = 0 \Rightarrow (m - 3)(m - 2) = 0$

$\therefore$ Complementary function $y_c = C_1 e^{2x} + \ldots$

$y_p = \frac{1}{(D^2 - 5D + 6)}e^{4x} = \frac{e^{4x}}{(D-3)(D-2)}$

let $\frac{1}{(D-3)(D-2)} = \frac{A}{(D-3)} + \frac{B}{(D-2)}$

$\Rightarrow 1 = A(D-2) + B(D-3)$ if $D = 2$

$B = -1$ and if $D = 3$

$A = 1$

$$\therefore y_p = \left[\frac{1}{(D-3)} - \frac{1}{(D-2)}\right]e^{4x}$$

$$= e^{3x} \int e^{-3x} e^{4x} dx - e^{2x} \int e^{-2x} e^{4x} dx$$

$$= e^{3x} \int e^{x} dx - e^{2x} \int e^{2x} dx$$

$$= e^{3x} \cdot e^{x} - e^{2x} \cdot \frac{e^{2x}}{2}$$

$$= e^{4x} - \frac{e^{4x}}{2} = \frac{e^{4x}}{2}$$

$$\therefore y = C_1 e^{2x} + C_2 e^{3x} + \frac{e^{4x}}{2}$$

Vedic method:

$(D^2 - 5D + 6) Y = e^{4x}$

$Y = ae^{4x}$

$Y' = 4ae^{4x}$
$y'' = 16 ae^{4x}$

$ae^{4x} [16 - 20 + 6] = e^{4x}$

$2a = 1$
$a = \frac{1}{2}$
$\therefore y = \frac{1}{2} e^{4x}$
$Y' = 2e^{4x}$
$y'' = 8e^{4x}$
$8e^{4x} - 10 e^{4x} + 3e^{4x} = \frac{e^{4x}}{2}$

8) $\frac{d^2y}{dx^2} - y = 2 + 5x$
⇨ $(D^2 - 1) Y = 2 + 5X$

Western method:
Auxiliary equation = $m^2 - 1 = 0 \Rightarrow m = 1, -1$
$\therefore y_c = C_1 e^{x} + C_2 e^{-x}$

$$y_p = \left[\frac{1}{(D^2-1)}\right](5X+2) = \left[\frac{1}{(D-1)(D+1)}\right](5X+2)$$

let $\left[\frac{1}{(D-1)(D+1)}\right] = \frac{A}{(D+1)} + \frac{B}{(D-1)}$

$\Rightarrow$ 1 = A (D − 1) + B (D + 1) if D = 1

1 = 2B or B = $\frac{1}{2}$ and if D = -1

1 = - 2A or A = -$\frac{1}{2}$

$$\therefore Y_P = \left[\frac{-1}{2(D+1)} + \frac{1}{2(D-1)}\right](5X+2)$$

or

$$y_p = \frac{1}{2}\left[\frac{1}{(D-1)} - \frac{1}{(D+1)}\right](5x+2)$$

$$= \frac{1}{2}\left[\frac{1}{(D-1)}(5x+2)\right] - \frac{1}{2}\left[\frac{1}{(D+1)}(5x+2)\right]$$

or $y_p = \frac{1}{2}e^x \int(5x+2)e^{-x}dx - \frac{1}{2}e^{-x}\int(5x+2)e^x dx$

consider $\int(5x+2)e^{-x}dx = 5\int xe^{-x}dx + 2\int e^{-x}dx$

$$= 5\left[-xe^{-x} - \int(-e^{-x})dx\right] - 2e^{-x}$$

$$= -5\,x\,e^{-x} - 5e^{-x} - 2e^{-x} = (-5\,xe^{-x} - 7e^{-x}) \quad \text{. (A)}$$

Now consider $\int(5x+2)e^x dx = 5\int xe^x dx + 2\int e^x dx$

$$= 5\left[xe^x - \int e^x dx\right] + 2e^x$$
$$= 5xe^x - 5e^x + 2e^x = 5xe^x - 3\,e^x \rightarrow \text{(B)}$$

By (A) & (B)

$$Y_P = \frac{1}{2}e^{x}\left[-5xe^{-x}-7e^{-x}\right] - \frac{1}{2}e^{-x}\left[5xe^{x}-3e^{x}\right]$$

$$= -\frac{5}{2}x-\frac{7}{2}-\frac{5}{2}x+\frac{3}{2}$$

$\therefore\ y_p = (-5x-2)$

$\therefore$ Solution $y = y_c + y_p = (C_1e^{-x} + C_2e^{x} - 2 - 5x)$

Vedic Method:

$y'' - y = 2 + 5x$

$y = a + bx + cx^2$

$Y' = b + 2cx$

$y'' = 2c$

$2c - a - bx - cx^2 = 2 + 5x$

$b = -5$

$2c - a = 2$

$\therefore\ a = -2,\ b = -5$

$y = -2 - 5x = -(2+5x)$

9) Western method:

$y^{11} + y^{1} - y = -10x^4 + 38x^3 + 126x^2 + 8x + 4$

$$\frac{1}{D^2+D-1} = [\frac{-1}{\sqrt{5}(D+\alpha)} + \frac{1}{\sqrt{5}(D+\beta)}] \text{Where}\alpha = \frac{(1+\sqrt{5})}{2} \text{ and} \beta = \frac{(1-\sqrt{5})}{2}$$

$$= [\frac{-1}{\sqrt{5}(D+\alpha)} + \frac{1}{\sqrt{5}(D+\beta)}][-10x^4 + 38x^3 + 126x^2 + 8x + 4]$$

I multiplying with $-10x^4$

$$= [\frac{10x^4}{\sqrt{5}(D+\alpha)} - \frac{10x^4}{\sqrt{5}(D+\beta)}]$$

$$= \frac{10}{\sqrt{5}}[\frac{x^4}{(D+\alpha)} - \frac{x^4}{(D+\beta)}]$$

$$= \frac{10}{\sqrt{5}}[e^{-\alpha x}\int x^4e^{\alpha x}dx - e^{-\beta x}\int x^4e^{\beta x}d..$$

We have $I_n = x^n\frac{e^{\alpha x}}{\alpha} - \frac{n}{\alpha}I_{n-1}$

Where $I_n = \int x^n e^{\alpha x}dx$

$$I_4 = \int x^4e^{\alpha x}dx = x^4\frac{e^{\alpha x}}{\alpha} - \frac{4}{\alpha}I_{4-1}$$

$$= x^4 \frac{e^{\alpha x}}{\alpha} - \frac{4}{\alpha}[\frac{x^3 e^{\alpha x}}{\alpha} - \frac{3}{\alpha} I_2]$$

$$= x^4 \frac{e^{\alpha x}}{\alpha} - \frac{4x^3 e^{\alpha x}}{\alpha^2} + \frac{12}{\alpha^2}[\frac{x^2 e^{\alpha x}}{\alpha} - \frac{2}{\alpha} I_1]$$

$$= x^4 \frac{e^{\alpha x}}{\alpha} - \frac{4x^3 e^{\alpha x}}{\alpha^2} + \frac{12x^2 e^{\alpha x}}{\alpha^3} - \frac{24}{\alpha^3} I_1$$

$$= x^4 \frac{e^{\alpha x}}{\alpha} - \frac{4x^3 e^{\alpha x}}{\alpha^2} + \frac{12x^2 e^{\alpha x}}{\alpha^3} - \frac{24}{\alpha^3}[\frac{xe^{\alpha x}}{\alpha} - \frac{1}{\alpha} I_0]$$

$$= x^4 \frac{e^{\alpha x}}{\alpha} - \frac{4x^3 e^{\alpha x}}{\alpha^2} + \frac{12x^2 e^{\alpha x}}{\alpha^3} - \frac{24xe^{\alpha x}}{\alpha^4} + \frac{24}{\alpha^4} \cdot \frac{e^{\alpha x}}{\alpha}$$

$$= x^4 \frac{e^{\alpha x}}{\alpha} - \frac{4x^3 e^{\alpha x}}{\alpha^2} + \frac{12x^2 e^{\alpha x}}{\alpha^3} - \frac{24xe^{\alpha x}}{\alpha^4} + \frac{24e^{\alpha x}}{\alpha^5}$$

$$\therefore e^{-\alpha x}\int x^4 e^{\alpha x}dx = \frac{x^4}{\alpha} - \frac{4x^3}{\alpha^2} + \frac{12x^2}{\alpha^3} - \frac{24x}{\alpha^4} + \frac{24}{\alpha^5} \rightarrow (1)$$

[Note that NR terms are successive derivatives of x^4 (-1^4)]

Similarly $e^{-\beta x}\int x^4 e^{\beta x}dx$ yields

$$\frac{x^4}{\beta} - \frac{4x^3}{\beta^2} + \frac{12x^2}{\beta^3} - \frac{24x}{\beta^4} + \frac{24}{\beta^5} \rightarrow (2)$$

$\therefore$ Equation I $\Rightarrow \frac{10}{\sqrt{5}}[(1) - (2)]$

$$= \frac{10}{\sqrt{5}}[\{\frac{x^4}{\alpha} - \frac{4x^3}{\alpha^2} + \frac{12x^2}{\alpha^3} - \frac{24x}{\alpha^4} + \frac{24}{\alpha^5}\} - \{\frac{x^4}{\beta} - \frac{4x^3}{\beta^2} + \frac{12x^2}{\beta^3} - \frac{24x}{\beta^4} + \frac{24}{\beta^5}\}]$$

$$= \frac{10}{\sqrt{5}}[\frac{x^4}{\alpha} - \frac{4x^3}{\alpha^2} + \frac{12x^2}{\alpha^3} - \frac{24x}{\alpha^4} + \frac{24}{\alpha^5} - \frac{x^4}{\beta} + \frac{4x^3}{\beta^2} - \frac{12x^2}{\beta^3} + \frac{24x}{\beta^4} - \frac{24}{\beta^5}]$$

$$= \frac{10}{\sqrt{5}}[x^4(\frac{1}{\alpha} - \frac{1}{\beta}) - 4x^3(\frac{1}{\alpha^2} - \frac{1}{\beta^2}) + 12x^2(\frac{1}{\alpha^3} - \frac{1}{\beta^3}) - 24x(\frac{1}{\alpha^4} - \frac{1}{\beta^4}) + 24(\frac{1}{\alpha^5} - \frac{1}{\beta^5})]$$

$$= \frac{10}{\sqrt{5}}[x^4(\frac{\beta - \alpha}{\alpha\beta}) - 4x^3(\frac{\beta^2 - \alpha^2}{(\alpha\beta)^2}) + 12x^2(\frac{\beta^3 - \alpha^3}{(\alpha\beta)^3}) - 24x(\frac{\beta^4 - \alpha^4}{(\alpha\beta)^4}) + 24(\frac{\beta^5 - \alpha^5}{(\alpha\beta)^5})]$$

$$= \frac{10}{\sqrt{5}}[x^4(\frac{-\sqrt{5}}{-1}) - 4x^3\frac{(-\sqrt{5})}{(-1)^2} + 12x^2 \text{[illegible]} + 24\frac{(-\sqrt{5})}{(-1)^5}]$$

$$= \frac{10}{\sqrt{5}}[x^4\sqrt{5} + 4x^3\sqrt{5} + 24x^2\sqrt{5} + 72x\sqrt{5} + 120\sqrt{5}]$$

$$= \frac{10}{\sqrt{5}}[x^4 + 4x^3 + 24x^2 + 72x + 120]\sqrt{5}$$

$$= [10x^4 + 40x^3 + 240x^2 + 720x + 1200] \rightarrow \text{I}$$

<u>II multiplying with 38</u> x^3:

$$[\frac{-1}{\sqrt{5}(D + \alpha)} + \frac{1}{\sqrt{5}(D + \beta)}]\, 38x^3 \rightarrow \text{II}$$

$$= \frac{38}{\sqrt{5}}\left[\frac{-x^3}{D+\alpha} + \frac{x^3}{D+\beta}\right]$$

$$= \frac{38}{\sqrt{5}}\left[e^{-\alpha x}\int -x^3 e^{\alpha x}dx + e^{-\beta x}\int x^3 e^{\beta x}dx\right]$$

$$= \frac{38}{\sqrt{5}}\left[-e^{-\alpha x}\int x^3 e^{\alpha x}dx + e^{-\beta x}\int x^3 e^{\beta x}dx\right]$$

Consider $-e^{-\alpha x}\int x^3 e^{\alpha x}dx \rightarrow (1)$

$$= -e^{-\alpha x}\left[\frac{x^3 e^{\alpha x}}{\alpha} - \frac{3}{\alpha}I_2\right] \quad \text{where } I_n = \int x^n e^{\alpha x}dx = x^n\frac{e^{\alpha x}}{\alpha} - \frac{n}{\alpha}I_{n-1}$$

$$= -e^{-\alpha x}\left[\frac{x^3 e^{\alpha x}}{\alpha} - \frac{3}{\alpha}\left(\frac{x^2 e^{\alpha x}}{\alpha} - \frac{2}{\alpha}I_1\right)\right]$$

$$= -e^{-\alpha x}\left[\frac{x^3 e^{\alpha x}}{\alpha} - \frac{3x^2 e^{\alpha x}}{\alpha^2} + \frac{6}{\alpha^2}I_1\right]$$

$$= -e^{-\alpha x}\left[\frac{x^3 e^{\alpha x}}{\alpha} - \frac{3x^2 e^{\alpha x}}{\alpha^2} + \frac{6}{\alpha^2}\left(\frac{xe^{\alpha x}}{\alpha} - \frac{1}{\alpha}I_0\right)\right]$$

$$= -e^{-\alpha x}\left[\frac{x^3 e^{\alpha x}}{\alpha} - \frac{3x^2 e^{\alpha x}}{\alpha^2} + \frac{6xe^{\alpha x}}{\alpha^3} - \frac{6}{\alpha^3}\cdot\frac{e^{\alpha x}}{\alpha}\right]$$

$$= \left[\frac{-x^3}{\alpha} + \frac{3x^2}{\alpha^2} - \frac{6x}{\alpha^3} + \frac{6}{\alpha^4}\right] \rightarrow (1)$$

Similarly $e^{-\beta x}\int x^3 e^{\beta x}dx$ is $\left[\frac{x^3}{\beta} - \frac{3x^2}{\beta^2} + \frac{6x}{\beta^3} - \frac{6}{\beta^4}\right] \rightarrow (2)$

$\therefore$ Equation II $= \frac{38}{\sqrt{5}}[(1)+(2)]$

$$= \frac{38}{\sqrt{5}}\left[\frac{-x^3}{\alpha} + \frac{3x^2}{\alpha^2} - \frac{6x}{\alpha^3} + \frac{6}{\alpha^4} + \frac{x^3}{\beta} - \frac{3x^2}{\beta^2} + \frac{6x}{\beta^3} - \frac{6}{\beta^4}\right]$$

$$= \frac{38}{\sqrt{5}}\left[x^3\left(\frac{1}{\beta} - \frac{1}{\alpha}\right) - 3x^2\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right) + 6x\left(\frac{1}{\beta^3} - \frac{1}{\alpha^3}\right) - 6\left(\frac{1}{\beta^4} - \frac{1}{\alpha^4}\right)\right]$$

$$= \frac{38}{\sqrt{5}}\left[x^3\left(\frac{\alpha-\beta}{\alpha\beta}\right) - 3x^2\left(\frac{\alpha^2-\beta^2}{(\alpha\beta)^2}\right) + 6x\left(\frac{\alpha^3-\beta^3}{(\alpha\beta)^3}\right) - 6\left(\frac{\alpha^4-\beta^4}{(\alpha\beta)^4}\right)\right]$$

$$= \frac{38}{\sqrt{5}}\left[x^3\left(\frac{\sqrt{5}}{-1}\right) - 3x^2\frac{(\sqrt{5})}{(-1)^2} + 6x\frac{(2\sqrt{5})}{(-1)^3} - 6\frac{(3\sqrt{5})}{(-1)^4}\right]$$

$$= \frac{38}{\sqrt{5}}\left[-x^3\sqrt{5} - 3x^2\sqrt{5} - 12x\sqrt{5} - 18\sqrt{5}\right]$$

$$= -38x^3 - 114x^2 - 456x - 684 \rightarrow \text{II}$$

<u>III multiplying with 126</u>x^2:

$$\left[\frac{-1}{\sqrt{5}(D+\alpha)} + \frac{1}{\sqrt{5}(D+\beta)}\right](126x^2) \rightarrow \text{III}$$

$$= \frac{126}{\sqrt{5}}\left[\frac{-x^2}{(D+\alpha)} + \frac{x^2}{(D+\beta)}\right]$$

$= \frac{126}{\sqrt{5}}[e^{-\alpha x}\int -x^2e^{\alpha x}dx + e^{-\beta x}\int x^2e^{\beta x}dx]$

$= \frac{126}{\sqrt{5}}[-e^{-\alpha x}\int x^2e^{\alpha x}dx + e^{-\beta x}\int x^2e^{\beta x}dx]$

Consider $-e^{-\alpha x}\int x^2e^{\alpha x}dx \rightarrow (1)$

$= -e^{-\alpha x}[\frac{x^2e^{\alpha x}}{\alpha} - \frac{2}{\alpha}I_1]$ where $I_n = x^n\frac{e^{\alpha x}}{\alpha} - \frac{n}{\alpha}I_{n-1}$

$= -e^{-\alpha x}[\frac{x^2e^{\alpha x}}{\alpha} - \frac{2}{\alpha}(x\frac{e^{\alpha x}}{\alpha} - \frac{1}{\alpha}I_0)]$

$= -e^{-\alpha x}[\frac{x^2e^{\alpha x}}{\alpha} - \frac{2xe^{\alpha x}}{\alpha^2} + \frac{2}{\alpha^2}.\frac{e^{\alpha x}}{\alpha}]$

$= -e^{-\alpha x}[\frac{x^2e^{\alpha x}}{\alpha} - \frac{2xe^{\alpha x}}{\alpha^2} + \frac{2e^{\alpha x}}{\alpha^3}]$

$= [-\frac{x^2}{\alpha} + \frac{2x}{\alpha^2} - \frac{2}{\alpha^3}] \rightarrow (1)$

Similarly $e^{-\beta x}\int x^2e^{\beta x}dx$ gives

$= [\frac{x^2}{\beta} - \frac{2x}{\beta^2} + \frac{2}{\beta^3}] \rightarrow (2)$

$\therefore$ Equation III $= \frac{126}{\sqrt{5}}[(1) + (2)]$

$= \frac{126}{\sqrt{5}}[-\frac{x^2}{\alpha} + \frac{2x}{\alpha^2} - \frac{2}{\alpha^3} + \frac{x^2}{\beta} - \frac{2x}{\beta^2} + \frac{2}{\beta^3}]$

$= \frac{126}{\sqrt{5}}[x^2(\frac{1}{\beta} - \frac{1}{\alpha}) - 2x(\frac{1}{\beta^2} - \frac{1}{\alpha^2}) + 2(\frac{1}{\beta^3} - \frac{1}{\alpha^3})]$

$= \frac{126}{\sqrt{5}}[x^2(\frac{\alpha-\beta}{\alpha\beta}) - 2x(\frac{\alpha^2-\beta^2}{(\alpha\beta)^2}) + 2(\frac{\alpha^3-\beta^3}{(\alpha\beta)^3})]$

$= \frac{126}{\sqrt{5}}[x^2(\frac{\sqrt{5}}{-1}) - 2x\frac{(\sqrt{5})}{(-1)^2} + 2.\frac{(4\sqrt{5})}{(-1)^3}]$

$= -126x^2 - 252x - 504 \rightarrow$ III

<u>IV multiplying with 8x:</u>

$[\frac{-1}{\sqrt{5}(D+\alpha)} + \frac{1}{\sqrt{5}(D+\beta)}](8x) \rightarrow$ IV

$= \frac{8}{\sqrt{5}}[\frac{-x}{(D+\alpha)} + \frac{x}{(D+\beta)}]$

$= \frac{8}{\sqrt{5}}[e^{-\alpha x}\int -xe^{\alpha x}dx + e^{-\beta x}\int xe^{\beta x}dx]$

Consider $e^{-\alpha x}\int -xe^{\alpha x}dx \rightarrow (1)$

$= -e^{-\alpha x}\int xe^{\alpha x}dx$

$= -e^{-\alpha x}[\frac{xe^{\alpha x}}{\alpha} - \frac{1}{\alpha}I_0]$ where $I_n = x^n\frac{e^{\alpha x}}{\alpha} - \frac{n}{\alpha}I_{n-1}$

$= -e^{-\alpha x}\left[\frac{xe^{\alpha x}}{\alpha} - \frac{e^{\alpha x}}{\alpha^2}\right]$

$= \left[-\frac{x}{\alpha} + \frac{1}{\alpha^2}\right] \rightarrow (1)$

Similarly $e^{-\beta x}\int xe^{\beta x}dx$ gives

$= \left[\frac{x}{\beta} - \frac{1}{\beta^2}\right] \rightarrow (2)$

$\therefore$ Equation IV $= \frac{8}{\sqrt{5}}[(1) + (2)]$

$$= \frac{8}{\sqrt{5}}\left[-\frac{x}{\alpha} + \frac{1}{\alpha^2} + \frac{x}{\beta} - \frac{1}{\beta^2}\right]$$

$$= \frac{8}{\sqrt{5}}\left[x\left(\frac{1}{\beta} - \frac{1}{\alpha}\right) - \left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)\right]$$

$$= \frac{8}{\sqrt{5}}\left[x\left(\frac{\alpha-\beta}{\alpha\beta}\right) - \left(\frac{\alpha^2-\beta^2}{(\alpha\beta)^2}\right)\right]$$

$$= \frac{8}{\sqrt{5}}\left[-x(\sqrt{5}) - \sqrt{5}\right] \qquad \because \alpha\beta = -1 \Rightarrow (\alpha\beta)^2 = 1$$

$$= -8x - 8 \rightarrow \text{IV}$$

<u>V multiplying with 4:</u>

$$\left[\frac{-1}{\sqrt{5}(D+\alpha)} + \frac{1}{\sqrt{5}(D+\beta)}\right](4) \rightarrow \text{V}$$

$$= \frac{4}{\sqrt{5}}\left[\frac{-1}{(D+\alpha)} + \frac{1}{(D+\beta)}\right]$$

$$= \frac{4}{\sqrt{5}}\left[-e^{-\alpha x}\int e^{\alpha x}dx + e^{-\beta x}\int e^{\beta x}dx\right]$$

$$= \frac{4}{\sqrt{5}}\left[-e^{-\alpha x}.\frac{e^{\alpha x}}{\alpha} + e^{-\beta x}.\frac{e^{\beta x}}{\beta}\right]$$

$$= \frac{4}{\sqrt{5}}\left[-\frac{1}{\alpha} + \frac{1}{\beta}\right]$$

$$= \frac{4}{\sqrt{5}}\left[\frac{\alpha-\beta}{\alpha\beta}\right]$$

$$= \frac{4}{\sqrt{5}}.\frac{\sqrt{5}}{-1} = -4 \rightarrow \text{V}$$

$\therefore$ PI = I + II + III + IV + V

$= 10x^4 + 40x^3 + 240x^2 + 720x + 1200 \rightarrow$ I

$= -38x^3 - 114x^2 - 456x - 684 \rightarrow$ II

$= -126x^2 - 252x - 504 \rightarrow$ III

$= -8x - 8 \rightarrow$ IV

$= -4 \rightarrow$ V

$= 10x^4 + 2x^3 + 0 + 4x + 0$

$\therefore$ PI is $y = 10x^4 + 2x^3 + 4x$

Vedic method:

$y'' + y' - y = -10x^4 + 38x^3 + 126x^2 + 8x + 4$
Let $y = a + bx + cx^2 + dx^3 + ex^4$
$y' = b + 2cx + 3dx^2 + 4ex^3$
$y'' = 2c + 6dx + 12ex^2$
$\therefore$ General equation is $(2c + b - a) + (6d + 2c - b)\,x + (12e + 3d - c)\,x^2 + (4e - d)\,x^3 - ex^4$
$= -10x^4 + 38x^3 + 126x^2 + 8x + 4$

Comparing the coefficient of like terms on both sides.

x^4 Coeff: $-e = -10$
$\therefore e = 10$
x^3 Coeff: $4e - d = 38$
$\therefore d = 2$
x^2 Coeff: $12e + 3d - c = 126$
$\therefore c = 0$
x Coeff: $6d + 2c - b = 8$
$\therefore b = 4$
Constant: $2c + b - a = 4$
$\therefore a = 0$
$\therefore$ Solution is $y = 4x + 2x^3 + 10x^4$

10) $y'' - 2y' + y = 3x^4 - 23x^3 + 30x^2 + 6x - 5$
$Y = 3x^4 + x^3 - 5$
$\Rightarrow y' = 12x^3 + 3x^2$
$-2y' = -24x^3 - 6x^2$
$y'' = 36x^2 + 6x$

$\therefore\ y'' - 2y' + y = 3x^4 - 23x^3 + 30x^2 + 6x - 5$

Equation in the symbolic form is
$(D^2 - 2D + 1)\,Y = 0$ or $(D-1)^2 = 0$
P.I:
$$\frac{1}{(D-1)^2}\left[3x^4 - 23x^3 + 30x^2 + 6x - 5\right]$$
Evaluating by part by part

<u>I $3x^4$:</u>
$$\frac{1}{(D-1)^2}\left[3x^4\right] = \frac{1}{(D-1)}\left[\frac{3x^4}{D-1}\right]$$

$$= \frac{1}{(D-1)}\left[3e^x \int x^4 e^{-x}\,dx\right] \rightarrow \text{I}$$

$$\left[Note: we\,have\,I_n = \int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}\right]$$

$$\therefore \quad \int x^4 e^{-x} dx = \frac{x^4 e^{-x}}{-1} - \frac{4}{-1} I_3$$

$$= -x^4 e^{-x} + 4\left[\frac{x^3 e^{-x}}{-1} - \frac{3}{-1} I_2\right]$$

$$= -x^4 e^{-x} - 4x^3 e^{-x} + 12\, I_2$$

$$= -x^4 e^{-x} - 4x^3 e^{-x} + 12\left[\frac{x^2 e^{-x}}{-1} - \frac{2}{-1} I_1\right]$$

$$= -x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} + 24\, I_1$$

$$= -x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} + 24\left[\frac{xe^{-x}}{-1} - \frac{1}{-1} I_0\right]$$

$$= -x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} - 24e^{-x} - 24e^{-x} = \int x^4 e^{-x} dx$$

Substituting the above in I

$$\frac{1}{(D-1)}\left[-3x^4 - 12x^3 - 36x^2 - 72x - 72\right] \rightarrow I$$

again evaluating part by part

(a) : - $3x^4$

$$\frac{-3x^4}{(D-1)} = -3e^x \int x^4 e^x dx \quad \text{from the above we have}$$

$$= -3\left[-x^4 - 4x^3 - 12x^2 - 24x + 24\right]$$

$$= 3x^4 + 12x^3 + 36x^2 + 72x + 72 \rightarrow (A)$$

(b): - -12 x^3

$$\frac{12x^3}{D-1} = -12e^x \int x^3 e^{-x} dx \quad \rightarrow (i)$$

$$\int x^3\ e^{-x} = \frac{x^3 e^{-x}}{-1} - \frac{3}{1} I_2$$

$$= -x^3 e^{-x} + 3\left[\frac{x^2 e^{-x}}{-1} - \frac{2}{-1} I_1\right]$$

$$= -x^3 e^{-x} - 3x^2 e^{-x} - 6\left[\frac{xe^{-x}}{-1} - \frac{1}{-1} I_0\right]$$

$$= -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x}$$

Substituting in (i)

we get

$$= -12e^x\left[-x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x}\right]$$

$$= 12x^3 + 36x^2 + 72x + 72 \quad \rightarrow (B)$$

(c)$36x^2$: -

$$\frac{-36x^2}{D-1} = -36e^x \int x^2 e^{-x}\,dx \quad \rightarrow \text{(i)}$$

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2\,I_1$$

$$= -x^2e^{-x} + 2\left(\frac{xe^{-x}}{-1} + I_0\right) = -x^2e^{-x} - 2xe^{-x} - 2e^{-x}$$

Substituting the above in (1) we get

$36x^2 + 72x + 72 \quad \rightarrow$ ©

(d) $-72x$:-

$$\frac{-72x}{D-1} = -72e^x \int xe^{-x}\,dx$$

$$= -72e^x \left[x(-e^{-x}) - \int (1.-e^{-x})dx\right]$$

$$= -72e^x \left[-xe^{-x} + \int e^{-x}\,dx\right]$$

$$= -72e^x \left[-xe^{-x} - e^{-x}\right] = 72x + 72 \quad \rightarrow \text{(d)}$$

(e) -72 :

$$\frac{-72}{D-1} = -72e^x \int e^{-x}\,dx$$

$$= -72e^x - e^{-x}$$

$$= 72 \quad \rightarrow \text{(e)}$$

$\therefore$ I = (a) + (b)+(c) + (d) + (e)

$3x^4 + 12x^3 + 36x^2 + 72x + 72 \rightarrow$ (a)

$12x^3 + 36x^2 + 72x + 72 \rightarrow$ (b)

$36x^2 + 72x + 72 \rightarrow$ (c)

$72x + 72 \rightarrow$ (d)

$72 \rightarrow$ (e)

i.e for $3x^4 \rightarrow 3x^4 + 24x^3 + 108x^2 + 288x + 36($

II $-23x^3$:

$$\frac{-23x^3}{(D-1)^2} = \frac{1}{(D-1)}\left[\frac{-23x^3}{(D-1)}\right]$$

$$= \frac{1}{(D-1)}\left[-23e^x \int x^3 e^{-x}\,dx\right] \quad \rightarrow \text{II}$$

$\int x^3 e^{-x} dx = -x^3 e^{-x} + 3I_2$

$= -x^3e^{-x} + 3(-x^2 e^{-x} + 2I_1)$

$= -x^3e^{-x} - 3x^2 e^{-x} + 6I_1$

$= -x^3e^{-x} - 3x^2 e^{-x} + 6(-xe^{-x} + I_0)$

$= -x^3e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x}$

Substituting the above in II we get

$$\frac{1}{(D-1)}\left[-23e^{x}(-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6\right]$$

$$\frac{1}{(D-1)}\left[23x^3 + 69x^2 + 138x + 138\right] \quad \rightarrow \text{II}$$

Again evaluating term by term

(a) $23x^3$:

$$\frac{1}{(D-1)}23x^3 = 23e^{x}\int x^3e^{-x}dx$$

$$= 23e^{x}\left[-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}\right]$$

$$= -23x^3 - 69x^2 - 138x - 138 \rightarrow \text{(a)}$$

(b) $69x^2$:

$$\frac{1}{(D-1)}69x^2 = 69e^{x}\int x^2e^{-x}dx$$

$$= 69e^{x}\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]$$

$$= -69x^2 - 138x - 138 \rightarrow \text{(b)}$$

(c)

$$\frac{138}{(D-1)}x = 138e^{x}\int xe^{-x}dx = -138x - 138 \rightarrow \text{©}$$

(d) 138:

$$\frac{138}{(D-1)} = 138e^{x}\int e^{-x}dx = -138 \rightarrow \quad (d)$$

$\therefore$ II = (a) + (b) + (c) + (d)

$$\begin{array}{rcl} -23x^3 - 69x^2 - 138x - 138 & \rightarrow & \text{(a)} \\ -69x^2 - 138x - 138 & \rightarrow & \text{(b)} \\ -138x - 138 & \rightarrow & \text{(c)} \\ -138 & \rightarrow & \text{(d)} \\ \hline -23x^3 - 138x^2 - 414x - 552 & \rightarrow & \text{II} \end{array}$$

III <u>$30x^2$:</u>

$$\frac{1}{(D-1)^2}30x^2$$

$$=\frac{1}{(D-1)}\left[\frac{30x^2}{(D-1)}\right]$$

$$=\frac{1}{(D-1)}\left[30e^x\int x^2e^{-x}dx\right]$$

$=\int x^2e^{-x}dx = -x^2e^{-x}-2xe^{-x}-2e^{-x}$ substituting this is III we get

$$=\frac{1}{(D-1)}\left[30x^2-60x-60\right]\rightarrow \text{III}$$

Evaluating again by term by term

(a) <u>$-30x^2$:</u>

$$\frac{-30x^2}{(D-1)}=-30e^x\int x^2e^{-x}dx$$

$$=-30e^x\left[-x^2e^{-x}-2xe^{-x}-2\right]$$

$$=30x^2+60x+60\rightarrow \text{(a)}$$

(b) <u>$-60x$:</u>

$$\frac{-60x}{(D-1)}=-60e^x\int xe^{-x}dx$$

$$=-60e^x\left[-xe^{-x}-e^{-x}\right]$$

$$=60x+60\rightarrow \text{(b)}$$

(c) <u>-60:</u>

$$\frac{-60}{(D-1)}=-60e^x\int e^{-x}dx=60\quad\rightarrow \text{(c)}$$

$\therefore$ III = (a) + (b) + (c)

$$\begin{array}{r} 30x^2+60x+60 \\ 60x+60 \\ 60 \\ \hline 30x^2+120x+180 \end{array}\rightarrow \text{III}$$

IV <u>$6x$:</u>

$$\frac{6x}{(D-1)^2}=\frac{1}{(D-1)}\left[\frac{6x}{(D-1)}\right]$$

$$=\frac{1}{(D-1)}\left[6e^x\int xe^{-x}dx\right]$$

$$= \frac{1}{(D-1)}\left[6e^{x}(-xe^{-x}-e^{-x})\right]$$

$$= \frac{1}{(D-1)}\left[-6x-6)\right] \rightarrow \text{IV}$$

(a) $\frac{-6x}{(D-1)} = -6e^{x}\int xe^{-x}dx = -6e^{x}(-xe^{-x}-e^{-x}) = 6x+6 \rightarrow$ (a)

$\frac{-6}{(D-1)} = -6e^{x}\int e^{-x}dx = 6 \rightarrow$ (b)

$\therefore$ IV = (a) + (b) = 6x + 12 $\rightarrow$ IV

V -5:

$$\frac{-5}{(D-1)^2} = \frac{1}{(D-1)}\left[\frac{-5}{(D-1)}\right]$$

$$= \frac{1}{(D-1)}\left[-5e^{x}\int e^{-x}dx\right] = \frac{5}{(D-1)}$$

$$= 5e^{x}\int e^{-x}dx$$

$$= -5$$

$\therefore$ P.I = I + II + III + IV + V

$3x^4 + 24x^3 + 108x^2 + 288x + 360 \rightarrow$ I
$-23x^3 - 138x^2 - 414x - 552 \rightarrow$ II
$30x^2 + 120x + 180 \rightarrow$ III
$6x + 12 \rightarrow$ IV
-5

$3x^4 + x^3 + 0 + 0 - 5$

$\therefore$ P.I is $y = 3x^4 + x^3 - 5$.

Vedic Method:

$y'' - 2y' + y = 3x^4 - 23x^3 + 30x^2 + 6x - 5$
let $y = a + bx + cx^2 + dx^3 + ex^4 \rightarrow (1)$
$y' = b + 2cx + 3dx^2 + 4ex^3$
$-2y' = -2b - 4cx - 6dx^2 - 8ex^3 \rightarrow (2)$
$y'' = 2c + 6dx + 12ex^2 \rightarrow (3)$
$\therefore y'' - 2y' + y$
$= (2c - 2b + a) + (6d - 4c + b)x + (12e - 6d + c)x^2 + (-8e + d)x^3 + ex^4$
$= 3x^4 - 23x^3 + 30x^2 + 6x - 5$
Equating the coefficients of like terms on both sides we have

Coeff of x^4:

$e = 3$

Coeff of x^3:

$-8e + d = -23 \Rightarrow -24 + d = -23$

$\because e = 3$

$\Rightarrow d = 1$

Coeff of x^2:

$12e - 6d + c = 30$

$36 - 6 + c = 0$ or $c = 0$

Coeff of x:

$6d - 4c + b = 6$

$\Rightarrow 6 + b = 6 \qquad \Rightarrow b = 0$

Constant:

$(2c - 2b + a) = -5$ since $c = 0$ & $b = 0$

$\Rightarrow a = -5$

Substituting these values in (1)

i.e. $y = a + bx + cx^2 + dx^3 + ex^4$ we get

$y = -5 + x^3 + 3x^4$

or $y = 3x^4 + x^3 - 5$

The following problems worked in Vedic method:

1) $(y^1)^3 + (y1)^2 + 2y^1 + y = 1728x^3 + 150x^2 + 24x + 6$

Let $y = a + bx + cx^2$

$y^1 = b + 2cx \Rightarrow 2y^1 = 2b + 4cx$

$(y1)^2 = b^2 + 4c^2x^2 + 4bcx$

$(y^1)^3 = (b + 2cx)^3 = b^3 + 8c^3x^3 + 6b^2cx + 6b^2x^2c^2$

$\therefore$ Given problem is

$(b^3 + b^2 + 2b + a) + (6b^2c + 4bc + 4c + b)x + (6bc^2 + 4c^2 + c)x^2 + (8c^3)x^3$

$= 1728x^3 + 150x^2 + 24x + 6$

Equating the coefficients of like terms on both sides

Coeff x^3: $8c^3 = 1728 \Rightarrow c^3 = 1728/8 = 216$

$c = \sqrt[3]{216} = 6 \quad \therefore c = 6$

Coeff x^2: $6bc^2 + 4c^2 + c = 150$

Substituting the value of c

$(6 \times 36)b + (4 \times 36) + 6 = 150$

$216b + 144 + 6 = 150$

Or $216b = 0 \Rightarrow b = 0$

Coeff x: $6b^2c + 4bc + 4c + b = 24$

Substituting the value of b and c we get

$0 + 0 + 4(b) + 0 = 24 \Rightarrow 24 = 24$

$\Rightarrow$ x coefficient is zero

Constant: $b^3 + b^2 + 2b + a = 6$

By substituting the value of b (=0)

$a = 6$

$\therefore$ Solution is $y = a + bx + cx^2$

$\Rightarrow y = 6 + 6x^2$

2) $y^1 + 2y + 6 = 4x + 16$

Let $y = a + bx + cx^2$

$\therefore y^1 = b + 2cx$

$2y = 2a + 2bx + 2cx^2$

$\therefore$ General Equation is $(b + 2cx) + (2a + 2bx + 2cx^2) + 6 = 4x + 16$

$\Rightarrow (2a + b + 6) + (2b + 2c)\, x + 2cx^2 = 4x + 16$

Comparing the coefficients of like terms on both sides

x^2 Coeff: $2c = 0 \Rightarrow c = 0$

x Coeff: $2b + 2c = 4 \Rightarrow b = 2$

Constant: $2a + b + 6 = 16 \Rightarrow a = 4$

$\therefore$ Solution is $y = 4 + 2x$

3) $y + (y^1)^2 = 20 + 108x + 495x^2 + 936x^3 + 1521x^4$

Let $y = a + bx + cx^2 + dx^3$

$y^1 = b + 2cx + 3dx^2$

$(y^1)^2 = b^2 + 4c^2x^2 + 9d^2x^4 + 4bcx + 12cdx^3 + 6bdx^2$

$\therefore$ G E is $y^1 + (y^1)^2 = (b + b^2) + (2c + 4bc)\, x + (3d + 4c^2 + 6bd)\, x^2 + 12cdx^3 + 9d2x^4$

Comparing the coefficient of like terms on both sides.

Constant: $b^2 + b = 20$

$\therefore b = -5, 4$

Let us assume $b = 4$

x Coeff: $2c + 4bc = 108$

$\therefore c = 6$

x^2 Coeff: $3d + 4c^2 + 6bd = 495$

$\therefore d = 13$

x^3 Coeff: $12cd = 936 \Rightarrow 936 = 936$

x^4 Coeff: $9d^2 = 1521 \Rightarrow 1521 = 1521$

b = -5 will not satisfy the given equation an... ... ...s the only value for 'b'

$\therefore$ Solution is $y = 4x + 6x^2 + 13x^3$

4) $y^{11} + 2y^1 + 6y = 14 + 24x$

Let $y = a + bx + cx^2$

$y^1 = b + 2cx \Rightarrow 2y^1 = 2b + 4cx$

$y^{11} = 2c$

$\therefore$ G.E is $2c + 2b + 4cx + 6a + 6bx + 6cx^2 = 14 + 24x$

$\Rightarrow (2c + 2b + 6a) + (4c + 6b)\, x + 6cx^2 = 4 + 24x$

Comparing the coefficient of like terms on both sides.

x^2 Coeff: $6c = 0$ $[\therefore c = 0]$

x Coeff: $4c + 6b = 24$ $[\therefore b = 4]$

Constant: $2c + 2b + 6a = 14 \Rightarrow 8 + 6a = 14$ $[\therefore a = 1]$

$\therefore$ Solution is $y = 1 + 4x$

5) $2y + 3y^1 + 6y^2 = 5e^x + 6e^{2x}$

Let $y = ae^x + be^{2x}$

$2y = 2ae^x + 2be^{2x}$

$3y^1 = 3ae^x + 6be^{2x}$

$y^2 = (ae^x + be^{2x})^2 = a^2e^{2x} + b^2e^{4x} + 2abe^{3x}$

$6y^2 = 6a^2e^{2x} + 6b^2e^{4x} + 12abe^{3x}$

$\therefore$ G E is $5ae^x + (8b + 6a^2)\, e^{2x} + 12abe^{3x} + 6b^2e^{4x} = 5e^x + 6e^{2x}$

Comparing the coefficient of like terms on both sides

e^x Coeff: $5a = 5$ $[\therefore a = 1]$

e^{4x} Coeff: $6b^2 = 0 \Rightarrow b = 0$

$\therefore$ Solution is $y = e^x$

6) $\int_0^x y\,dx + 3\int_0^x y\,dx = 16x + 6x^2$

$$\Rightarrow 4\int_0^x y\,dx = 16x + 6x^2$$

Let $y = a + bx \Rightarrow 4\int_0^x y\,dx = 4(ax + \frac{b}{2}x^2)$

$\therefore 4(ax + \frac{b}{2}x^2) = 16x + 6x^2$

Comparing the coefficient of like terms on both sides.

$4a = 16 \Rightarrow a = 4$ and $4\frac{b}{2} = 6 \Rightarrow b = 3$

$\therefore$ Solution is $y = 4 + 3x$

7) $\int_0^x\int_0^x y^2\,dx + \int_0^x y\,dx = 2x + \frac{7}{2}x^2 + 2x^3 + \frac{3}{4}x^4$

Let $y = a + bx + cx^2$

Or $y^2 = a^2 + b^2x^2 + c^2x^4 + 2abx + 2bcx^3 + 2cax^2$

$$\int_0^x y^2dx = a^2x + \frac{b^2}{3}x^3 + \frac{c^2}{5}x^5 + abx^2 + \frac{bc}{2}x^4 + \frac{2}{3}cax^3$$

$$\int_0^x\int_0^x y^2\,dx = \frac{a^2}{2}x^2 + \frac{b^2}{12}x^4 + \frac{c^2}{30}x^6 + \frac{ab}{3}x^3 + \frac{bc}{10}x^5 + \frac{ca}{6}x^4 \quad \to (1)$$

$$\int_0^x y\,dx = ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 \quad \to (2)$$

General equation is (1) + (2)

$$= ax + \frac{(a^2+b)}{2}x^2 + \frac{(ab+c)}{3}x^3 + \frac{(b^2+2ca)}{12}x^4 + \frac{bc}{10}x^5 + \frac{c^2}{30}x^6$$

$$= 2x + \frac{7}{2}x^2 + 2x^3 + \frac{3}{4}x^4$$

Comparing the coefficient of like terms on both sides.

x Coeff: a = 2

x^2 Coeff: $\frac{a^2+b}{2} = \frac{7}{2} \Rightarrow b = 3$

x^3 Coeff: $\frac{ab+c}{3} = 2 \Rightarrow c = 0$

$\therefore$ The solution is $y = 2 + 3x$

8) $$y^{II} + \left(\frac{dy}{dx}\right)^2 + \int_0^x y^2\,dx = 13 + 28x + 22x^2 + \frac{17}{3}x^3 + 3x^4 + \frac{4}{5}x^5$$

Let $y = a + bx + cx^2$

$y^2 = a^2 + b^2x^2 + c^2x^4 + 2abx + 2bcx^3 + 2cax^2$

$y^I = b + 2cx \Rightarrow y^{II} = 2c \quad \to (1)$

$(y^I)^2 = b^2 + 4c^2x^2 + 4bcx \quad \to (2)$

$$\int_0^x y^2\,dx = a^2x + \frac{b^2}{3}x^3 + \frac{c^2}{5}x^5 + abx^2 + \frac{bc}{2}x^4 + \frac{2ca}{3}x^3 \quad \to (3)$$

$\therefore$ G.E. is (1) + (2) + (3)

$$\Rightarrow (2c+b^2) + (4bc+a^2)x + (4c^2+ab)x^2 + (b^2 \ldots \text{[illegible]} + \frac{bc}{2}x^4 + \frac{c^2}{5}x^5$$

$$= 1\ldots + \text{[illegible]} + \frac{17}{3}x^3 + 3x^4 + \frac{4}{5}x^5$$

Comparing the Coefficients of like terms.

x^5 Coeff: $\frac{c^2}{5} = \frac{4}{5}$ $\therefore$ c = ± 2 but c = -2 is not vali[illegible] [illegible] = 2

x^4 Coeff: $\frac{bc}{2} = 3 \Rightarrow b = 3$

x^3 Coeff: $\frac{b^2+2ca}{3} = \frac{17}{3} \Rightarrow a = 2$

$\therefore$ Solution is $y = 2 + 3x + 2x^2$

9) $(y')^2 + y'' + x\int_1^2 y\,dx = 4x^2 + \frac{43}{3}x + 6$

Let $y = a + bx + cx^2$

$$x\int_1^2 y\,dx = x\left[a(x)_1^2 + \frac{b}{2}(x^2)_1^2 + \frac{c}{3}(x^3)_1^2\right]$$

$$= \left[a + \frac{3b}{2} + \frac{7c}{3}\right]x \qquad \rightarrow (1)$$

$$(y')^2 = (b+2cx)^2 = \left(b^2 + 4c^2x^2 + 4bcx\right) \qquad \rightarrow (2)$$

and $y'' = 2c$ $\rightarrow (3)$

G.E. is (1) + (2) + (3) $\Rightarrow \left(b^2 + 2c\right) + \left(a + \frac{3b}{2} + \frac{7c}{3} + 4bc\right)x + 4c^2x^2 = 6 + \frac{43}{3}x + 4x^2$

Comparing the Coefficients of like terms

x^2 Coeff: $4c^2 = 4 \Rightarrow [c = 1,-1]$ c is taken as 1 since c = -1 is not valid

x Coeff: $\frac{6a + 9b + 14c + 24bc}{6} = \frac{43}{3}$

$\Rightarrow 6a + 9b + 14c + 24bc = 86$

Substituting c = 1

$6a + 33b = 72$

Or

$2a + 11b = 24 \rightarrow A$

Constant: $b^2 + 2c = 6$

$b^2 = 4$ or $b = 2,-2$ here b = 2 since b = -2 is not valid

Substituting b = 2 in equation A.

$2a + 22 = 24 \Rightarrow a = 1$

$\therefore$ Solution is $y = 1 + 2x + x^2$

10) $y' + \int_0^1\int_1^x y''\,dx = 9x^2 + 2x - 5$

Let $y = a + bx + cx^2 + dx^3 + ex^4$

$y' = b + 2cx + 3dx^2 + 4ex^3 \longrightarrow (1)$

$y'' = 2c + 6dx + 12ex^2$

$$\int_1^x y^{11}dx = 2c(x)_1^x + 6d\left(\frac{x^2}{2}\right)_1^x + 12e\left(\frac{x^3}{3}\right)_1^x$$

$$= -2c - 3d - 4e + 2cx + 3dx^2 + 4ex^3 \qquad \rightarrow (1)$$

$$\int_0^1\int_1^x y'' dx = -2c-3d-4e+2c\left(\frac{x^2}{2}\right)_0^1+3d\left(\frac{x^3}{3}\right)_0^1+4e\left(\frac{x^4}{4}\right)_0^1$$

$= (-c - 2d - 3e)$ $\rightarrow (2)$

G.E is (1) + (2) $\Rightarrow$ $(b - c - 2d - 3e) + 2cx + 3dx^2 + 4ex^3 = -5 + 2x + 9x^2$

Comparing the coefficient terms on both sides

x^3 Coeff: $4e = 0$ $\Rightarrow e = 0$

x^2 Coeff: $3d = 9$ $\Rightarrow d = 3$

x Coeff: $2c = 2$ $\Rightarrow c = 1$

Constant: $b - c - 2d - 3e = -5$

$\Rightarrow b = 2$

$\therefore$ Solution is $y = 2x + x^2 + 3x^3$

11) $y'' + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{x} y''' = 20x^3 - 5$

Let $y = a + bx + cx^2 + dx^3 + fx^4 + gx^5 + hx^6 + jx^7$

$Y' = b + 2cx + 3dx^2 + 4fx^3 + 5gx^4 + 6hx^5 + 7jx^6$

$Y'' = 2c + 6dx + 12fx^2 + 20gx^3 + 30hx^4 + 42jx^5$

$Y''' = 6d + 24\,fx + 60gx^2 + 120hx^3 + 210jx^4$

$Y'''' = 24f + 120gx + 360hx^2 + 840jx^3$

$$\int_{\frac{1}{2}}^{x} y''' = \left[24fx + 60gx^2 + 120hx^3 + 210jx^4\right]_{\frac{1}{2}}^{x}$$

$$= 24fx + 60gx^2 + 120hx^3 + 210jx^4 - 12f - 15g - 15h - \frac{210}{16}j$$

$$\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{x} y''' = \left[12fx^2 + 20gx^3 + 30hx^4 + 42jx^5 - 12f \qquad - \frac{105}{8}jx\right]_0^{\frac{1}{2}}$$

$$= 3f + \frac{5}{2}g + \frac{15}{8}h + \frac{21}{16}j - 6f - \frac{15}{2}g - \frac{15}{2}h - \frac{105}{16}j$$

$$= \left(-3f - 5g - \frac{45}{8}h - \frac{84}{16}j\right)$$

$\therefore$ General Equation

$$\Rightarrow \left[2c - 3f - 5g - \frac{45}{8}h - \frac{84}{16}j\right] + 6dx + 12fx^2 + 20gx^3 + 30hx^4 + 42jx^5$$

Equating the coefficients of like terms on both sides

$x^5 : 42j = 0 \qquad j = 0$
$x^4 : 30h = 0 \qquad h = 0$
$x^3 : 20g = 20 \qquad g = 1$
$x^2 : 12f = 0 \qquad f = 0$
$x : 6d = 0 \qquad d = 0$

Constant: $2c - 3f - 5g - \frac{45}{h} - \frac{84}{16} j = -5$

$\Rightarrow 2c - 5g = -5 \quad \text{or } c = 0$

P. I $= x^5$

PARTIAL DIFFERENTIAL EQUATION

Linear Differential Equation:

A differential equation is said to be linear if

(1) Every dependent variable and every derivative involved occurs in the first degree.

(2) No products of dependent variables and / or derivatives occurred.

A differential equation that is not linear as per the above is said to be non linear.

An Equation involving partial differential coefficients of function of two or more variables is known as Partial Differential Equation. Such equations are said to be linear if it is of 1^{st} degree in the dependent variable and its partial derivatives (i. e powers or products of the dependent variable and its partial derivatives must be absent). A differential equation, which is not linear, is called non linear differential equations.

Consider the given problem (**Example 1**): $u_x3 + u_y2 + u_{xy} = 12$

To solve for 'u 'as a function of (x, y)

If boundary conditions (B. C) are given, they first to be consider. For example,

In this case (B. C)

Boundary Conditions are:

$U_{(0,0)} = 4$, $U_x(0, 0) = 3$, $u_x2(0, 0) = 4$

$u_x3(0,0) = 6$, $u_x4(0,0)$ ------------------- $= 0$

$U_y(0, 0) = 2$, $u_y2(0, 0) = 2$, $u_y3(0, 0)$ ------------ $= 0$

$U_{xy}(0, 0) = 4$, $u_x2_y(0, 0)$ ------------ $= 0$

The solution U is written in the general form of ... on of x and y in power series as given below

$$
\begin{aligned}
U = \; & a_0 + b_0x + c_0x^2 + \ldots + e_0x^4 + \ldots \\
& + a_1y + b_1xy + c_1x^2y + d_1x^3y + e_1x^4y + \ldots \\
& + a_2y^2 + b_2xy^2 + c_2x^2y^2 + d_2x^3y^2 + e_2x^4y^2 + \ldots \\
& + a_3y^3 + b_3xy^3 + c_3x^2y^3 + d_3x^3y^3 + e_3x^4y^3 + \ldots \\
& + a_4y^4 + b_4xy^4 + c_4x^2y^4 + d_4x^3y^4 + e_4x^4y^4 + \ldots
\end{aligned}
$$

LHS $u_x 3 + u_y 3 + u_{xy}$

x^0 Coefficient $\Rightarrow 6d_0 + 2a_2 + b_1 = 12$

$6 + 2 + 4 = 12$

x^1 Coefficient $\Rightarrow 24e_0 + 2b_2 + 2c_1 = 0$

$\Rightarrow b2 = 0$

x^2 Coefficient $\Rightarrow 2c_2 + 3c_1 = 0$

$\Rightarrow c_2 = 0$

Similarly $d_2 = 0$, $e_2 = 0$

xy Coefficient $\Rightarrow 24e_1 + 6b_3 + 2c_1 = 0$

x^2y Coefficient $\Rightarrow 6c_3 + 3d_1 = 0$

$\Rightarrow c3 = 0$

Similarly $d_3 = 0$, $e_3 = 0$

y^2 Coefficient $\Rightarrow 6d_2 + 24a_4 + 3b_3 = 0$

$\Rightarrow b3 = 0$

y^3 Coefficient $\Rightarrow 6d3 + b4 = 0$

Similarly $c4 = 0$, $d4 = 0$

U =	4		3		2		1		0	
	a_0	+	b_0x	+	c_0x^2	+	d_0x^3	+	e_0x^4	+ ...
	2		4		0		0		0	
+	a_1y	+	b_1xy	+	c_1x^2y	+	d_1x^3y	+	e_1x^4y	+ ...
	1		0		0		0		0	
+	a_2y^2	+	b_2xy^2	+	$c_2x^2y^2$	+	$d_2x^3y^2$	+	$e_2x^4y^2$	+ ...
	0		0		0		0		0	
+	a_3y^3	+	b_3xy^3	+	$c_3x^2y^3$	+	$d_3x^3y^3$	+	$e_3x^4y^3$	+ ...
	0		0		0				0	
+	a_4y^4	+	b_4xy^4	+	$c_4x^2y^4$	+	$d_4x^3y^4$	+	$c_4x^4y^4$	+ ...

$U = 4 + 3x + 2x^2 + x^3 + 2y + 4xy + y^2$

The coefficients a_0, b_0 .... a_1, b_1........c_0,c_1

Are to be evaluated with the help of Boundary Conditions and using the method of argumentation.

Now by substitutions the Boundary Condition U (0, 0) = 4 in to the general expansion of U. We get $a_0 = 4$ for obtaining the remaining coefficients, one has to sort out the differentiation. Contributions from the various terms pertain to the different partial differentiated terms given in the problem. In addition to this one has to also include the Boundary Conditions appropriately getting the value of the coefficients.

A few working details one elaborated .The LHS of the given example is

$U_x3 + U_y3 + U_{xy}$

In order to evaluate the coefficients of x0 one considers the corresponding differentials of the terms in the expansion for U for example x_0 coefficient from U_{x^3} gives only 6 d_0 the x_0 coefficient from U_{y^2} is $2_{a2} \cdot {}_{a0}$ Coefficient from u_{xy} is b, The sum is equal to 12(RHS) all the remaining coefficients in the given expansion can be similarly obtain by working out the corresponding contributions and applying the argumentation .The full details are shown completely to get the final result.

Example (2):

$u_{x^2} + u_{y^2} = 16$

$U_{(0,0)} = 8$, $U_y\,(0,0) = 6$, $u_y2\,(0,0) = 4$

$u_y3(0,0)$ ---- $= 0$, $U_{xy}\,(0,0) = 3$, $u_x2_y\,(0,0)$ -------------- $= 0$

$U_x\,(0,0) = 4$, $u_x2\,(0,0) = 12$, $U_x3(0,0)$ -------------- $= 0$

U =	8		4		6		0		0
	a_0	+	b_0x	+	c_0x^2	+	d_0x^3	+	e_0x^4
	6		3		0		0		0
+	a_1y	+	b_1xy	+	c_1x^2y	+	d_1x^3y	+	e_1x
	2		0		0		0		0
+	a_2y^2	+	b_2xy^2	+	$c_2x^2y^2$	+	d		$_2x^4y^2$
	0		0		0		0		0
+	a_3y^3	+	b_3xy^3	+	$c_3x^2y^3$	+	$d_3x^3y^3$	+	$e_3x^4y^3$
	0		0		0				
+	a_4y^4	+	b_4xy^4	+	$c_4x^2y^4$	+	$d_4x^3y^4$	+	$e_4x^4y^4$

L.H.S: $U_x^2 + U_y^2$

x^0 Coefficient $\Rightarrow 2c_0 + 2a_2 = 12 + 4 = 16$

x^1 Coefficient $\Rightarrow 6d_0 + 2b_2 = 0$

x^2 Coefficient $\Rightarrow 24e_0 + 2c_2 = 0 \Rightarrow c_2 = 0$

x^3 Coefficient $\Rightarrow 2d_2 = 0 \Rightarrow d_2 = 0$

Similarly $e_2 = 0$

xy Coefficient $\Rightarrow 6d_1 + 6b_3 = 0 \Rightarrow b_3 = 0$

x^2y Coefficient $\Rightarrow 12e_1 + 12c_3 = 0 \Rightarrow c_3 = 0$

Similarly $d_3 = e_3 = 0$---------------- $= 0$

xy^2 Coefficient $\Rightarrow 6d_2 + 12b_4 = 0 \Rightarrow b_4 = 0$

x^2y^2 Coefficient $\Rightarrow 12e_2 + 12c_4 = 0 \Rightarrow c_4 = 0$

Similarly $d_4 = e_4 =$ ---------------- $= 0$

$$\therefore U = 8 + 4x + 6x^2 + 6y + 3xy + 2y^2$$

Example: (3)

$U_x - 6U^{1/2} = 0$

Boundary conditions are:

$U_x^2(0, 0) = 18, U(0, 0) = U_x(0, 0), U_x^3(0, 0) =$ -------- 0

$U_y^2(0, 0) = 32, U_y = (0, 0), U_y3$ -------$= 0$

$U_{xy}(0, 0) = 24, U_{x^2y}(0, 0) = 0, U_{x^3y}$------$= 0$

$$U = \overset{0}{a_0} + \overset{0}{b_0 x} + \overset{9}{c_0 x^2} + \overset{0}{d_0 x^3} + \dots$$

$$+ \overset{0}{a_1 y} + \overset{24}{b_1 xy} + \overset{0}{c_1 x^2 y} + \dots$$

$$+ \overset{16}{a_2 y^2} + b_2 xy^2 + c_2 x^2 y^2 + d_2 x^3 y^2 + \dots$$

$$+ \overset{0}{a_3 y^3} + b_3 xy^3 + c_3 x^2 y^3 + d_3 x^3 y^3 + \dots$$

$$\therefore U = 9x^2 + 24xy + 16y^2$$

$$
\begin{array}{lllll}
U^{1/2} = & 0 & 3 & 0 & 0 \\
& A_0 \quad + & B_0x \quad + & C_0x^2 \quad + & D_0x^3 \quad +\ldots \\
& 4 & & & \\
& + A_1y \quad + & B_1xy \quad + & C_1x^2y + & D_1x^3y + \ldots \\
& + A_2y^2 \quad + & B_2xy^2 + & C_2x^2y^2 + & D_2x^3y^2 + \ldots \\
& + A_3\,y^3 \quad + & B_3x\,y^3 + & C_3x^2\,y^3 + & D_3x^3\,y^3 + \ldots
\end{array}
$$

Example: (4)

$y\,U_x^{\,2} + xU_Y^{\,2} + U_{xy} - 2U^{1/2} = 0$

Boundary Conditions:

$U(0, 0) = 1 \Rightarrow a_0 = 1$, $U_x(0, 0) = 2 \Rightarrow b_0 = 2$

$U_x^{\,2}(0, 0) = 2 \Rightarrow c_0 = 1$, $U_x^{\,3}(0, 0) = 0 \Rightarrow d_0 = 0$

$U_y(0, 0) = 2 \Rightarrow a_1 = 2$, $U_y^{\,2}(0, 0) = 2 \Rightarrow a_2 = 1$

$U_y^{\,3}(0, 0) = 0 \Rightarrow a_3 = 0$, $U_{XY}(0, 0) = 2 \Rightarrow b_1 = 2$

$$
\begin{array}{lllll}
U = & 1 & 2 & 1 & 0 \\
& a_0 \quad + & b_0x \quad + & c_0x^2 \quad + & d_0x^3 \quad + \ldots \\
& 2 & 2 & 0 & 0 \\
& + a_1y \quad + & b_1xy \quad + & c_1x^2y \quad + & d_1x^3 \\
& 1 & 0 & 0 & \\
& + a_2y^2 \quad + & b_2xy^2 + & c_2x^2y^2 + & d_2xy + \ldots \\
& 0 & 0 & 0 & 0 \\
& + a_3\,y^3 \quad + & b_3xy^3 + & c_3x^2y^3 + & d_3x^3y^3 + \ldots
\end{array}
$$

$\therefore\ U = x^2+2x+y^2+2y+2xy+1$

The given relation can be verified as $2y+2x+2-(2x+2y+2)=0$

$$U^{1/2} = \overset{1}{A_0} + \overset{1}{B_0 x} + \overset{0}{C_0 x^2} + \overset{0}{D_0 x^3} + \ldots$$

$$+ \overset{1}{A_1 y} + B_1 xy + C_1 x^2 y + D_1 x^3 y + \ldots$$

$$+ A_2 y^2 + B_2 xy^2 + C_2 x^2 y^2 + D_2 x^3 y^2 + \ldots$$

$$+ A_3 y^3 + B_3 x y^3 + C_3 x^2 y^3 + D_3 x^3 y^3 + \ldots$$

$$\therefore U^{1/2} = (x+y+1)$$

Example (5):

$U^{1/2}+2xU_y=16x^2+26x+16xy+2y+3$

$U^{1/2} = 2x+2y+3$

$U(0, 0) = 9 \Rightarrow a_0 = 9,\ U_x(0, 0) = 12 \Rightarrow b_0 = 12$

$U_x^{\,2} = 8 \Rightarrow c_0 = 4,\ U_x^{\,3} = 0 \Rightarrow d_0 = 0$

$U_y = 12 \Rightarrow a_1 = 2,\ U_y^{\,2} = 8 \Rightarrow a_2 = 4$

$U_y^{\,3} = 0 \Rightarrow a_3 = 0,\ U_{XY} = 8 \Rightarrow b_1 = 8$

$Ux^2y = 0 \Rightarrow c_1 = 0$

$$U^{1/2} + 2xU_y = 16x^2 + 26x + 16xy + 2y + 3$$

$$U = \overset{9}{a_0} + \overset{12}{b_0 x} + \overset{4}{c_0 x^2} + \overset{0}{d_0 x^3} + \overset{0}{e_0 x} + \ldots$$

$$+ \overset{12}{a_1 y} + \overset{8}{b_1 xy} + \overset{0}{c_1 x^2 y} + \overset{0}{} \quad \overset{0}{a_1 x^4 y} + \ldots$$

$$+ \overset{4}{a_2 y^2} + \overset{0}{b_2 xy^2} + \overset{0}{c_2 x^2 y^2} + d_2 x^3 y^2 + e_2 x^4 y^2 + \ldots$$

$$+ \overset{0}{a_3 y^3} + b_3 x y^3 + c_3 x^2 y^3 + d_3 x^3 y^3 + e_3 x^4 y^3 + \ldots$$

$$U = 4x^2+12x+4y^2+12y+8xy+9$$

$$
\begin{array}{lccccccccc}
U^{1/2} = & \overset{3}{A_0} & + & \overset{2}{B_0x} & + & \overset{0}{C_0x^2} & + & \overset{0}{D_0x^3} & + & \overset{0}{E_0x^4}\ldots \\
& + \overset{2}{A_1y} & + & \overset{0}{B_1xy} & + & \overset{0}{C_1x^2y} & + & \overset{0}{D_1x^3y} & + & \overset{0}{E_1x^4y}\ldots \\
& + A_2y^2 & + & B_2xy^2 & + & C_2x^2y^2 & + & D_2x^3y^2 & + & E_2x^4y^2\ldots \\
& + A_3y^3 & + & B_3xy^3 & + & C_3x^2y^3 & + & D_3x^3y^3 & + & E_3x^4y^3\ldots
\end{array}
$$

$U(0, 0) = 9$, $U_X = 12$, $U_x^2 = 8$, $U_x^3 = 0$, $U_y = 12$

$U^{1/2}$

$a_0 = 9 \Rightarrow A_0 = \pm 3$ 1st case with $A_0 = {}_{+}3$

$B_0 = \dfrac{b_0}{2A_0} = \dfrac{12}{6} = 2$

$C_0 = \dfrac{c_0 - B_0^2}{2A_0} = \dfrac{4-4}{6} = 0$

$D_0 = \dfrac{d_0 - 2B_0C_0}{2A_0} = 0$

$A_1 = \dfrac{a_1}{2A_0} = \dfrac{12}{6} = 2$

$B_1 = \dfrac{b_1 - 2A_1B_0}{2A_0} = 0$

$C_1 = \dfrac{c_1 - (2A_1C_0 + 2B_1B_0)}{2A_0} = 0$

Similarly $D_1 = E_1 = 0$

$U^{\frac{1}{2}} = (3 + 2x + 2y), U = (3 + 2x + 2y)^2 = 9 + 4x^2 + 4y^2 + 12x + 12v + 8xy$

2^{nd} row

Coefficient	$U^{1/2}$	$2x.U_y$	R.H.S	
1	3	-		
x	2	$2a_1$	26	$a_1 = 12$
x^2	0	$2b_1$	16	$b_1 = 8$
x^3	0	$2c_1$	0	$c_1 = 0$
x^4	0	$2d_1$	0	$d_1 = 0$

Similarly $e_1 = 0$

3^{rd} row

Coefficient	$U^{1/2}$	$2x.U_y$	R.H.S	
xy	0	$4a_2$	16	$a_2 = 4$
x^2y	0	$4b_2$	0	$b_2 = 0$
x^3y	0	$4c_2$	0	$c_2 = 0$

One need not evaluate beyond this.

2^{nd} Case with $A_0 = -3$

9	12	4	0	0	
a_0 +	b_0x +	c_0x^2 +	d_0x^3 +	e_0x^4 +	…
12	8	0	0	0	
+ a_1y +	b_1xy +	c_1x^2y +	d_1x^3y +	e_1x^4y +	…
4					
+ a_2y^2 +	b_2xy^2 +	$c_2x^2y^2$ +	$d_2x^3y^2$ +	$e_2x^4y^2$ +	…
-3	-2	0	0	0	
A_0 +	B_0x +	C_0x^2 +	D_0x^3 +	E_0x4…	
-2	0	0			
+ A_1y +	B_1xy +	C_1x^2y +	D_1x^3y +	E_1x4y…	
+ A_2y^2 +	B_2xy^2 +	$C_2x^2y^2$ +	$D_2x^3y^2$ +	E_2x4y2…	

$A_0 = -3$

$B_0 = \frac{b_0}{2A_0} = \frac{12}{-6} = -2$

$C_0 = \frac{c_0 - {B_0}^2}{2A_0} = \frac{4-4}{-6} = 0$

$D_0 = \frac{d_0 - 2B_0C_0}{2A_0} = 0$

$A_1 = \frac{a_1}{2A_0} = \frac{12}{-6} = -2$

$B_1 = \frac{b_1 - 2A_1B_0}{2A_0} = 0$

$C_1 = \frac{c_1 - (2A_1C_0 + 2B_1B_0)}{2A_0} = 0$

Similarly $D_1 = E_1 = 0$

Considering $A_0 = +3$, $U^{1/2} = (3+2x+2y)$ and with $A_0 = -3$, $U^{1/2} = -(3+2x+2y)$

Hence U is perfect square and $U = 9 + 12x + 4x^2 + 12y + 8xy + 4y^2$.

Example (6):

$u^{\frac{1}{2}} + u_y = 1 + 2x + xy + 2x^2y$

Boundary conditions: $U(0,0) = 1, U_X(0,0)=1$, $U_X^2\,(0,0) = U_X^3(0,0) = 0$

…………..=0

U =

$$\boxed{1} \quad \boxed{0} \quad \boxed{0} \quad \boxed{0} \quad \boxed{0}$$

$$a_0 + b_0x + c_0x^2 + d_0x^3 + e_0x^4 + \text{--------}$$

$$\boxed{2} \quad \boxed{2} \quad \boxed{0} \quad \boxed{0} \quad \boxed{0}$$

$$+ a_1y + b_1xy + c_1x^2y + d_1x^3y + e_1x^4y + \text{--------}$$

$$\boxed{1/2} \quad \boxed{1} \quad \boxed{1} \quad \boxed{0} \quad \boxed{0}$$

$$+ a_2y^2 + b_2xy^2 + c_2x^2y^2 + d_2x^3y^2 + e_2x^4y^2 + \text{-------}$$

$$\boxed{-1/12} \quad \boxed{-1/6} \quad \boxed{0} \quad \boxed{0} \quad \boxed{0}$$

$$+ a_3y^3 + b_3xy^3 + c_3x^2y^3 + d_3x^3y^3 + e_3x^4y^3 + \text{-------}$$

First Row:

$U^{\frac{1}{2}} =$

$$\boxed{-1} \quad \boxed{0} \quad \boxed{0} \quad \boxed{0} \quad \boxed{0}$$

$$A_0 + B_0x + C_0x^2 + D_0x^3 + E_0x^4 + \text{-------}$$

$$\boxed{-1} \quad \boxed{-1} \quad \boxed{0} \quad \boxed{0} \quad \boxed{0}$$

$$+ A_1y + B_1xy + C_1x^2y + D_1x^3y + E_1y' \quad \text{--}$$

$$\boxed{1/4} \quad \boxed{1/2} \quad \boxed{0} \quad \boxed{0} \quad \boxed{0}$$

$$+ A_2y^2 + B_2xy^2 + C_2x^2y^2 + D_2x^3y^2 + E_2x^4y^2 \quad \text{-----}$$

$$\boxed{} \quad \boxed{} \quad \boxed{} \quad \boxed{} \quad \boxed{}$$

$$+ A_3y^3 + B_3xy^3 + C_3x^2y^3 + D_3x^3y^3 + E_3x^4y^3 + \text{-------}$$

$$B_0 = \frac{b_0}{A_0}, C_0 = \frac{c_0}{A_0}, D_0 = \frac{d_0}{A_0}, E_0 = \frac{e_0}{A_0}$$

$$U^{\frac{1}{2}} + U_Y = 1 + 2x + xy + 2x^2 y$$

Second row of U

Here we successively equating coefficients of $1, x, x^2, x^3$ ____ etc. it is convenient to put down the contributions of the terms $u^{\frac{1}{2}}$, u_Y and the RHS of equation (1) in a table.

Coefficient of	$u^{\frac{1}{2}}$	u_Y	RHS of Equation (1)	Coefficient evaluation from equation (1)
1	-1	a_1	1	$-1 + a_1 = 1 \Rightarrow a_1 = 2$
x	0	b_1	2	$b_1 = 2$
x^2	0	c_1	0	$c_1 = 0$
x^3	0	d_1	0	$d_1 = 0$
x^4	0	e_1	0	$e_1 = 0$

$$A_1 = \frac{a_1}{2A_0} = \frac{2}{-2} = -1$$

$$B_1 = \frac{b_1 - 2A_1B_0}{2A_0} = \frac{2-0}{-2} = -1$$

$$C_1 = \frac{(C_1 - 2A_1C_0 - 2B_1B_0)}{2A_0} = \frac{0-0}{-2} = 0$$

$$D_1 = \frac{d_1 - (2B_0C_1 + 2B_1C_0 + 2D_0A_1)}{2A_0} = 0$$

$$E_1 = \frac{e_1 - (2B_0D_1 + 2C_0C_2 + 2D_0B_2 + 2E_0A_1)}{2A_0} = 0$$

Third row of U

Coefficient of	$u^{\frac{1}{2}}$	u_y	RHS of Equation (1)	Coefficient evaluation from equation (1)
y	-1	2a	0	$2a_2 = 1 \Rightarrow a_2 = 1/2$
xy	-1	$2b_2$	1	$2b_2 = 2 \Rightarrow b_2 = 1$
x^2y	0	$2c_2$	2	$2c_2 = 2 \Rightarrow c_2 = 1$
x^3y	0	$2d_2$	0	$d_2 = 0$
x^4y	0	$2e_2$	0	$e_2 = 0$

$$A_2 = \frac{a_2 - A_1^2}{2A_0} = \frac{\frac{1}{2} - 1}{-2} = \frac{-\frac{1}{2}}{-2} = +\frac{1}{4}$$

$$B_2 = \frac{(b_2 - 2A_2B_0 - 2A_1B_1)}{2A_0} = \frac{1-0-2}{-2} = +\frac{1}{2}$$

$$C_2 = \frac{(c_2 - 2A_1C_1 - 2A_2C_0 - 2B_2B_0 - B_1^2)}{2A_0} = \frac{1-0-0-0-1}{-2} = 0$$

$$D_2 = \frac{d_2 - (2B_0C_2 + 2C_0B_2 + 2D_0A_2 + 2A_1D_1 + 2B_1C_1)}{2A_0} = \frac{0-(0)}{-2} = 0$$

$$E_2 = \frac{e_2 - (2B_0D_2 + 2C_0C_2 + 2D_0B_2 + 2E_0A_2 + 2A_1E_1 + 2B_1D_1 + 2C_1^2)}{2A_0} = \frac{0-0}{-2} = 0$$

Fourth row of u

Coefficient of	$u^{\frac{1}{2}}$	u_Y	RHS of Equation (1)	Coefficient evaluation from equation (1)
y^2	$\frac{1}{4}$	$3a_3$	0	$a_3 = -\frac{1}{12}$
xy^2	$\frac{1}{2}$	$3b_3$	0	$b_3 = -\frac{1}{6}$
x^2y^2	0	$3c_3$	0	$c_3 = 0$
x^3y^2	0	$3d_3$	0	$d_3 = 0$
x^4y^2	0	$3e_3$	0	$e_3 = 0$

2nd solution of u is

$$1 + 2y + 2xy + \frac{1}{2}y^2 + xy^2 + x^2y^2 - \frac{1}{12}y^3 - \frac{1}{6}xy^3 + \text{--------}$$

Example (7):

$$U^{\frac{1}{2}} + U_y = 1 + 2x + xy + 2x^2y$$

This is the example given in B.A. book applying B.C. to the 1st row only. Altered problem by changing the relation (given problem) by applying the new B.C. as

$$U = U^{\frac{1}{2}} + U_y = 3 + 3x + xy + 2x^2y + x^3$$

Boundary conditions

$U(0, 0) = 1, U_x(0, 0) = U_x{}^2(0, 0) = U_x{}^3(0, 0) =$

$U_y(0, 0) = 2$, $U_{(xy)}(0, 0) = 3$, $U_{(x^3y)}(0, 0) = 6$

U as per B.C $\quad a_0=1$ $a_1=2$ $b_1=3$ $d_1=1$

$U =$

1	0	0	0	0	0
a_0 +	b_0x +	c_0x^2 +	d_0x^3 +	e_0x^4 +	f_0x^5
2	3	0	1	0	0
+ a_1y +	b_1xy +	c_1x^2y +	d_1x^3y +	e_1x^4y +	f_1x^5y
$\frac{-1}{2}$	$\frac{-1}{4}$	1	$\frac{-1}{4}$	0	0
+ a_2y^2 +	b_2xy^2 +	$c_2x^2y^2$ +	$d_2x^3y^2$ +	$e_2x^4y^2$ +	$f_2\,x^5y^2$
$\frac{1}{4}$	$\frac{13}{24}$	$\frac{5}{24}$	$\frac{5}{24}$	$\frac{1}{4}$	0
+ a_3y^3 +	b_3xy^3 +	$c_3x^2y^3$ +	$d_3x^3y^3$ +	$e_3x^4y^3$ +	$f_3x^5y^3$
$\frac{-1}{32}$	$\frac{-145}{192}$	$\frac{-19}{24}$	$\frac{-17}{48}$	$\frac{-21}{32}$	$\frac{-23}{64}$
+ a_4y^4 +	b_4xy^4 +	$c_4x^2y^4$ +	$d_4x^3y^4$ +	$e_4x^4y^4$ +	$f_4x^5y^4$

$U^{1/2} =$

1	0	0	0	0	0
A_0 +	B_0x +	C_0x^2 +	D_0x^3 +	E_0x^4 +	F_0x^5
1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	0
+ A_1y +	B_1xy +	C_1x^2y +	D_1x^3y +	E_1x^4y +	F_1x^5y
$\frac{-3}{4}$	$\frac{-13}{8}$	$\frac{-5}{8}$	$\frac{-5}{8}$	$\frac{-3}{4}$	0
+ A_2y^2 +	B_2xy^2 +	$c_2x^2y^2$ +	D_2x^3	$E_2x^4y^2$ +	$F_2\,x^5y^2$
$\frac{1}{8}$	$\frac{145}{48}$	$\frac{19}{6}$	$\frac{17}{12}$	$\frac{21}{8}$	$\frac{23}{16}$
+ A_3y^3 +	B_3xy^3 +	$C_3x^2y^3$	+$D_3x^3y^3$	+$E_3x^4y^3$	+$F_3x^5y^3$
+ A_4y^4 +	B_4xy^4 +	$C_4x^2y^4$+	$D_4x\,y$ +	$E_4x^4y^4$+	$F_4x^5y^4$

1^{st} row of U

$a_0 = 1$

$b_0 = c_0 = d_0 = e_0 =$ --------------- $= 0$

1^{st} row of $U^{\frac{1}{2}}$

$A_0 = \pm a_0 \Rightarrow A_0 = \pm 1$

Let $A_0 = 1$

2^{nd} row of U

Coefficient of	$U^{\frac{1}{2}}$	U_y	R.H.S	Coefficient evaluated
1	1	$a_1 = 2$	3	$a_1 = 2$
X	0	$b_1 = 3$	3	$b_1 = 3$
x^2	0	c_1	0	$c_1 = 0$
x^3	0	$d_1 = 1$	1	$d_1 = 1$
x^4	0	e_1	0	$e_1 = 0$

2^{nd} row of $U^{\frac{1}{2}}$

$$A_1 = \frac{a_1}{2A_0} = 1$$

$$B_1 = \frac{B_1 - (2A_1 B_0)}{2A_0} = \frac{3}{2}$$

$$C_1 = \frac{c_1 - (2B_1B_0 + 2A_1C_0)}{2A_0} = 0$$

$$D_1 = \frac{d_1 - (2A_1D_0 + 2B_1C_0 + 2B_0C_1)}{2A_0} = [illegible]$$

$$E_1 = \frac{e_1 - (2A_1E_0 + 2B_1D_0 + 2C_0C_1) + 2B_0D_1}{2A_0} \quad 0$$

Similarly $F_1 = 0$

3^{rd} row of U

Coefficient of	$U^{\frac{1}{2}}$	U_y	R.H.S	Coefficient evaluated
Y	1	$2a_2$	0	$a_2 = \frac{-1}{2}$
xy	$\frac{3}{2}$	$2b_2$	1	$b_2 = \frac{-1}{4}$
X^2y	0	$2c_2$	2	$c_2 = 1$
X^3y	$\frac{1}{2}$	$2d_2$	0	$d_2 = \frac{-1}{4}$
X^4y	0	$2e_2$	0	$e_2 = 0$

3^{rd} row of $U^{\frac{1}{2}}$

$$A_2 = \frac{A_2 - A_1^2}{2A_0} = \frac{-3}{4}$$

$$B_2 = \frac{b_2 - (2B_0A_2 + 2A_1B_1)}{2A_0} = \frac{-13}{8}$$

$$C_2 = \frac{C_2 - (2B_0B_2 + 2C_0A_2 + 2A_1C_1 + B_1^2)}{2A_0} = \frac{-5}{8}$$

$$D_2 = \frac{d_2 - (2B_0C_2 + 2B_2C_0 + 2D_0A_2 + 2A_1D_1 + 2B_1C_1)}{2A_0} = \frac{-5}{8}$$

$$E_2 = \frac{e_2 - (2B_0D_2 + 2C_0C_2 + 2D_0B_2 + 2E_0A_2 + 2B_1D_1 + 2A_1E_0 + C_1^2)}{2A_0} = \frac{0 - (3/2)}{2}$$

$$= \frac{-3}{4}$$

$$F_2 = \frac{f2 - (2B_0E_2 + C_0D_2 + 2D_0C_2 + 2E_0B_2 + 2F_0A_2 + 2A_1F_1 + B_1E_0 + 2C_1D_0)}{2A_0}$$

$$= 0$$

4^{th} row of U

Coefficient of	$U^{\frac{1}{2}}$	U	R.H.S	Coefficient evaluated
y^2	$\frac{-3}{4}$	$3a_3$	0	$a_3 = \frac{3}{12} = \frac{1}{4}$
xy^2	$\frac{-13}{8}$	$3b_3$	0	$b_3 = \frac{13}{24}$
x^2y^2	$\frac{-5}{8}$	$3c_3$	0	$c_3 = \frac{5}{24}$
x^3y^2	$\frac{-5}{8}$	$3d_3$	0	$d_3 = \frac{5}{24}$
x^4y^2	$\frac{-3}{4}$	$3e_3$	0	$e_3 = \frac{3}{12}$

4^{th} row of $U^{\frac{1}{2}}$

$$A_3 = \frac{a_3}{2A_0} = \frac{1/4}{2} = \frac{1}{8}$$

$$B_3 = \frac{b_3 - (2B_0A_3 + 2A_1B_2 + 2B_1A_2)}{2A_0}$$

$$= \frac{\frac{13}{24} - (\frac{-13}{4} + \frac{-9}{4})}{2A_0} = \frac{145}{} = \frac{145}{48}$$

$$C_3 = \frac{C_3 - (2B_0B_3 + 2C_0A_3 + 2A_1C_2 + 2B_1B_2 + 2C_1A_2)}{2A_0}$$

$$= \frac{\frac{5}{24} - (0 + 0\frac{-5}{4} + \frac{-39}{8} + 0)}{2A_0}$$

$$= \frac{\frac{5}{24} - \left(\frac{-49}{8}\right)}{2A_0}$$

$$= \frac{152}{48} = \frac{76}{24} = \frac{19}{6}$$

$$D_3 = \frac{d_3 - (2B_0C_3 + 2C_0B_3 + 2D_0A_3 + 2A_1D_2 + 2B_1C_2 + 2C_1B_2 + 2D_1A_2)}{2A_0}$$

$$= \frac{\frac{5}{29} - \left(0+0+0-\frac{5}{4} + \frac{-15}{8} + 0 + \frac{-3}{4}\right)}{2}$$

$$= \frac{\frac{5}{24} - \left(\frac{-31}{8}\right)}{2} = \frac{68}{48} = \frac{34}{24} = \frac{17}{12}$$

$$E_3 = \frac{e_3 - (2B_0D_3 + 2C_0C_3 + 2D_0B_3 + 2E_0A_3 + 2A_1E_2 + 2B_1D_2 + 2C_1C_2 + 2D_1B_2 + 2E_1A_2)}{2A_0}$$

$$= \frac{\frac{1}{4} - \left(0 + 0 + 0 + 0 - \frac{3}{2} + \frac{-15}{8} + 0 + \frac{-13}{8} + 0\right)}{2}$$

$$= \frac{\frac{1}{4} + \left(\frac{40}{8}\right)}{2}$$

$$= \frac{\frac{1}{4} + 5}{2} = \frac{21}{8}$$

$$F_3 = \frac{F_3 - (2B_0E_3+2C_0D_3+2D_0C_3+2E_0B_3+2A_1F_2+2B_1E_2+2C_1D_2+2D_1C_2+2E_1B_2+2F_{1.}}{2A_0}$$

$$= \frac{0-(0+0+0+0+0+\frac{-9}{4}+\frac{-5}{8})}{2} = \frac{23}{16}$$

5th row of U

Coefficient	$U^{\frac{1}{2}}$	U	R H S	Coefficient evaluation
y^3	$\frac{1}{8}$	$4a_4$	0	$a_4 = \frac{-1}{32}$
xy^3	$\frac{145}{48}$	$4b_4$	0	$b_4 = \frac{-145}{192}$
x^2y^3	$\frac{19}{6}$	$4c_4$	0	$c_4 = \frac{-19}{24}$
x^3y^3	$\frac{17}{12}$	$4d_4$	0	$d_4 = \frac{-17}{48}$
x^4y^3	$\frac{21}{8}$	$4e_4$	0	$e_4 = \frac{-21}{32}$
x^5y^3	$\frac{23}{16}$	$4f_4$	0	$f_4 = \frac{-23}{64}$

$\therefore U = 1 + 2y + 3xy + x^3y + \frac{-1}{2}y^2 - \frac{1}{4}xy^2 + x^2y^2 + \frac{-1}{4}x^3y^2 + \frac{1}{4}y^3 + \frac{13}{24}xy^3 +$
$\frac{5}{24}x^2y^3 + \frac{5}{24}x^3y^3 + \frac{1}{4}x^4y^3 - \frac{1}{32}y^4 - \frac{145}{192}xy^4 - {}^{10} \quad {}^{7}x^3y^4 - \frac{21}{32}x^4y^4 - \frac{23}{64}x^5y^4 +$
....

Example (8):

$-U^{\frac{1}{3}} + U_y = 1 + 2x + xy + 2x^2y$

Boundary conditions:

$U(0,0) = 1,\ U_x(0,0) = 0 \ldots\ldots\ldots$

$U(0,0) = 1 \Rightarrow a_0 = 1$

$U =$

1	0	0	0
$+\ a_0$ +	b_0x +	c_0x^2 +	d_0x^3
0	2	0	0
$+\ a_1y$ +	b_1xy +	c_1x^2y +	d_1x^3y
0	$\frac{1}{6}$	1	0
$+\ a_2y^2$ +	b_2xy^2 +	$c_2x^2y^2$ +	$d_2x^3y^2$
0	$\frac{-1}{54}$	$\frac{-1}{9}$	0
$+\ a_3y^3$ +	b_3xy^3 +	$c_3x^2y^3$ +	$d_3x^3y^3$

First row:

$U^{\frac{1}{3}} =$

1	0	0	0	0
A_0 +	B_0x +	C_0x^2 +	D_0x^3 +	E_0x^4
0	$\frac{2}{3}$	0	0	0
$+\ A_1y$ +	B_1xy +	C_1x^2y +	D_1x^3y +	E_1x^4y
0	$\frac{1}{18}$	$\frac{1}{3}$	0	0
$+\ A_2y^2$ +	B_2xy^2 +	$C_2x^2y^2$ +	$D_2x^3y^2$ +	$E_2x^4y^2$
$+\ A_3y^3$ +	B_3xy^3 +	$C_3x^2y^2$ +	$D_3x^3y^3$ +	$E_3x^4y^3$

Second row:

Coefficient	$U^{\frac{1}{3}}$	U_y	R.H.S	Evaluation of Expansion
1	1	a_1	1	$a_1 = 0$
X	0	b_1	2	$b_1 = 2$
x^2	0	c_1	0	$c_1 = 0$
x^3	0	d_1	0	$d_1 = 0$

$$A_1 = \frac{a1}{3A_0^2} = 0$$

$$B_1 = \frac{b_1 - 6A_0A_1B_0}{3A_0^2} = \frac{2-0}{3} = \frac{2}{3}$$

$$C_1 = \frac{c_1 - (3B_0^2 A_1 + 6A_0B_0B_1 + 6A_0C_0A_1)}{3A_0^2} = \frac{0-0}{3} = 0$$

$$D_1 = \frac{d_1 - (6A_0D_0A_1 + 3B_0^2B_1 + 6A_0C_0B_1 + 6A_0B_0C_1 + 6A_0A_1C_1)}{3A_0^2} \quad \frac{0}{3} = 0$$

Third row:

Coefficient	$U^{\frac{1}{3}}$	U_y	R.H.S	Evaluation of Expansion
y	0	$2a_2$	0	$a_2 = 0$
xy	$\frac{2}{3}$	$2b_2$	0	$b_2 = \frac{1}{6}$
x^2y	0	$2c_2$	0	$c_2 = 1$
x^3y	0	$2d_2$	0	$d_3 = 0$

$$A_2 = \frac{a_2 - 3A_0A_1^2)}{3A_0^2} = 0$$

$$B_2 = \frac{b_2 - (3B_0A_1^2 + 6A_0B_0^2A_2) + 6A_0A_1B_1)}{3} = \frac{1}{18}$$

$$C_2 = c_2 - (3C_0A_1^2 + 6A_0C_0A_2 + 6A_0A_1C_1 + 6A_0B_0B_2) = \frac{1}{3}$$

$$D_2 = d_2 - (\,) = 0$$

$$U^{\frac{1}{3}} = 1 + \frac{2}{3}xy + \frac{1}{18}xy^2 + \frac{1}{3}x^2y^2$$

$$+$$

$$U_y = 2x + \frac{1}{3}xy + 2x^2y$$

$$\overline{1 + 2x + xy + \frac{1}{18}xy^2 + 2x^2y + \frac{1}{3}x^2y^2}$$

Fourth row:

Coefficient	$U^{\frac{1}{3}}$	U_y	R.H.S	Evaluation of Expansion
Y^2	0	$3a_3$	0	$a_3 = 0$
Xy^2	$\frac{1}{18}$	$3b_3$	0	$B_3 = \frac{-1}{54}$
x^2y^2	$\frac{1}{3}$	$3c_3$	0	
x^3y^2	0	$3d_3$	0	$D_3 = 0$

$$U^{\frac{1}{3}} = 1 + \frac{2}{3}xy + \frac{1}{18}xy^2 + \frac{1}{3}x^2y^2$$

$$+$$

$$U_y = 2x + \frac{1}{3}xy + 2x^2y - \frac{1}{18}xy^2 - \frac{1}{3}x^2y^2$$

$$\overline{1 + 2x + xy + 2x^2y}$$

Section - 9
H.C.F

In the Vedic Method, the following Sutras are applied to find out the H.C.F.

1. Lopana–Sthapana Upasutra, Sankalana–Vyavakalana Process
2. Adyamadyena sub-sutra and
3. In spection and Argumentation (Vilomana)

Factorization as and when it is necessary.

To explain the first Sutram – Lopana Sthapana, the elimination or retention or in other words alternate destruction of the highest and lowest powers.

For example let us consider $12x^2 + x - 1$ and $15x^2 + 8x + 1$
Here x^2 is the highest one and x is the lowest one. In the first step elimination is carried out for x^2, and in the second step elimination is carried out for x^0.

Eg. $12x^2 + x - 1$, $15x^2 + 8x + 1$

$60x^2 + 5x - 5$	$12x^2 + x - 1$
$\underline{60x^2 + 32x + 4}$	$\underline{15x^2 + 8x + 1}$
$-27x - 9 = -9(1 + 3x)$	$27x^2 + 9x = 9x(1 + 3x)$

$\therefore 1 + 3x$ is the HCF

At every step, a common factor or factorization is also tested. While destroying the lowest power of x, i.e., by adding the two given expressions, we can write the answer as $9x(1 + 3x)$. On examination, of the two answers $1 + 3x$ is common, therefore, $1 + 3x$ is the Highest Common Factor. This is the application of Lopana-Sthapana and Sankalana -Vyavakalana.

By means of factorization also which is carried out in Vedic Method, by applying two of the expressions $15x^2 + 8x + 1$, 2x, 3[illegible] asutras, Anurupyena and Adyamadyenaantyamantyena one gets the [illegible] [illegible] tained as follows.

The middle term is split into two parts. The ratio of the coefficient of x^2 to that of the first part of the middle term, i.e., $12x^2 / 4x = -3x/1$ which is equal to the ratio of the second part of the middle term to the last term $-3x / -1 = 3x/1$.
So one of the factors is $1 + 3x$

The next factor is obtained by Adyamadyenantyamantyena of this factor with reference to the expression,

$12x^2+x-1 = (1+3x)(4x-1)$

Applying the same method to the second expression, we get $(1 + 3x)(1+5x)$

Factors are $(1+3x)(1+5x)$ $\therefore$ The HCF is $1 + 3x$

In some of the cases application of Lopana-Sthapana is successively carried out.

Eg. $2x^4 - 2x^3 + x^2 + 3x - 6$, $4x^4 - 2x^3 + 3x - 9$

$$\begin{array}{l}
4x^4 - 2x^3 \qquad + 3x - 9 \\
\underline{4x^4 - 4x^3 + 2x^2 + 6x - 12} \\
\quad 2x^3 - 2x^2 - 3x + 3 \\
\quad \underline{2x^3 + 2x^2 - 3x - 3} \\
\qquad \underline{-4x^2 \qquad + 6} \\
= 2(2x^2 - 3)
\end{array}$$

$$\begin{array}{l}
8x^4 - 4x^3 + 6x - 18 \\
\underline{-6x^4 + 6x^3 - 3x^2 - 9x + 18} \\
\underline{2x^4 + 2x^3 - 3x^2 - 3x} \\
\quad x(2x^3 + 2x^2 - 3x - 3) \\
\qquad \underline{2x^3 \qquad - 3x} \\
\qquad \quad 2x^2 \qquad - 3 \\
\text{HCF is } 2x^2 - 3
\end{array}$$

In certain cases it is found that the application of Lopana-Sthapana is not giving the HCF value as for the procedure. This is observed in case where the result is showing an expression equivalent to the order of the one of the given expressions and hence a factorization is considered to be available till one gets the final answer. It is also finally that when once HCF is declared then, that should be necessarily common factor to both the given expressions.

Highest Common Factor To Numbers

Eg(1) Find the H.C.F of 480 and 780

Current Method

1	480	780	1
	<u>300</u>	<u>480</u>	
1	180	300	1
	<u>120</u>	<u>180</u>	
	60	120	2
		<u>120</u>	
		<u>0</u>	

∴ H.C.F is 60

Vedic Method

By addition we get 1260 and by subtraction we get 300.

1260 – 4 x 300 = 60

∴ H.C.Fis60

Eg (2) Find the H.C.F of 108 and 252

Current Method

3	108	252	2
	<u>108</u>	<u>216</u>	
	0	<u>36</u>	

∴ H.C.F is 36

Vedic Method

252 – 2 x 108 = 36

∴ H.C.F is 36

Eg (3) Find the H.C.F of 2288, 4301

Current Method

1	2288	4301	1
	<u>2013</u>	<u>2288</u>	
3	275	2013	7
	<u>264</u>	<u>1925</u>	
	<u>11</u>	88	8
		<u>88</u>	
		<u>0</u>	

∴ H.C.F is 11

Vedic Method

2288 x 2 – 4301 = 275

5 x 8 = 88

x 3 = 11

∴ H.C.Fis11

Eg (4)
Find the H.C.F of 8505 and 13545

Current Method

1	8505	13545	1
	2013	8505	
2	3465	5040	1
	3150	3465	
	315	1575	5
		1575	
		0	

∴ H.C.F is 315

Vedic Method

13545 x2 – 8505 x 3
= 27090 - 2515
= 1575

25515 – 1575 x 16 = 315

∴ H.C.F is 315

Eg (5) Find the H.C.F of 308 and 420

Current Method

2	308	420	1
	224	308	
3	84	112	1
	84	84	
	0	28	

Vedic Method

420 – 308 = 112
112 x 4 – 420 = 28
∴ H.C.F is 28
∴ H.C.F is 28

Eg (6) Find the H.C.F of 559 and 728

Current Method

3	559	728	1
	509	559	
4	52	169	3
	52	156	
	0	13	

∴ H.C.F is 13

Vedic Method

728 – 559 = 169
559 – 3 x 169 =52
169 – 3 x 52 = 13

∴ H.C.F is 1

The following problems can be solved by Lopana Sthapana method.

1) $a^3x - a^2bx - 6ab^2x$

$a^2bx^2 - 4ab^2x^2 + 3b^3x^2$

Current G.C.M Method

$a^3x - a^2bx - 6ab^2x$
$= ax(a^2 - ab - 6b^2)$
$a^2bx^2 - 4ab^2x^2 + 3b^3x^2$
$= bx^2(a^2 - 4ab + 3b^2)$

$$\begin{array}{c|r|r|c} a & a^2 - 1ab - 6b^2 & a^2 - 4ab + 3b^2 & 1 \\ & \underline{a^2 - 3ab} & \underline{a^2 - ab - 6b^2} & \\ & \underline{2ab - 6b^2} & \underline{-3ab + 9b^2} & \\ & 2b(a - 3b) & -3b(a - 3b) & 1 \\ & & \underline{a - 3b} & \\ & & 0 & \end{array}$$

$\therefore x(a - 3b)$ is the Highest Common Factor.

Vedic Method

$a^3x - a^2bx - 6ab^2x = ax(a^2 - ab - 6b^2)$
$a^2bx^2 - 4ab^2x^2 + 3b^3x^2$
$= bx^2(a^2 - 4ab + 3b^2)$

$$\begin{array}{l} a^2 - ab - 6b^2 \\ \underline{a^2 - 4ab + 3b^2} \\ \quad 3ab - 9b^2 \\ \quad 3b(a - 3b) \end{array}$$

$$\begin{array}{l} a^2 - ab - 6b^2 \\ \underline{2a^2 - 8ab + 6b^2} \\ 3a^2 - 9ab \\ 3a(a - 3b) \end{array}$$

$\therefore x(a - 3b)$ is the Highest Common Factor.

2) $2x^2 + 9x + 4,\ 2x^2 + 11x + 5,\ 2x^2 - 3x - 2$

Current G.C.M Method

$$\begin{array}{c|r|r|c} x & 2x^2 + 9x + 4 & 2x^2 + 11x + 5 & 1 \\ & \underline{2x^2 + 1x} & \underline{2x^2 + 9x + 4} & \\ & \underline{8x + 4} & 2x + 1 & 1 \\ & 4(2x + 1) & \underline{2x + 1} & \\ & & 0 & \end{array}$$

$$\begin{array}{c|r|r|c} 1 & 2x + 1 & 2x^2 - 3x - 2 & x \\ & \underline{2x + 1} & \underline{2x^2 + x} & \\ & 0 & \underline{-4x - 2} & \\ & & -2(2x + 1) & \end{array}$$

$\therefore 2x + 1$ is the Highest Common Factor.

Vedic Method

$$\begin{array}{l} 2x^2 + 9x + 4 \\ \underline{2x^2 + 11x + 5} \\ \underline{\quad -2x - 1} \\ \quad -1(2x + 1) \end{array}$$

$$\begin{array}{l} +10x^2 + 45x + 20 \\ \underline{-8x^2 - 44x - 20} \\ + \underline{2x^2 + x} \\ \quad x(2x + 1) \end{array}$$

$$\begin{array}{l} 2x^2 + 9x + 4 \\ \underline{2x^2 - 3x - 2} \\ \underline{\quad 12x + 6} \\ \quad 6(2x + 1) \end{array}$$

$$\begin{array}{l} 2x^2 + 9x + 4 \\ \underline{4x^2 - 6x - 4} \\ \underline{6x^2 + 3x} \\ 3x(2x + 1) \end{array}$$

$\therefore 2x + 1$ is the Highest Common Factor.

3) $3x^4 + 8x^3 + 4x^2,\ 3x^5 + 11x^4 + 6x^3,\ 3x^4 - 16x^3 - 12x^2$

Current G.C.M Method

$3x^4 + 8x^3 + 4x^2 = x^2(3x^2 + 8x + 4)$
$3x^5 + 11x^4 + 6x^3 = x^3(3x^2 + 11x + 6)$
$3x^4 - 16x^3 - 12x^2 = x^2(3x^2 - 16x - 12)$

$$\begin{array}{c|r|r|c}
x & 3x^2 + 11x + 6 & 3x^2 - 16x - 12 & 1 \\
 & \underline{3x^2 + 02x} & \underline{3x^2 + 11x + 6} & \\
 & 09x + 6 & \underline{-27x - 18} & \\
 & 3(3x + 2) & -9(3x + 2) & 1 \\
 & & \underline{3x + 2} & \\
 & & 0 &
\end{array}$$

$$\begin{array}{c|r|r|c}
1 & 3x + 2 & 3x^2 + 8x + 4 & x \\
 & \underline{3x + 2} & \underline{3x^2 + 2x} & \\
 & 0 & \underline{6x + 4} & \\
 & & 2(3x + 2) &
\end{array}$$

$\therefore 3x + 2$ is the Highest Common Factor.

Vedic Method

$$\begin{array}{r}
3x^2 + 8x + 4 \\
\underline{3x^2 - 16x - 12} \\
\underline{24x + 16} \\
8(3x + 2)
\end{array}$$

$$\begin{array}{r}
9x^2 + 24x + 12 \\
\underline{3x^2 - 16x - 12} \\
\underline{12x^2 + 8x} \\
4x(3x + 2)
\end{array}$$

$$\begin{array}{r}
3x^2 + 11x + 6 \\
\underline{3x^2 - 16x - 12} \\
\underline{27x + 18} \\
9(3x + 2)
\end{array}$$

$$\begin{array}{r}
6x^2 + 22x + 12 \\
\underline{3x^2 - 16x - 12} \\
\underline{9x^2 + 6x} \\
3x(3x + 2)
\end{array}$$

$\therefore 3x + 2$ is the Highest Common Factor.

4) $3(a - b)^3,\ a^2 - 2ab + b^2$

Current Method by Factorization

$3(a - b)^3$
$a^2 - 2ab + b^2 = (a - b)^2$
$\therefore (a - b)^2$ is the Highest Common Factor

Current G.C.M. Method

$$\begin{array}{l|c|l}
a^2 - 2ab + b^2 & & \\
= (a - b)^2 & (a - b)^3 & a - b \\
 & \underline{(a - b)^3} & \\
 & 0 &
\end{array}$$

$\therefore (a - b)^2$ is the Highest Common Factor.

Vedi[illegible]thod

$3(a - b)^3 = 3a^3 - 9a^2b + 9ab^2 - 3b^3$

$$\begin{array}{r|r}
3a^3 - [illegible] & 3a^3 - 9a^2b + 9ab^2 - 3b^3 \\
\underline{3a^3 - [illegible]} & \underline{3a^2b - 6ab^2 + 3b^3} \\
\underline{-3a^2b + 6ab^2 - 3b^3} & \underline{3a^3 - 6a^2b + 3ab^2} \\
-3b(a^2 - 2ab + b^2) & 3a(a^2 - 2ab + b^2)
\end{array}$$

$\therefore (a - b)^2$ [illegible] Highest Common Factor

5). $x^3y^3 - y^6,\ y^2(xy - y^2)^2$

Current Method by Factorization

$$x^3y^3 - y^6 = y^3(x^3 - y^3)$$
$$= y^3(x - y)(x^2 + xy + y^2)$$
$$y^2(xy - y^2)^2 = y^2[y(x - y)]^2$$
$$= y^4(x - y)^2$$

$\therefore y^3(x - y)$ is the Highest Common Factor.

Current G.C.M. Method

$$x^3y^3 - y^6 = y^3(x^3 - y^3)$$
$$y^2(xy - y^2)^2 = y^4(x - y)^2$$
$$= y^3(x^2y - 2xy^2 + y^3)$$

$$\begin{array}{c|c|c|c}
y & x^2y - 2xy^2 + y^3 & x^3 - y^3 & -1 \\
 & \underline{x^2y + xy^2 - 2y^3} & \underline{-x^2y + 2xy^2 - y^3} & \\
 & \underline{-3xy^2 + 3y^3} & \underline{x^3 + x^2y - 2xy^2} & \\
1 & -3y^2(x - y) & x(x^2 + xy - 2y^2) & x \\
 & \underline{(x - y)} & \underline{x^2 - xy} & \\
 & 0 & 2xy - 2y^2 & \\
 & & 2y(x - y) &
\end{array}$$

$\therefore y^3(x - y)$ is the Highest Common Factor.

Vedic Method

$$x^3y^3 - y^6 = y^3(x^3 - y^3)$$
$$y^2(xy - y^2)^2 = y^4(x - y)^2$$
$$= y^3(x^2y - 2xy^2 + y^3)$$

$x^3y - y^4$
$\underline{x^2y - 2x^2y^2 + xy^3}$
$\underline{-y^4 + 2x^2y^2 - xy^3}$
$y^2(-y^2 + 2x^2 - xy)$
$\underline{-2y^2 + x^2 + xy}$
$\underline{-3y^2 + 3x^2}$
$\underline{3(x^2 - y^2)}$
$\underline{3(x - y)(x + y)}$

$x^3 - y^3$
$\underline{x^2y - 2xy^2 + y^3}$
$\underline{x^2y - 2xy^2 + x^3}$
$x(xy - 2y^2 + x^2)$
$\underline{y^2 - x^2}$
$\underline{xy - y^2}$
$\underline{y(x - y)}$

$\therefore y^3(x - y)$ is the Highest Common Factor.

6) $a^3 - 36a,\ a^3 + 2a^2 - 48a$

Current Method by Factorization

$$a^3 - 36a = a(a^2 - 36)$$
$$= a(a - 6)(a + 6)$$
$$a^3 + 2a^2 - 48a = a(a^2 + 2a - 48)$$
$$= a(a^2 + 8a - 6a - 48)$$
$$= a[a(a + 8) - 6(a + 8)]$$
$$= a(a + 8)(a - 6)$$

$\therefore a(a - 6)$ is the Highest Common Factor

Vedic Method by Factorization

$$a^3 - 36a = a(a^2 - 36)$$
$$= a(a - 6)(a + 6)$$
$$a^2 - 48a = a(a^2 + 2a - 48)$$
$$= a(a + 8)(a - 6)$$

$\therefore a(a - 6)$ is the Highest Common Factor

Current G.C.M. Method

$a^3 - 36a = a(a^2 - 36)$
$a^3 + 2a^2 - 48a = a(a^2 + 2a - 48)$

a	$a^2 - 36$	$a^2 + 2a - 48$	1
	$\underline{a^2 - 6a}$	$\underline{a^2 \quad - 36}$	
	$\underline{-36 + 6a}$	$\underline{2a - 12}$	
	$6(a - 6)$	$2(a - 6)$	1
		$\underline{(a - 6)}$	
		0	

$\therefore a(a - 6)$ is the Highest Common Factor

Vedic Method

$a^3 - 36a = a(a^2 - 36)$
$a^3 + 2a^2 - 48a = a(a^2 + 2a - 48)$

$a^2 + 2a - 48$	$3a^2 + 6a - 144$
$\underline{a^2 \quad - 36}$	$\underline{-4a^2 \quad + 144}$
$\underline{2a - 12}$	$\underline{6a - a^2}$
$2(a - 6)$	$a(6 - a)$

$\therefore a(a - 6)$ is the Highest Common Factor

7) $4m^4 - 9m^2, 6m^3 - 5m^2 - 6m, 6m^4 + 5m^3 - 6m^2$

Current G.C.M. Method

$4m^4 - 9m^2 = m^2(4m^2 - 9)$
$6m^3 - 5m^2 - 6m = m(6m^2 - 5m - 6)$
$6m^4 + 5m^3 - 6m^2 = m^2(6m^2 + 5m - 6)$

2	$4m^2 - 9$	$6m^2 - 5m - 6$	1
	$\underline{4m^2 - 10m + 6}$	$\underline{4m^2 \quad - 9}$	
	$\underline{10m - 15}$	$2m^2 - 5m + 3$	m
-1	$5(2m - 3)$	$\underline{2m^2 - 3m}$	
	$\underline{2m - 3}$	$-2m + 3$	
	0		

2	$4m^2 - 9$	$6m^2 + 5m - 6$	1
	$\underline{4m^2 + 10m + 6}$	$\underline{4m^2 - 9}$	
	$\underline{-10m - 15}$	$2m^2 + 5m + 3$	m
1	$-5(2m+3)$	$\underline{2m^2 + 3m}$	
	$\underline{(2m + 3)}$	$2m + 3$	
	0		

$6m^2 - 5m - 6$	$6m^2 + 5m - 6$
1	$\underline{6m^2 - 5m - 6}$
	$\underline{10m}$

$\therefore$ There is no further Common Factor.
$\therefore$ m is the Highest Common Factor.

Vedic Method

$4m^4 - 9m^2 = m^2(4m^2 - 9)$
$6m^3 - 5m^2 - 6m = m(6m^2 - 5m - 6)$
$6m^4 + 5m^3 - 6m^2 = m^2(6m^2 + 5m - 6)$

$6m^2 - 5m - 6$	$6m^2 - 5m - 6$
$\underline{6m^2 + 5m - 6}$	$\underline{-6m^2 - 5m + 6}$
$\underline{-10m}$	$\underline{-10m}$

$12m^2 \quad - 27$	$8m^2 \quad - 18$
$\underline{12m^2 - 10m - 12}$	$\underline{-18m^2 + 15m + 18}$
$\underline{10m - 15}$	$\underline{-10m^2 + 15m}$
$5(2$ [illegible]	$-5m(2m - 3)$

$12m^2 \quad - 27$	$8m^2 \quad - 18$
[illegible] $\underline{0m - 12}$	$\underline{-18m^2 - 15m + 18}$
[illegible] $\underline{n - 15}$	$\underline{-10m^2 - 15m}$
[illegible] $m + 3)$	$-5m(2m + 3)$

$\therefore$ There is no further Common Factor.
$\therefore$ m is the Highest Common Factor.

8) $3a^4x^3 - 8a^3x^3 + 4a^2x^3,\ 3a^5x^2 - 11a^4x^2 + 6a^3x^2,\ 3a^4x^3 + 16a^3x^3 - 12a^2x^3$

Current G.C.M. Method

$3a^4x^3 - 8a^3x^3 + 4a^2x^3 = a^2x^3(3a^2 - 8a + 4)$
$3a^5x^2 - 11a^4x^2 + 6a^3x^2 = a^3x^2(3a^2 - 11a + 6)$
$3a^4x^3 + 16a^3x^3 - 12a^2x^3 = a^2x^3(3a^2 + 16a - 12)$

$-a$	$3a^2 - 8a + 4$	$3a^2 - 11a + 6$	1
	$\underline{3a^2 - 2a}$	$\underline{3a^2 - 8a + 4}$	
	$\underline{-6a + 4}$	$-3a + 2$	1
	$2(-3a + 2)$	$\underline{-3a + 2}$	
		0	

a	$3a^2 - 8a + 4$	$3a^2 + 16a - 12$	1
	$\underline{3a^2 - 2a}$	$\underline{3a^2 - 8a + 4}$	
	$\underline{-6a + 4}$	$\underline{24a - 16}$	
1	$-2(3a - 2)$	$8(3a - 2)$	
	$\underline{(3a - 2)}$		
	0		

a	$3a^2 - 11a + 6$	$3a^2 + 16a - 12$	1
	$\underline{3a^2 - 2a}$	$\underline{3a^2 - 11a + 6}$	
	$\underline{-9a + 6}$	$\underline{+27a - 18}$	
1	$-3(3a - 2)$	$9(3a - 2)$	
	$\underline{(3a - 2)}$		
	0		

$\therefore a^2x^2(3a - 2)$ is the Highest Common Factor.

Vedic Method

$3a^4x^3 - 8a^3x^3 + 4a^2x^3$
$= a^2x^3(3a^2 - 8a + 4)$
$3a^5x^2 - 11a^4x^2 + 6a^3x^2$
$= a^3x^2(3a^2 - 11a + 6)$
$3a^4x^3 + 16a^3x^3 - 12a^2x^3$
$= a^2x^3(3a^2 + 16a - 12)$

$3a^2 - 8a + 4$	$9a^2 - 24a + 12$
$\underline{3a^2 - 11a + 6}$	$\underline{-6a^2 + 22a - 12}$
$\underline{+3a - 2}$	$\underline{3a^2 - 2a}$
	$a(3a - 2)$

$3a^2 - 11a + 6$	$6a^2 - 22a + 12$
$\underline{3a^2 + 16a - 12}$	$\underline{3a^2 + 16a - 12}$
$\underline{-27a + 18}$	$\underline{9a^2 - 6a}$
$-9(3a - 2)$	$3a(3a - 2)$

$3a^2 - 8a + 4$	$9a^2 - 24a + 12$
$\underline{3a^2 + 16a - 12}$	$\underline{3a^2 + 16a - 12}$
$\underline{-24a + 16}$	$\underline{12a^2 - 8a}$
$-8(3a - 2)$	$4a(3a - 2)$

$\therefore a^2x^2(3a - 2)$ is the Highest Common Factor

9) $x^3 + 4x^2 - 5x - 20,\ x^3 + 6x^2 - 5x - 30$

Current G.C.M Method

$$
\begin{array}{c|c|c|c}
x & x^3 + 4x^2 - 5x - 20 & x^3 + 6x^2 - 5x - 30 & 1 \\
 & \underline{x^3 \qquad - 5x} & \underline{x^3 + 4x^2 - 5x - 20} & \\
 & \underline{4x^2 \qquad - 20} & \underline{2x^2 \qquad - 10} & \\
 & 4(x^2 - 5) & 2(x^2 - 5) & 1 \\
 & & \underline{x^2 - 5} & \\
 & & 0 &
\end{array}
$$

∴ Highest Common Factor is $x^2 - 5$

Vedic Method

$$
\begin{array}{cc}
x^3 + 6x^2 - 5x - 30 & 2x^3 + 12x^2 - 10x - 60 \\
\underline{x^3 + 4x^2 - 5x - 20} & \underline{-3x^3 - 12x^2 + 15x + 60} \\
\underline{2x^2 \qquad - 10} & \underline{-x^3 \qquad + 5x} \\
2(x^2 - 5) & -x(x^2 - 5)
\end{array}
$$

∴ Highest Common Factor is $x^2 - 5$.

10) $x^3 - x^2 - 5x - 3,\ x^3 - 4x^2 - 11x - 6$

Current G.C.M Method

$$
\begin{array}{c|c|c|c}
x & x^3 - x^2 - 5x - 3 & x^3 - 4x^2 - 11x - 6 & 1 \\
 & \underline{x^3 + 2x^2 + x} & \underline{x^3 - x^2 - 5x - 3} & \\
 & -3x^2 - 6x - 3 & \underline{-3x^2 - 6x - 3} & \\
 & -3(x^2 + 2x + 1) & -3(x^2 + 2x + 1) & 1 \\
 & & \underline{x^2 + 2x + 1} & \\
 & & 0 &
\end{array}
$$

∴ Highest Common Factor is $x^2 + 2x + 1$.

Vedic Method

$$
\begin{array}{cc}
x^3 - x^2 + 5x - 3 & x^3 - 4x^2 - 11x - 6 \\
\underline{x^3 - 4x^2 - 11x - 6} & \underline{-2x^3 + 2x^2 + 10x + 6} \\
\underline{3x^2 + 6x + 3} & \underline{-x^3 - 2x^2 - x} \\
-3(x^2 + 2x + 1) & -x(x^2 + 2x + 1)
\end{array}
$$

∴ Highest Common Factor is $x^2 + 2x + 1$.

11) $x^3 + 3x^2 - 8x - 24,\ x^3 + 3x^2 - 3x - 9$

Current G.C.M. Method

$$
\begin{array}{c|c|c|c}
x^2 & x^3 + 3x^2 - 3x - 9 & x^3 + 3x^2 - 8x - 24 & 1 \\
 & \underline{x^3 + 3x^2} & \underline{x^3 + 3x^2 - 3x - 9} & \\
 & \underline{-3x - 9} & -5x - 15 & \\
 & -3(x + 3) & -5(x + 3) & 1 \\
 & & \underline{x + 3} & \\
 & & 0 &
\end{array}
$$

∴ Highest Common Factor is $x + 3$.

Vedic Method

$$
\begin{array}{cc}
x^3 + 3x^2 - x - 9 & + \ [illegible]^3 + 24x^2 - 24x - 72 \\
\underline{x^3 + 3x^2 - 8x - 24} & \underline{-3x^3 - 9x^2 + 24x + 72} \\
\underline{5x + 15} & \underline{+5x^3 + 15x^2} \\
5(x + 3) & 5x(x + 3)
\end{array}
$$

…n Factor is $x + 3$.

12) $a^3 - 3ax^2 + 2x^3$, $a^3 - 5a^2x + 7ax^2 - 3x^3$

Current G.C.M Method

$$
\begin{array}{r|r|r|l}
a & a^3 \qquad - 3ax^2 + 2x^3 & a^3 - 5a^2x + 7ax^2 - 3x^3 & 1 \\
 & \underline{a^3 - 2a^2x + ax^2} & \underline{a^3 \qquad - 3ax^2 + 2x^3} & \\
 & \underline{2a^2x - 4ax^2 + 2x^3} & \underline{-5a^2x + 10ax^2 - 5x^3} & \\
 & 2x(a^2 - 2ax + x^2) & -5x(a^2 - 2ax + x^2) & 1 \\
 & & \underline{a^2 - 2ax + x^2} & \\
 & & 0 &
\end{array}
$$

Highest Common Factor is $a^2 - 2ax + x^2$

Vedic Method

$$
\begin{array}{rr}
a^3 \qquad - 3ax^2 + 2x^3 & 3a^3 - 9ax^2 + 6x^3 \\
\underline{a^3 - 5a^2x + 7ax^2 - 3x^3} & \underline{2a^3 - 10a^2x + 14ax^2 - 6x^3} \\
\underline{5a^2x - 10ax^2 + 5x^3} & \underline{5a^3 - 10a^2x + 5ax^2} \\
5x(a^2 - 2ax + x^2) & 5a(a^2 - 2ax + x^2)
\end{array}
$$

$\therefore$ Highest Common Factor is $a^2 - 2ax + x^2$.

13) $x^4 - 2x^3 - 4x - 7$, $x^4 + x^3 - 3x^2 - x + 2$

Current G.C.M. Method

$$
\begin{array}{r|r|r|l}
x & x^4 + x^3 - 3x^2 - x + 2 & x^4 - 2x^3 \qquad - 4x - 7 & \\
 & \underline{x^4 - x^3 + x^2 + x} & \underline{x^4 + x^3 - 3x^2 - x + 2} & \\
 & \underline{2x^3 - 4x^2 - 2x + 2} & \underline{-3x^3 + 3x^2 - 3x - 9} & \\
x & 2(x^3 - 2x^2 - 2x + 1) & -3(x^3 - x^2 + x + 3) & 1 \\
 & \underline{x^3 + 3x^2 + 2x} & \underline{-2x^2 - 2x + 1} & \\
-5x & -5x^2 - 4x + 1 & \underline{x^3 + 3x^2 + 2} & \\
 & \underline{-5x^2 - 5x} & -5x^2 - 15x - 10 & 1 \\
 & x + 1 & \underline{-5x^2 - 4x + 1} & \\
 & & \underline{-11x - 11} & \\
 & & -11(x + 1) & 1 \\
 & & \underline{x + 1} & \\
 & & 0 &
\end{array}
$$

$\therefore$ Highest Common Factor is $x + 1$.

Vedic Method

$$
\begin{array}{rr}
x^4 + x^3 - 3x^2 - x + 2 & 7x^4 + 7x^3 - 21x^2 - 7x + 14 \\
\underline{x^4 - 2x^3 \qquad - 4x - 7} & \underline{2x^4 - 4x^3 \qquad - 8x - 14} \\
\underline{3x^3 - 3x^2 + 3x + 9} & \underline{9x^4 + 3x^3 - 21x^2 - 15x} \\
\underline{3(x^3 - x^2 + x + 3)} & 3x(3x^3 + x^2 - 7x - 5) \\
2x^3 - 2x^2 + 2x + 6 & \underline{3x^3 - 3x^2 + 3x + 9} \\
\underline{2x^3 - 5x^2 - 7x} & \underline{4x^2 - 10x - 14} \\
\underline{3x^2 + 9x + 6} & 2(2x^2 - 5x - 7) \\
3(x^2 + 3x + 2) & \underline{2x^2 + 6x + 4} \\
\underline{x^2 + x} & \underline{-11x - 11} \\
\underline{2x + 2} & -11(x + 1) \\
2(x + 1) &
\end{array}
$$

$\therefore$ Highest Common Factor is $x+1$

15) $3x^4 - 3x^3 - 2x^2 - x - 1,\ 9x^4 - 3x^3 - x - 1$

Current G.C.M. Method

$$
\begin{array}{r|r|r|r}
x & 3x^4 - 3x^3 - 2x^2 - x - 1 & 9x^4 - 3x^3 \quad - x - 1 & 3 \\
 & \underline{3x^4 + 3x^3 + x^2 + x} & \underline{9x^4 - 9x^3 - 6x^2 - 3x - 3} & \\
-2x & -6x^3 - 3x^2 - 2x - 1 & \underline{6x^3 + 6x^2 + 2x + 2} & \\
 & \underline{-6x^3 \quad - 2x} & \underline{2(3x^3 + 3x^2 + x + 1)} & \\
 & \underline{-3x^2 \quad - 1} & 6x^3 + 6x^2 + 2x + 2 & -1 \\
 & -(3x^2 + 1) & \underline{6x^3 + 3x^2 + 2x + 1} & \\
 & & 3x^2 + 1 & 1 \\
 & & \underline{3x^2 + 1} & \\
 & & 0 &
\end{array}
$$

$\therefore$ Highest Common Factor is $3x^2 + 1$.

Vedic Method

$$
\begin{array}{r}
9x^4 - 3x^3 \quad - x - 1 \\
\underline{9x^4 - 9x^3 - 6x^2 - 3x - 3} \\
\underline{6x^3 + 6x^2 + 2x + 2} \\
2(3x^3 + 3x^2 + x + 1) \\
\underline{3x^3 \quad + x} \\
3x^2 \quad + 1
\end{array}
\qquad
\begin{array}{r}
9x^4 - 3x^3 \quad - x - 1 \\
\underline{-3x^4 + 3x^3 + 2x^2 + x + 1} \\
\underline{6x^4 \quad + 2x^2} \\
2x^2(3x^2 +
\end{array}
$$

$\therefore$ Highest Common Factor is $3x^2 + 1$.

16) $2x^4 - 2x^3 + x^2 + 3x - 6,\ 4x^4 - 2x^3 + 3x - 9$

Current G.C.M. Method

$$
\begin{array}{l|r|r|l}
x & 2x^4 - 2x^3 + x^2 + 3x - 6 & 4x^4 - 2x^3 \qquad + 3x - 9 & 2 \\
 & \underline{2x^4 - 2x^3 - 3x^2 + 3x} & \underline{4x^4 - 4x^3 + 2x^2 + 6x - 12} & \\
 & 4x^2 \qquad - 6 & 2x^3 - 2x^2 - 3x + 3 & x \\
1 & 2(2x^2 \qquad - 3) & \underline{2x^3 \qquad - 3x} & \\
 & \underline{2x^2 \qquad - 3} & \underline{-2x^2 + 3} & \\
 & 0 & -(2x^2 - 3) &
\end{array}
$$

$\therefore$ Highest Common Factor is $2x^2 - 3$.

Vedic Method

$$
\begin{array}{rr}
4x^4 - 2x^3 \qquad + 3x - 9 & 8x^4 - 4x^3 \qquad + 6x - 18 \\
\underline{4x^4 - 4x^3 + 2x^2 + 6x - 12} & \underline{-6x^4 + 6x^3 - 3x^2 - 9x + 18} \\
2x^3 - 2x^2 - 3x + 3 & \underline{2x^4 + 2x^3 - 3x^2 - 3x} \\
\underline{2x^3 + 2x^2 - 3x - 3} & x(2x^3 + 2x^2 - 3x - 3) \\
\underline{-4x^2 \qquad + 6} & \underline{2x^3 \qquad - 3x} \\
-2(2x^2 \qquad - 3) & 2x^2 \qquad - 3
\end{array}
$$

$\therefore$ Highest Common Factor is $2x^2 - 3$.

17) $3x^3 - 3ax^2 + 2a^2x - 2a^3,\ 3x^3 + 12ax^2 + 2a^2x + 8a^3$

Current G.C.M. Method

$$
\begin{array}{l|r|r|l}
x & 3x^3 - 3ax^2 + 2a^2x - 2a^3 & 3x^3 + 12ax^2 + 2a^2x + 8a^3 & 1 \\
 & \underline{3x^3 \qquad + 2a^2x} & \underline{3x^3 - 3ax^2 + 2a^2x - 2a^3} & \\
 & \underline{-3ax^2 \qquad - 2a^3} & \underline{15ax^2 \qquad + 10a^3} & \\
 & -a(3x^2 \qquad + 2a^2) & 5a(3x^2 + 2a^2) & 1 \\
 & & \underline{3x^2 + 2a^2} & \\
 & & 0 &
\end{array}
$$

Vedic Method

$$
\begin{array}{rr}
3x^3 + 12ax^2 + 2a^2x + 8a^3 & 12x^3 - 12ax^2 + 8a^2x - 8a^3 \\
\underline{3x^3 - 3ax^2 + 2a^2x - 2a^3} & \underline{3x^3 + 12ax^2 + 2a^2x + 8a^3} \\
\underline{15ax^2 \qquad + 10a^3} & \underline{15x^3 \quad + 10a^2x} \\
5a(3x^2 + 2a^2) & 5x(3x^2 + 2a^2)
\end{array}
$$

$\therefore$ Highest Common Factor is $3x^2 + 2a^2$.

18) $10x^3 + 25ax^2 - 5a^3,\ 4x^3 + 9ax^2 - 2a^2x - a^3$

Current G.C.M. Method

$10x^3 + 25ax^2 - 5a^3 = 5(2x^2 + 5ax^2 - a^3)$

$$
\begin{array}{r|r|r|l}
2x & 2x^3 + 5ax^2 \qquad - a^3 & 4x^3 + 9ax^2 - 2a^2x - a^3 & 2 \\
 & \underline{2x^3 + 4ax^2 - 2a^2x} & \underline{4x^3 + 10ax^2 \qquad - 2a^3} & \\
 & \underline{ax^2 + 2a^2x - a^3} & \underline{-\ ax^2 - 2a^2x + a^3} & \\
 & a(x^2 + 2ax - a^2) & -a(x^2 + 2ax - a^2) & 1 \\
 & & \underline{x^2 + 2ax - a^2} & \\
 & & 0 &
\end{array}
$$

$\therefore$ Highest Common Factor is $x^2 + 2ax - a^2$.

Vedic Method

$$
\begin{array}{rr}
4x^3 + 9ax^2 - 2a^2x - a^3 & 4x^3 + 9ax^2 - 2a^2x - a^3 \\
\underline{4x^3 + 10ax^2 \qquad - 2a^3} & \underline{-2x^3 - 5ax^2 \qquad + a^3} \\
\underline{-ax^2 - 2a^2x + a^3} & \underline{2x^3 + 4ax^2 - 2a^2x} \\
-a(x^2 + 2ax - a^2) & 2x(x^2 + 2ax - a^2)
\end{array}
$$

$\therefore$ Highest Common Factor is $x^2 + 2ax - a^2$

19) $4x^5 + 14x^4 + 20x^3 + 70x^2,\ 8x^7 + 28x^6 - 8x^5 - 12x^4 + 56x^3$

Current G.C.M. Method

$4x^5 + 14x^4 + 20x^3 + 70x^2 = 2x^2(2x^3 + 7x^2 + 10x + 35)$
$8x^7 + 28x^6 - 8x^5 - 12x^4 + 56x^3 = 4x^3(2x^4 + 7x^3 - 2x^2 - 3x + 14)$

	$\underline{2x^3 + 7x^2 + 10x + 35}$	$2x^4 + 7x^3 - 2x^2 - 3x + 14$	x
x	$6x^3 + 21x^2 + 30x + 105$	$\underline{2x^4 + 7x^3 + 10x^2 + 35x}$	
	$\underline{6x^3 + 21x^2 + 7x}$	$\underline{-12x^2 - 38x + 14}$	
x	$2x^2 + 37x + 105$	$-2(6x^2 + 19x - 7)$	3
	$\underline{2x^2 + 7x}$	$\underline{6x^2 + 111x + 315}$	
	$\underline{30x + 105}$	$\underline{-92x - 322}$	
	$15(2x + 7)$	$-46(2x + 7)$	1
		$\underline{2x + 7}$	
		0	

$\therefore$ Highest Common Factor is $2x + 7$.

Vedic Method

$4x^5 + 14x^4 + 20x^3 + 70x^2 = 2x^2(2x^3 + 7x^2 + 10x + 35)$
$8x^7 + 28x^6 - 8x^5 - 12x^4 + 56x^3 = 4x^3(2x^4 + 7x^3 - 2x^2 - 3x + 14)$

$2x^4 + 7x^3 + 10x^2 + 35x$	$10x^4 + 35x^3 - 10x^2 - 15x + 70$
$\underline{2x^4 + 7x^3 - 2x^2 - 3x + 14}$	$\underline{-\ 4x^3 - 14x^2 - 20x - 70}$
$\underline{12x^2 + 38x - 14}$	$\underline{10x^4 + 31x^3 - 24x^2 - 35x}$
$\underline{2(6x^2 + 19x - 7)}$	$\underline{x(10x^3 + 31x^2 - 24x - 35)}$
$\underline{30x^2 + 95x - 35}$	$30x^3 + 93x^2 - 72x - 105$
$6x^2 + 19x - 7$	$\underline{30x^3 + 95x^2 - 35x}$
$\underline{6x^2 + 111x + 315}$	$\underline{-\ 2x^2 - 37x - 105}$
$\underline{-92x - 322}$	$-(2x^2 + 37x + 105)$
$-46(2x + 7)$	$\underline{2x^2 + 7x}$
	$\underline{30x + 105}$
	$15(2x + 7)$

$\therefore$ Highest Common Factor is $2x + 7$.

20) $72x^3 - 12ax^2 + 72a^2x - 420a^3$, $18x^3 + 42ax^2 - 282a^2x + 270a^3$

Current Method by Factorization

$72x^3 - 12ax^2 + 72a^2x - 420a^3$
$= 12(6x^3 - ax^2 + 6a^2x - 35a^3)$

Current G.C.M. Method

$72x^3 - 12ax^2 + 72a^2x - 420a^3$
$= 12(6x^3 - ax^2 + 6a^2x - 35a^3)$

$18x^3 + 42ax^2 - 282a^2x + 270a^3$
$= 6(3x^3 + 7ax^2 - 47a^2x + 45a^3)$

$$
\begin{array}{c|r|r|c}
x & 3x^3 + 7ax^2 - 47a^2x + 45a^3 & 6x^3 - ax^2 + 6a^2x - 35a^3 & 2 \\
 & \underline{3x^3 - 20ax^2 + 25a^2x} & \underline{6x^3 + 14ax^2 - 94a^2x + 90a^3} & \\
 & \underline{27ax^2 - 72a^2x + 45a^3} & \underline{-15ax^2 + 100a^2x - 125a^3} & \\
x & 9a(3x^2 - 8ax + 5a^2) & -5a(3x^2 - 20ax + 25a^2) & 1 \\
 & \underline{3x^2 - 5ax} & \underline{3x^2 - 8ax + 5a^2} & \\
 & \underline{-3ax + 5a^2} & \underline{-12ax + 20a^2} & \\
 & -a(3x - 5a) & -4a(3x - 5a) & 1 \\
 & & \underline{3x - 5a} & \\
 & & 0 &
\end{array}
$$

∴ Highest Common Factor is $6(3x - 5a)$

Vedic Method

$72x^3 - 12ax^2 + 72a^2x - 420a^3$
$= 12(6x^3 - ax^2 + 6a^2x - 35a^3)$
$18x^3 + 42ax^2 - 282a^2x + 270a^3$
$= 6(3x^3 + 7ax^2 - 47a^2x + 45a^3)$

$$
\begin{array}{r}
6x^3 + 14ax^2 - 94a^2x + 90a^3 \\
\underline{6x^3 - ax^2 + 6a^2x - 35a^3} \\
15ax^2 - 100a^2x + 125a^3 \\
5a(3x^3 - 20ax + 25a^2) \\
\underline{3x^2 - 5ax} \\
\underline{-15ax + 25a^2} \\
-5a(3x - 5a)
\end{array}
\qquad
\begin{array}{r}
54x^3 - 9ax^2 + 54a^2x - 315a^3 \\
\underline{21x^3 + 49ax^2 - 329a^2x + 315a^3} \\
75x^3 + 40ax^2 - 275a^2x \\
5x(15x^2 + 8ax - 55a^2) \\
\underline{15x^2 - 100ax + 125a^2} \\
\underline{108ax - 180a^2} \\
36a(3x - 5a)
\end{array}
$$

∴ Highest Common Factor is $3x - 5a$.

21) $9x^4 + 2x^2y^2 + y^4$, $3x^4 - 8x^3y + 5x^2y^2 - 2xy^3$.

Current G.C.M Method

	$\underline{3x^4 - 8x^3y + 5x^2y^2 - 2xy^3}$	$9x^4 + 2x^2y^2 + y^4$
3		
x	$x(3x^3 - 8x^2y + 5xy^2 - 2y^3)$	$\underline{9x^4 - 24x^3y + 15x^2y^2 - 6xy^3}$
	$\underline{3x^3 - 2x^2y + xy^2}$	$\underline{24x^3y - 13x^2y^2 + 6xy^3 + y^4}$
8		
	$\underline{-6x^2y + 4xy^2 - 2y^3}$	$y(24x^3 - 13x^2y + 6xy^2 + y^3)$
	$-2y(3x^3 - 2xy + y^2)$	$\underline{24x^3 - 64x^2y + 40xy^2 - 16y^3}$
		$\underline{51x^2y - 34xy^2 + 17y^3}$
		$17y(3x^2 - 2xy + y^2)$
1		$\underline{3x^2 - 2xy + y^2}$
		0

$\therefore$ Highest Common Factor is $3x^2 - 2xy + y^2$.

Vedic Method

$9x^4 + 2x^2y^2 + y^4$

$\underline{9x^4 - 24x^3y + 15x^2y^2 - 6xy^3}$

$\underline{24x^3y - 13x^2y^2 + 6xy^3 + y^4}$

$\underline{y(24x^3 - 13x^2 + 6xy^2 + y^3)}$

$72x^3 - 39x^2 + 18xy^2 + 3y^3$

$\underline{72x^3 + 12x^2y - 16xy^2 + 20y^3}$

$\underline{5y^3}$

$\underline{-51x^2y + 34xy^2 - 17y^3}$

$-17y(3x^2 - 2xy + y^2)$

$18x^4 + 4x^2y^2 + 2y^4$

$\underline{3x^3y - 8x^2y^2 + 5xy^3 - 2y^4}$

$\underline{18x^4 + 3x^3y - 4x^2y^2 + 5xy^3}$

$x(18x^3 + 3x^2y - 4xy^2 + 5y^3)$

$\underline{18x^3 - 12x^2y + 6xy^2}$

$\underline{15x^2y - 10xy^2 +}$

$5y(3x^2 - 2xy + y^2)$

$\therefore$ Highest Common Factor is $3x^2 - 2xy + y^2$.

The following problems can be solved by Adyamadyena and Anurupayana method

1) $a^3x - a^2bx - 6ab^2x$
$a^2bx^2 - 4ab^2x^2 + 3b^3x^2$

Current Method by Factorization

$a^3x - a^2bx - 6ab^2x$
$= ax(a^2 - ab - 6b^2)$
$= (ax(a^2 - 3ab + 2ab) - 6b^2)$
$= ax[a(a - 3b) + 2b(a - 3b)]$
$= ax(a + 2b)(a - 3b)$

$a^2bx^2 - 4ab^2x^2 + 3b^3x^2$
$bx^2(a^2 - 4ab + 3b^2)$
$= bx^2(a^2 - ab - 3ab + 3b^2)$
$= bx^2[a(a - b) - 3b(a - b)]$
$= bx^2(a - b)(a - 3b)$

$\therefore x(a - 3b)$ is the Highest Common Factor.

Vedic Method by Factorization

$a^3x - a^2bx - 6ab^2x$
$= ax(a^2 - ab - 6b^2)$
$= ax(a + 2b)(a - 3b)$
$a^2bx^2 - 4ab^2x^2 + 3b^3x^2$
$= bx^2(a^2 - 4ab + 3b^2)$
$= bx^2(a - b)(a - 3b)$

$\therefore x(a-3b)$ is the Highest Common Factor.

2) $2x^2 + 9x + 4$, $2x^2 + 11x + 5$, $2x^2 - 3x - 2$

Current Method by Factorization

$2x^2 + 9x + 4 = 2x^2 + x + 8x + 4$
$= x(2x + 1) + 4(2x + 1)$
$= (2x + 1)(x + 4)$

$2x^2 + 11x + 5 = 2x^2 + x + 10x + 5$
$= x(2x + 1) + 5(2x + 1)$
$= (2x + 1)(x + 5)$

$2x^2 - 3x - 2 = 2x^2 - 4x + x - 2$
$= 2x(x - 2) + 1(x - 2)$
$= (2x + 1)(x - 2)$

$\therefore (2x + 1)$ is the Highest Common Factor.

Vedic Method by Factorization

$2x^2 + 9x + 4$
By Adyamadyena and Anurupyena Sutras
$= 2x^2 + x + 8x + 4$
$= (2x + 1)(x + 4)$
$2x^2 + 11x + 5$

By Adyamadyena and Anurupyena Sutras
$= 2x^2 + x + 10x + 5$
$= (2x + 1)(x + 5)$
$2x^2 - 3x - 2$

By Adyamadyena and Anurupyena Sutras
$2x^2 + x - 4x - 2 = (2x + 1)(x - 2)$

$\therefore 2x + 1$ is the Highest Common Factor.

3) $3x^4 + 8x^3 + 4x^2$, $3x^5 + 11x^4 + 6x^3$, $3x^4 - 16x^3 - 12x^2$

Current Method by Factorization

$3x^4 + 8x^3 + 4x^2$
$= x^2(3x^2 + 6x + 2x + 4)$
$= x^2[3x(x + 2) + 2(x + 2)]$
$= x^2(3x + 2)(x + 2)$

$3x^5 + 11x^4 + 6x^3$
$= x^3(3x^2 + 9x + 2x + 6)$
$= x^3[3x(x + 3) + 2(x + 3)]$
$= x^3(x + 3)(3x + 2)$

$3x^4 - 16x^3 - 12x^2$
$= x^2(3x^2 - 16x - 12)$
$= x^2(3x^2 - 18x + 2x - 12)$
$= x^2[3x(x - 6) + 2(x - 6)]$
$= x^2(3x + 2)(x - 6)$

$\therefore x^2(3x + 2)$ is the Highest Common Factor.

Vedic Method by Factorization

$3x^4 + 8x^3 + 4x^2$
$= x^2(3x^2 + 8x + 4)$
ByAdyamadyena and Anurupyena Sutras
$= x^2(3x^2 + 6x + 2x + 4)$
$= x^2(x + 2)(3x + 2)$

$3x^5 + 11x^4 + 6x^3$
$= x^3(3x^2 + 11x + 6)$
By Adyamadyena and Anurupyena Sutras
$= x^3(3x^2 + 9x + 2x + 6)$
$= x^3(x + 3)(3x + 2)$

$3x^4 - 16x^3 - 12x^2$
$= x^2(3x^2 - 16x - 12)$
By Adyamadyena and Anurupyena Sutras
$= x^2(3x^2 - 18x + 2x - 12)$
$= x^2(x - 6)(3x + 2)$

$\therefore x^2(3x + 2)$ is the Highest Common Factor.

4) $3a^2 + 7a - 6$, $2a^2 + 7a + 3$

Current Method by Factorization

$3a^2 + 7a - 6 = 3a^2 + 9a - 2a - 6$
$= 3a(a + 3) - 2(a + 3)$
$= (a + 3)(3a - 2)$
$2a^2 + 7a + 3 = 2a^2 + a + 6a + 3$
$= a(2a + 1) + 3(2a + 1)$
$= (2a + 1)(a + 3)$

$\therefore (a + 3)$ is the Highest Common Factor

Vedic Method by Factorization

By Adyamadyena and Anurupyena Sutras

$3a^2 + 7a - 6 = 3a^2 + 9a - 2a - 6$
$= (a + 3)(3a - 2)$
$2a^2 + 7a + 3 = 2a^2 + a + 6a + 3$
$= (2a + 1)(a + 3)$

$\therefore (a + 3)$ is the Highest Common Factor

5) $4m^4 - 9m^2, 6m^3 - 5m^2 - 6m, 6m^4 + 5m^3 - 6m^2$

Current Method by Factorization

$4m^4 - 9m^2 = m^2(4m^2 - 9)$
$= m^2(2m - 3)(2m + 3)$
$6m^3 - 5m^2 - 6m = m(6m^2 - 5m - 6)$
$= m(6m^2 - 9m + 4m - 6)$
$= m[3m(2m - 3) + 2(2m - 3)]$
$= m(2m - 3)(3m + 2)$
$6m^4 + 5m^3 - 6m^2 = m^2(6m^2 + 5m - 6)$
$= m^2(6m^2 + 9m - 4m - 6)$
$= m^2[3m(2m + 3) - 2(2m + 3)]$
$= m^2(2m + 3)(3m - 2)$

$\therefore$ m is the Highest Common Factor.

Vedic Method by Factorization

$4m^4 - 9m^2 = m^2(4m^2 - 9)$
$= m^2(2m - 3)(2m + 3)$
By Adyamadyena and Anurupyena Sutras
$6m^3 - 5m^2 - 6m = m(6m^2 - 5m - 6)$
$= m(6m^2 - 9m + 4m - 6)$
By Adyamadyena and Anurupyena Sutras
$= m(2m - 3)(3m + 2)$
$6m^4 + 5m^3 - 6m^2 = m^2(6m^2 + 5m - 6)$
$= m(6m^2 + 9m - 4m - 6)$

By Adyamadyena and Anurupyena Sutras
$= m(2m + 3)(3m - 2)$

$\therefore$ m is the Highest Common Factor.

6) $3a^4x^3 - 8a^3x^3 + 4a^2x^3, 3a^5x^2 - 11a^4x^2 + 6a^3x^2, 3a^4x^3 + 16a^3x^3 - 12a^2x^3$

Current Method by Factorization

$3a^4x^3 - 8a^3x^3 + 4a^2x^3 = a^2x^3(3a^2 - 8a + 4)$
$= a^2x^3(3a^2 - 6a - 2a + 4)$
$= a^2x^3[3a(a - 2) - 2(a - 2)]$
$= a^2x^3(a - 2)(3a - 2)$
$3a^5x^2 - 11a^4x^2 + 6a^3x^2 = a^3x^2(3a^2 - 11a + 6)$
$= a^3x^2(3a^2 - 9a - 2a + 6)$
$= a^3x^2[3a(a - 3) - 2(a - 3)]$
$= a^3x^2(3a - 2)(a - 3)$
$3a^4x^3 + 16a^3x^3 - 12a^2x^3 = a^2x^3(3a^2 + 16a - 12)$
$= a^2x^3(3a^2 + 18a - 2a - 12)$
$= a^2x^3[3a(a + 6) - 2(a + 6)]$
$a^2x^3(a + 6)(3a - 2)$

$\therefore a^2x^2(3a - 2)$ is the highest Common Factor.

Vedic Method by Factorization

By Adyamadyana and Anurupyana Sutras:
$3a^4x^3 - 8a^3x^3 + 4a^2x^3 = a^2x^3(3a^2 - 8a + 4)$
$= a^2x^2(3a^2 - 6a - 2a + 4)$
$= a^2x^2(a - 2)(3a - 2)$
$3a^5x^2 - 11a^4x^2 + 6a^3x^2 = a^3x^2(3a^2 - 11a + 6)$
$= a^3x^2(3a^2 - 9a - 2a + 6)$
$= a^3x^2(a - 3)(3a - 2)$
$3a^4x^3 + 16a^3x^3 - 12a^2x^3 = a^2x^3(3a^2 + 16a - 12)$
$= a^2x^3(3a^2 + 18a - 2a - 12)$
$= a^2x^3(a + 6)(3a - 2)$

$\therefore a^2x^2(3a - 2)$ is the Highest Common Factor.

7) $x^3 + 4x^2 - 5x - 20,\ x^3 + 6x^2 - 5x - 30$

Current Method by Factorization

$x^3 + 4x^2 - 5x - 20$

If $x = -4$

$(-4)^3 + 4(-4)^2 - 5(-4) - 20 = 0$

$\therefore x + 4$ is one factor.

$$
\begin{array}{l}
x + 4)\ x^3 + 4x^2 - 5x - 20\ (x^2 - 5 \\
\qquad \underline{x^3 + 4x^2} \\
\qquad\qquad -5x - 20 \\
\qquad\qquad \underline{-5x - 20} \\
\qquad\qquad\quad 0
\end{array}
$$

$\therefore x^3 + 4x^2 - 5x - 20 = (x^2 - 5)(x + 4)$

$x^3 + 6x^2 - 5x - 30$

If $x = -6$

$(-6)^3 + (6(-6)^2 - 5(-6) - 30 = 0$

$\therefore x + 6$ is one factor.

$$
\begin{array}{l}
x + 6)\ x^3 + 6x^2 - 5x - 30\ (x^2 - 5 \\
\qquad \underline{x^3 + 6x^2} \\
\qquad\qquad -5x - 30 \\
\qquad\qquad \underline{-5x - 30} \\
\qquad\qquad\quad 0
\end{array}
$$

$\therefore x^3 + 6x^2 - 5x - 30 = (x^2 - 5)(x + 6)$

$\therefore$ Highest Common Factor is $x^2 - 5$

Vedic Method by Factorization

$x^3 + 4x^2 - 5x - 20$

By Adyamadyantyana and Anurupyena Sutras

$x^3 + 4x^2 - 5x - 20 = (x^2 - 5)(x + 4)$

$x^3 + 6x^2 - 5x - 30$

By Adyamadyantyana and Anurupyena Sutras

$\therefore x^3 + 6x^2 - 5x - 30 = (x^2 - 5)(x + 6)$

$\therefore$ Highest Common Factor is $x^2 - 5$

8) $x^3 + 3x^2 - 8x - 24, x^3 + 3x^2 - 3x - 9$

Current Method by Factorization

$x^3 + 3x^2 - 8x - 24$
If $x = -3$
$(-3)^3 + 3(-3)^2 - 8(-3) - 24 = 0$
$\therefore x + 3$ is one factor.

$$
\begin{array}{r l}
x+3) & x^3 + 3x^2 - 8x - 24\ (x^2 - 8 \\
 & \underline{x^3 + 3x^2\qquad\qquad} \\
 & \qquad -8x - 24 \\
 & \qquad \underline{-8x - 24} \\
 & \qquad\quad 0
\end{array}
$$

$\therefore x^3 + 3x^2 - 8x - 24 = (x^2 - 8)(x + 3)$
$x^3 + 3x^2 - 3x - 9$
If $x = -3$
$(-3)^3 + 3(-3)^2 - 3(-3) - 9 = 0$
$\therefore x + 3$ is one factor.

$$
\begin{array}{r l}
x+3) & x^3 + 3x^2 - 3x - 9\ (x^2 - 3 \\
 & \underline{x^3 + 3x^2\qquad\qquad} \\
 & \qquad -3x - 9 \\
 & \qquad \underline{-3x - 9} \\
 & \qquad\quad 0
\end{array}
$$

$\therefore x^3 + 3x^2 - 3x - 9 = (x^2 - 3)(x + 3)$
$\therefore$ Highest Common Factor is $(x+3)$.

Vdic Method by Factorization

$x^3 + 3x^2 - 8x - 24$
By Adyamadyenanatyamantyana and Anurupyena Sutras
$x^3 + 3x^2 - 8x - 24 = (x + 3)(x^2 - 8)$
$x^3 + 3x^2 - 3x - 9$
By Adyamadyenantyamantyana and Anurupyena Sutras
$x^3 + 3x^2 - 3x - 9 = (x + 3)(x^2 - 3)$
$\therefore$ Highest Common Factor is $x + 3$.

9) $2x^3 + 4x^2 - 7x - 14, 6x^3 - 10x^2 - 21x + 35$

Current Method by Factorization

$2x^3 + 4x^2 - 7x - 14$
If $x = -2$
$2(-2)^3 + 4(-2)^2 - 7(-2) - 14 = 0$
$\therefore x + 2$ is one factor.

$$
\begin{array}{r l}
x+2) & 2x^3 + 4x^2 - 7x - 14\ (2x^2 - 7 \\
 & \underline{2x^3 + 4x^2\qquad\qquad} \\
 & \qquad -7x - 14 \\
 & \qquad \underline{-7x - 14} \\
 & \qquad\quad 0
\end{array}
$$

$\therefore 2x^3 + 4x^2 - 7x - 14 = (2x^2 - 7)(x + 2)$
$6x^3 - 10x^2 - 21x + 35$
$= 2x^2(3x - 5) - 7(3x - 5)$
$= (2x^2 - 7)(3x - 5)$
$\therefore$ Highest Common Factor is $2x^2 - 7$.

Vedic Method by Factorization

$2x^3 + 4x^2 - 7x - 14$
By Adyamadyena and Anurupyena Sutras
$= (x + 2)(2x^2 - 7)$
$6x^3 - 10x^2 - 21x + 35$
By Adyamadyena and Anurupyena Sutras
$= (3x - 5)(2x^2 - 7)$
$\therefore$ Highest Common Factor is $2x^2 - 7$.

10) $3x^4 - 3x^3 - 2x^2 - x - 1,\ 9x^4 - 3x^3 - x - 1$

Current Method by Factorization

Let $E = 3x^4 - 3x^3 - 2x^2 - x - 1$
$3E = 9x^4 - 9x^3 - 6x^2 - 3x - 3$
$(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$
$\therefore 3E = (3x^2 + 1)^2 - 9x^3 - 12x^2 - 3x - 4$
$= (3x^2 + 1)^2 - (9x^3 + 12x^2 + 3x + 4)$
$= (3x^2 + 1)^2 - [3x(3x^2 + 1) + 4(3x^2 + 1)]$
$= (3x^2 + 1)^2 - [(3x^2 + 1)(3x + 4)]$
$= (3x^2 + 1)(3x^2 + 1 - 3x - 4)$
$= (3x^2 + 1)(3x^2 - 3x - 3)$

$\therefore E = (3x^2 + 1)(x^2 - x - 1)$

Let $F = 9x^4 - 3x^3 - x - 1$
$= (3x^2 + 1)^2 - 3x^3 - 6x^2 - x - 2$
$= (3x^2 + 1)^2 - (3x^3 + 6x^2 + x + 2)$
$= (3x^2 + 1)^2 - [3x^2(x + 2) + 1(x + 2)]$
$= (3x^2 + 1)^2 - [(3x^2 + 1)(x + 2)]$
$= (3x^2 + 1)(3x^2 + 1 - x - 2)$
$= (3x^2 + 1)(3x^2 - x - 1)$

$\therefore$ Highest Common Factor is $3x^2 - 1$

Vedic Method by Factorization

1st Exp.
$3x^4 - 3x^3 - 2x^2 - x - 1$
$(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$
$9x^4 - 9x^3 - 6x^2 - 3x - 3 = (3x^2 + 1)^2 - 9x^3 - 12x^2 - 3x - 4$
$= (3x^2 + 1)^2 - (9x^3 + 12x^2 + 3x + 4)$

By Adyamadyena and Anurupyena Sutras
$= (3x^2 + 1)^2 - (3x + 4)(3x^2 + 1)$
$= (3x^2 + 1)(3x^2 + 1 - 3x - 4)$
$= (3x^2 + 1)(3x^2 - 3x - 3)$
$\therefore 3x^4 - 3x^3 - 2x^2 - x - 1 = (3x^2 + 1)(x^2 - x - 1)$

2nd Exp.
$9x^4 - 3x^3 - x - 1 = (3x^2 + 1)^2 - 3x^3 - 6x^2 - x - 2$
$= (3x^2 + 1)^2 - (3x^3 + 6x^2 + x + 2)$
By Adyamadyena and Anurupyena Sutras
$= (3x^2 + 1)^2 - (x + 2)(3x^2 + 1)$
$= (3x^2 + 1)(3x^2 + 1 - x - 2)$
$\therefore$ 2nd Exp $= (3x^2 + 1)(3x^2 - x - 1)$

$\therefore$ Highest Common Factor is $3x^2 + 1$.

11) $2x^4 - 2x^3 + x^2 + 3x - 6,\ 4x^4 - 2x^3 + 3x - 9$

Current Method by Factorization

Let $E = 2x^4 - 2x^3 + x^2 + 3x - 6$
$2E = 4x^4 - 4x^3 + 2x^2 + 6x - 12$
$(2x^2 - 3)^2 = 4x^4 - 12x^2 + 9$
$2E = (2x^2 - 3)^2 - 4x^3 + 14x^2 + 6x - 21$
$= (2x^2 - 3)^2 - (4x^3 - 14x^2 - 6x + 21)$
$= (2x^2 - 3)^2 - [2x^2(2x - 7) - 3(2x - 7)]$
$= (2x^2 - 3)^2 - (2x^2 - 3)(2x - 7)$
$= (2x^2 - 3)(2x^2 - 3 - 2x + 7)$
$= (2x^2 - 3)(2x^2 - 2x + 4)$
$E = (2x^2 - 3)(x^2 - x + 2)$
$4x^4 - 2x^3 + 3x - 9$
$= (2x^2 - 3)^2 - 2x^3 + 12x^2 + 3x - 18$
$= (2x^2 - 3)^2 - (2x^3 - 12x^2 - 3x + 18)$
$= (2x^2 - 3)^2 - [2x^2(x - 6) - 3(x - 6)]$
$= (2x^2 - 3)^2 - (2x^2 - 3)(x - 6)$
$= (2x^2 - 3)(2x^2 - 3 - x + 6)$
$= (2x^2 - 3)(2x^2 - x + 3)$

$\therefore$ Highest Common Factor is $2x^2 - 3$.

Vedic Method by Factorization

$2x^4 - 2x^3 + x^2 + 3x - 6$
$(2x^2 - 3)^2 = 4x^4 - 12x^2 + 9$
$4x^4 - 4x^3 + 2x^2 + 6x - 12$
$= (2x^2 - 3)^2 - 4x^3 + 14x^2 + 6x - 21$
$= (2x^2 - 3)^2 - (4x^3 - 14x^2 - 6x + 21)$

By Adyamadyena and Anurupyena Sutras
$= (2x^2 - 3)^2 - (2x - 7)(2x^2 - 3)$
$= (2x^2 - 3)(2x^2 - 3 - 2x + 7)$
$= (2x^2 - 3)(2x^2 - 2x + 4)$

$\therefore 2x^4 - 2x^3 + x^2 + 3x - 6 = (2x^2 - 3)(x^2 - x + 2)$
$4x^4 - 2x^3 + 3x - 9$
$= (2x^2 - 3)^2 - 2x^3 + 12x^2 + 3x - 18$
$= (2x^2 - 3)^2 - (2x^3 - 12x^2 - 3x + 18)$

By Adyamadyena and Anurupyena Sutras
$= (2x^2 - 3)^2 - (x - 6)(2x^2 - 3)$
$= (2x^2 - 3)(2x^2 - 3 - x + 6)$
$= (2x^2 - 3)(2x^2 - x + 3)$

$\therefore$ Highest Common Factor is $2x^2 - 3$.

12) $3x^3 - 3ax^2 + 2a^2x - 2a^3$, $3x^3 + 12ax^2 + 2a^2x + 8a^3$

Current Method by Factorization

$3x^3 - 3ax^2 + 2a^2x - 2a^3$
$= 3x^2(x - a) + 2a^2(x - a)$
$= (x - a)(3x^2 + 2a^2)$
$3x^3 + 12ax^2 + 2a^2x + 8a^3$
$= 3x^2(x + 4a) + 2a^2(x + 4a)$
$= (x + 4a)(3x^2 + 2a^2)$

$\therefore$ Highest Common Factor is $3x^2 + 2a^2$.

Vedic Method by Factorization

$3x^3 - 3ax^2 + 2a^2x - 2a^3$
By Adyamadyena and Anurupyena Sutras
$= (x - a)(3x^2 + 2a^2)$
$3x^3 + 12ax^2 + 2a^2x + 8a^3$
By Adyamadyena and Anurupyena Sutras
$= (x + 4a)(3x^2 + 2a^2)$

$\therefore$ Highest Common Factor is $3x^2 + 2a^2$

The following problems can be solved by Argumentation method

Find the Highest Common Factor of

1. $2x^3 + 3x^2 + x + 6, 2x^3 + x^2 + 2x + 3$

Current Method by Factorization

$2x^3 + 3x^2 + x + 6$
If $x = -2$
$2(-2)^3 + 3(-2)^2 + (-2) + 6$
$= -16 + 12 - 2 + 6 = 0$
$\therefore x + 2$ is one factor.

$$
\begin{array}{l}
x + 2)\ 2x^3 + 3x^2 + x + 6\ (2x^2 - x + 3 \\
\qquad\quad \underline{2x^3 + 4x^2} \\
\qquad\qquad\quad -x^2 + x + 6 \\
\qquad\qquad\quad \underline{-x^2 - 2x} \\
\qquad\qquad\qquad\quad 3x + 6 \\
\qquad\qquad\qquad\quad \underline{3x + 6} \\
\qquad\qquad\qquad\qquad 0
\end{array}
$$

$2x^3 + 3x^2 + x + 6 = (x + 2)(2x^2 - x + 3)$
$2x^3 + x^2 + 2x + 3$
If $x = -1$
$2(-1)^3 + (-1)^2 + 2(-1) + 3$
$= -2 + 1 - 2 + 3 = 0$
$\therefore (x + 1)$ is one factor.

$$
\begin{array}{l}
x + 1)\ 2x^3 + x^2 + 2x + 3\ (2x^2 - x + 3 \\
\qquad\quad \underline{2x^3 + 2x^2} \\
\qquad\qquad\quad -x^2 + 2x + 3 \\
\qquad\qquad\quad \underline{-x^2 - x} \\
\qquad\qquad\qquad\quad 3x + 3 \\
\qquad\qquad\qquad\quad \underline{3x + 3} \\
\qquad\qquad\qquad\qquad 0
\end{array}
$$

$2x^3 + x^2 + 2x + 3 = (x + 1)(2x^2 - x + 3)$
$\therefore 2x^2 - x - 3$ is the Highest Common Factor.

V.M. by Factorization

$2x^3 + 3x^2 + x + 6$

$S_c = 12$; Factors are 1, 2, 3, 4, 6, 12
Last term = 6
Whose factors are 1, 2, 3, 6
Possible Factors are 1, ±2, ±3
$(x + 2)$ is one factor.

By Paravartya Division, we get another factor as $2x^2 - x + 3$

$2x^3 + x^2 + 2x + 3$
$S_e = S_o$
$\therefore (x + 1)$ is one factor
By Paravartya Division, we get another factor as $2x^2 - x + 3$
$\therefore 2x^2 - x + 3$ is the Highest Common Factor.

2. $a^3 + 3a^2 - 16a + 12,\ a^3 + a^2 - 10a + 8$

Current Method by Factorization

$a^3 + 3a^2 - 16a + 12$
If $a = 1$
$1 + 3 - 16 + 12 = 0$
$\therefore (a - 1)$ is one factor.

$$\begin{array}{r}
a - 1)\ a^3 + 3a^2 - 16a + 12\ (a^2 + 4a - 12 \\
\underline{a^3 - a^2} \qquad\qquad\qquad \\
4a^2 - 16a + 12 \\
\underline{4a^2 - 4a} \qquad\quad \\
-12a + 12 \\
\underline{-12a + 12} \\
0
\end{array}$$

$a^3 + 3a^2 - 16a + 12 = (a - 1)(a^2 + 4a - 12)$
$= (a - 1)(a^2 + 6a - 2a - 12)$
$= (a - 1)[a(a + 6) - 2(a + 6)]$
$= (a - 1)(a + 6)(a - 2)$

$a^3 + a^2 - 10a + 8$
If $a = 1$
$1 + 1 - 10 + 8 = 0$
$\therefore (a - 1)$ is one factor.

$$\begin{array}{r}
a - 1)\ a^3 + a^2 - 10a + 8\ (a^2 + 2a - 8 \\
\underline{a^3 - a^2} \qquad\qquad\quad \\
2a^2 - 10a \\
\underline{2a^2 - 2a} \\
-8a + 8 \\
\underline{-8a + 8} \\
0
\end{array}$$

$a^3 + a^2 - 10a + 8 = (a - 1)(a^2 + 2a - 8)$
$= (a - 1)(a^2 + 4a - 2a - 8)$
$= (a - 1)[a(a + 4) - 2(a + 4)]$
$= (a - 1)(a + 4)(a - 2)$

$\therefore (a - 1)(a - 2)$ is the Highest Common Factor.

Vedic Method by Factorization

$a^3 + 3a^2 - 16a + 12$
$S_c = 0$
$\therefore (a - 1)$ is a factor

By Paravartya Division
$a^2 + 4a - 12$ is another factor
$\therefore (a + 6)(a - 2)$ are factors.

$a^3 + 3a^2 - 16a + 12 = (a - 1)(a + 6)(a - 2)$
$a^3 + a^2 - 10a + 8$
$S_c = 0$
$\therefore (a - 1)$ is a factor

By Paravartya Division
$a^2 + 2a - 8$ is another factor
$\therefore (a + 4)(a - 2)$ are factors.

$a^3 + a^2 - 10a + 8 = (a - 1)(a - 2)(a + 4)$

$\therefore (a - 1)(a - 2)$ is the Highest Common Factor.

3. $q^3 - 3q + 2,\ q^3 - 5q^2 + 7q - 3$

Current Method by Factorization

$q^3 - 3q + 2$
If $q = 1$
then $1 - 3 + 2 = 0$
$\therefore (q - 1)$ is one factor.

$$
\begin{array}{l}
q - 1)\ q^3 - 3q + 2\ (q^2 + q - 2 \\
\qquad\quad \underline{q^3 - q^2} \\
\qquad\qquad q^2 - 3q + 2 \\
\qquad\qquad \underline{q^2 - q} \\
\qquad\qquad\quad -2q + 2 \\
\qquad\qquad\quad \underline{-2q + 2} \\
\qquad\qquad\qquad 0
\end{array}
$$

$q^3 - 3q + 2 = (q - 1)(q^2 + q - 2)$
$= (q - 1)(q^2 + 2q - q - 2)$
$= (q - 1)[q(q + 2) - 1(q + 2)]$
$= (q - 1)(q - 1)(q + 2)$
$q^3 - 5q^2 + 7q - 3$
If $q = 1$
$1 - 5 + 7 - 3 = 0$
$\therefore (q - 1)$ is one factor.

$$
\begin{array}{l}
q - 1)\ q^3 - 5q^2 + 7q - 3\ (q^2 - 4q + 3 \\
\qquad\quad \underline{q^3 - q^2} \\
\qquad\quad -4q^2 + 7q - 3 \\
\qquad\quad \underline{-4q^2 + 4q} \\
\qquad\qquad\quad 3q - 3 \\
\qquad\qquad\quad \underline{3q - 3} \\
\qquad\qquad\qquad 0
\end{array}
$$

$q^3 - 5q^2 + 7q - 3 = (q - 1)(q^2 - 4q + 3)$
$= (q - 1)(q^2 - q - 3q + 3)$
$= (q - 1)[q(q - 1) - 3((q - 1)]$
$= (q - 1)(q - 1)(q - 3)$

$\therefore (q - 1)^2 = q^2 - 2x + 1$ is the Highest Common Factor.

Vedic Method by Factorization

$q^3 - 3q + 2$
$S_c = 0$
$\therefore (q - 1)$ is one factor

By Paravartya Division

$q^2 + q - 2$ is another factor.
$= (q + 2)(q - 1)$
$\therefore q^3 - 3q + 2 = ((q - 1)^2(q + 2)$
$q^3 - 5q^2 + 7q - 3$
$S_c = 0$
$\therefore (q - 1)$ is one factor

By Paravartya Division
$(q^2 - 4q + 3)$ is another factor
$= (q - 1)(q - 3)$
$\therefore q^3 - 5q^2 + 7q - 3 = (q - 1)^2(q - 3)$
$\therefore q^2 - 2x + 1$ is the Highest Common Factor.

4. $3y^4 - 3y^3 - 15y^2 - 9y,\ 4y^5 - 16y^4 - 44y^3 - 24y^2$

Current Method by Factorization

$3y^4 - 3y^3 - 15y^2 - 9y$
$= 3y(y^3 - y^2 - 5y - 3)$
$y^3 - y^2 - 5y - 3$

If $y = -1$
$(-1)^3 - (-1)^2 - 5(-1) - 3 = 0$
$\therefore (y + 1)$ is one factor.

$y + 1)\ y^3 - y^2 - 5y - 3\ (y^2 - 2y - 3$
$\underline{y^3 + y^2}$
$-2y^2 - 5y - 3$
$\underline{-2y^2 - 2y}$
$-3y - 3$
$\underline{-3y - 3}$
0

$\therefore y^3 - y^2 - 5y - 3 = (y + 1)(y^2 - 2y - 3)$
$= (y + 1)(y^2 - 3y + y - 3)$
$= (y + 1)[y(y - 3) + 1(y - 3)]$
$= (y + 1)(y + 1)(y - 3)$

$4y^5 - 16y^4 - 44y^3 - 24y^2$
$= 4y^2(y^3 - 4y^2 - 11y - 6)$

$y^3 - 4y^2 - 11y - 6$
If $y = -1$
$(-1)^3 - 4(-1)^2 - 11(-1) - 6$
$= -1 - 4 + 11 - 6 = 0$
$\therefore (y + 1)$ is one factor.

$y + 1)\ y^3 - 4y^2 - 11y - 6\ (y^2 - 5y - 6$
$\underline{y^3 + y^2}$
$-5y^2 - 11y - 6$
$\underline{-5y^2 - 5y}$
$-6y - 6$
$\underline{-6y - 6}$
0

$y^3 - 4y^2 - 11y - 6 = (y + 1)(y^2 - 5y - 6)$
$= (y + 1)(y^2 - 6y + y - 6)$
$= (y + 1)[y(y - 6) + 1(y - 6)]$
$= (y + 1)((y + 1)(y - 6)$
$\therefore y(y + 1)^2 = y^3 + 2y^2 + y$ is the Highest Common Factor.

Vedic Method by Factorization

$3y^4 - 3y^3 - 15y^2 - 9y = 3y(y^3 - y^2 - 5y - 3)$
$y^3 - y^2 - 5y - 3$
$S_e = S_o$
$\therefore (y + 1)$ is one factor.

By Paravartya Division:
$y^2 - 2y - 3$ is another factor
$= (y - 3)(y + 1)$
$\therefore y^3 - y^2 - 5y - 3 = (y + 1)^2(y - 3)$

$4y^5 - 16y^4 - 44y^3 - 24y^2$
$= 4y^2(y^3 - 4y^2 - 11y - 6)$
$y^3 - 4y^2 - 11y - 6$
$S_e = S_o$
$\therefore (y + 1)$ is one factor.

By Paravartya Division:
$y^2 - 5y - 6$ is another factor
$= (y - 6)(y + 1)$
$\therefore y^3 - 4y^2 - 11y - 6 = (y + 1)^2(y - 6)$
$\therefore y(y + 1)^2$ is the Highest Common Factor.

5. $15x^4 - 15x^3 + 10x^2 - 10x$, $30x^5 + 120x^4 + 20x^3 + 80x^2$

Current Method by Factorization

$5x(3x^3 - 3x^2 + 2x - 2)$
$10x^2(3x^3 + 12x^2 + 2x + 8)$

$3x^3 - 3x^2 + 2x - 2$
If $x = 1$
$3 - 3 + 2 - 2 = 0$
$\therefore (x - 1)$ is one factor.

$$\begin{array}{r|l} x-1) & 3x^3 - 3x^2 + 2x - 2\ (3x^2 + 2 \\ & \underline{3x^3 - 3x^2} \\ & \qquad\qquad 2x - 2 \\ & \qquad\qquad \underline{2x - 2} \\ & \qquad\qquad\quad 0 \end{array}$$

$\therefore 3x^3 - 3x^2 + 2x - 2 = (x - 1)(3x^2 + 2)$
$3x^3 + 12x^2 + 2x + 8$
If $x = -4$
$3(-4)^3 + 12(-4)^2 + 2(-4) + 8$
$3(-64) + 12 \times 16 - 8 + 8 = 0$
$\therefore (x + 4)$ is one factor

$$\begin{array}{r|l} x+4) & 3x^3 + 12x^2 + 2x + 8\ (3x^2 + 2 \\ & \underline{3x^3 + 12x^2} \\ & \qquad\qquad 2x + 8 \\ & \qquad\qquad \underline{2x + 8} \\ & \qquad\qquad\quad 0 \end{array}$$

$\therefore 3x^3 + 12x^2 + 2x + 8 = (x + 4)(3x^2 + 2)$
$\therefore 5x(3x^2 + 2)$ is the Highest Common Factor.

Vedic Method by Factorization

$15x^4 - 15x^3 + 10x^2 - 10x = 5x(3x^3 - 3x^2 + 2x - 2)$
$3x^3 - 3x^2 + 2x - 2$
$S_c = 0$
$\therefore (x - 1)$ is one factor.

By Paravartya Division:
$3x^2 + 2$ is another factor.

$30x^5 + 120x^4 + 20x^3 + 80x^2 = 10x^2(3x^3 + 12x^2 + 2x + 8)$
$S_c = 25$
Factors are 1, 5, 25
Last term is 8
Factors are 1, 2, 4, 8
$\therefore$ Possible factor is 4
$\therefore x + 4$ is one factor.

By Paravartya Division:
$3x^2 + 2$ is another factor.

$\therefore 5x(3x^2 + 2)$ is the Highest Common Factor.

6. $3x^4 - 9x^3 + 12x^2 - 12x$, $6x^3 - 6x^2 - 15x + 6$

Current Method by Factorization

$3x^4 - 9x^3 + 12x^2 - 12x$
$= 3x(x^3 - 3x^2 + 4x - 4)$
$x^3 - 3x^2 + 4x - 4$
If $x = 2$
$8 - 3(4) + 4(2) - 4 = 8 - 12 + 8 - 4 = 0$
$\therefore (x - 2)$ is one factor.

$$
\begin{array}{r|l}
x-2) & x^3 - 3x^2 + 4x - 4\ (x^2 - x + 2 \\
 & \underline{x^3 - 2x^2} \\
 & \quad -\ x^2 + 4x - 4 \\
 & \quad \underline{-\ x^2 + 2x} \\
 & \qquad\quad 2x - 4 \\
 & \qquad\quad \underline{2x - 4} \\
 & \qquad\qquad 0
\end{array}
$$

$x^3 - 3x^2 + 4x - 4 = (x - 2)(x^2 - x + 2)$
$3x^4 - 9x^3 + 12x^2 - 12x$
$= 3x(x - 2)(x^2 - x + 2)$
$6x^3 - 6x^2 - 15x + 6$
$= 3(2x^3 - 2x^2 - 5x + 2)$
$2x^3 - 2x^2 - 5x + 2$
If $x = 2$
$2(8) - 2(4) - 5(2) + 2 = 16 - 8 - 10 + 2 = 0$
$\therefore x - 2$ is one factor.

$$
\begin{array}{r|l}
x-2) & 2x^3 - 2x^2 - 5x + 2\ (2x^2 + 2x - 1 \\
 & \underline{2x^3 - 4x^2} \\
 & \qquad 2x^2 - 5x + 2 \\
 & \qquad \underline{2x^2 - 4x} \\
 & \qquad\qquad -x + 2 \\
 & \qquad\qquad \underline{-x + 2} \\
 & \qquad\qquad\quad 0
\end{array}
$$

Therefore,
$6x^3 - 6x^2 - 15x + 6 = 3(x - 2)(2x^2 + 2x - 1)$
$\therefore 3(x - 2)$ is the Highest Common Factor.

Vedic Method by Factorization

$3x^4 - 9x^3 + 12x^2 - 12x$
$= 3x(x^3 - 3x^2 + 4x - 4)$
$x^3 - 3x^2 + 4x - 4$
Last term is $- 4$
Factors are $\pm 1, \pm 2, \pm 4$
$S_c = -2$
Factors are $\pm 1, \pm 2$
Possible factor is -2
$\therefore (x - 2)$ is one factor.

By Paravartya Division, we get,
$x^2 - x + 2$ as another factor.
$\therefore 3x^4 - 9x^3 + 12x^2 - 12x$
$= 3x(x - 2)(x^2 - x + 2)$
$6x^3 - 6x^2 - 15x + 6$
$= 3(2x^3 - 2x^2 - 5x + 2)$
$2x^3 - 2x^2 - 5x + 2$
Last term is 2
Factors are $\pm 1, \pm 2$
$S_c = -3$
Factors are $\pm 1, \pm 3$
Possible values are ± 2
$\therefore (x - 2)$ is one factor.

By Paravartya Division:
We get $2x^2 + 2x - 1$ as another factor.
$\therefore 6x^3 - 6x^2 - 15x + 6$
$= 3(x - 2)(2x^2 + 2x - 1)$
$\therefore 3(x - 2)$ is the Highest Common Factor.

7. $2a^5 - 4a^4 - 6a,\ a^5 + a^4 - 3a^3 - 3a^2$

Current Method by Factorization

$2a^5 - 4a^4 - 6a = 2a(a^4 - 2a^3 - 3)$

$a^4 - 2a^3 - 3$

If $a = -1$

$(-1)^4 - 2(-1)^3 - 3 = 0$

$\therefore (a + 1)$ is one factor.

$a + 1)\ a^4 - 2a^3 - 3\ (a^3 - 3a^2 + 3a - 3$

$\underline{a^4 + a^3}$

$-3a^3 - 3$

$\underline{-3a^3 - 3a^2}$

$3a^2 - 3$

$\underline{3a^2 + 3a}$

$-3a - 3$

$\underline{-3a - 3}$

0

$a^4 - 2a^3 - 3 = (a + 1)(a^3 - 3a^2 + 3a - 3)$

$\therefore 2a^5 - 4a^4 - 6a = 2a(a+1)(a^3 - 3a^2 + 3a - 3)$

$a^5 + a^4 - 3a^3 - 3a^2 = a^2(a^3 + a^2 - 3a - 3)$

$a^3 + a^2 - 3a - 3$

If $a = -1$

$(-1)^3 + (-1)^2 - 3(-1) - 3 = 0$

$\therefore (a + 1)$ is one factor.

$a + 1)\ a^3 + a^2 - 3a - 3\ (a^2 - 3$

$\underline{a^3 + a^2}$

$-3a - 3$

$\underline{-3a - 3}$

0

$\therefore a^5 + a^4 - 3a^3 - 3a^2 = a^2(a + 1)(a^2 - 3)$

$\therefore a(a + 1)$ is the Highest Common Factor.

Vedic Method by Factorization

$2a^5 - 4a^4 - 6a = 2a(a^4 - 2a^3 - 3)$

$a^4 - 2a^3 - 3$

$S_o = S_e$

$\therefore (a + 1)$ is one factor.

By Paravartya Division, we get,

$a^3 - 3a^2 + 3a - 3$ as another factor.

$2a^5 - 4a^4 - 6a = 2a(a + 1)(a^3 - 3a^2 + 3a - 3)$

$a^5 + a^4 - 3a^3 - 3a^2 = a^2(a^3 + a^2 - 3a - 3)$

$a^3 + a^2 - 3a - 3$

$S_o = S_e$

$\therefore (a + 1)$ is one factor.

By Paravartya Division, we get,

$a^2 - 3$ as another factor.

$a^5 + a^4 - 3a^3 - 3a^2 = a^2(a + 1)(a^2 - 3)$

$\therefore a(a + 1)$ is the highest Common Factor.

8. $x^3 + 4x^2 - 2x - 15,\ x^3 - 21x - 36$

Current Method by Factorization

$x^3 + 4x^2 - 2x - 15$

If $x = -3$

$(-3)^3 + 4(-3)^2 - 2(-3) - 15 = -27 + 36 + 6 - 15 = 0$

$\therefore (x + 3)$ is one factor.

$$
\begin{array}{l}
x + 3)\ x^3 + 4x^2 \quad - 2x - 15\ (x^2 + x - 5 \\
\qquad\quad \underline{x^3 + 3x^2} \\
\qquad\qquad x^2 - 2x \\
\qquad\qquad \underline{x^2 + 3x} \\
\qquad\qquad\quad -5x - 15 \\
\qquad\qquad\quad \underline{-5x - 15} \\
\qquad\qquad\qquad 0
\end{array}
$$

$\therefore x^3 + 4x^2 - 2x - 15 = (x + 3)(x^2 + x - 5)$

$x^3 - 21x - 36$ if $x = -3$

$(-3)^3 - 27x - 36 = 0$

$(x + 3)$ is one factor.

$$
\begin{array}{l}
x + 3)\ x^3 - 21x - 36\ (x^2 - 3x - 12 \\
\qquad\quad \underline{x^3 + 3x^2} \\
\qquad\quad - 3x^2 - 21x - 36 \\
\qquad\quad \underline{- 3x^2 - 9x} \\
\qquad\qquad -12x - 36 \\
\qquad\qquad \underline{-12x - 36} \\
\qquad\qquad\quad 0
\end{array}
$$

$x^3 - 21x - 36 = (x + 3)(x^2 - 3x - 12)$

$\therefore x + 3$ is the Highest Common Factor.

Vedic Method by Factorization

$x^3 + 4x^2 - 2x - 15$

$S_c = -12$

Factors are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

Last term is -15

Factors are $\pm1, \pm3, \pm5, \pm15$

Possible Values are $1, \pm3, \pm5$

$\therefore (x + 3)$ is one factor.

By Paravartya Division, we get

$x^2 + x - 5$ as another factor.

$\therefore x^3 + 4x^2 - 2x - 15 = (x+3)(x^2 + x - 5)$

$x^3 - 21x - 36$

$S_c = -56$

Factors are $\pm1, \pm2, \pm4, \pm7, \pm8, \pm14, \pm28, \pm56$

Last term is -36

Factors are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm9, \pm12, \pm18, \pm36$

Possible values are $1, -2, \pm3, 6, -9$

$\therefore (x + 3)$ is one factor.

By Paravartya Division, we get

$x^2 - 3x - 12$ as another factor.

$x^3 - 21x - 36 = (x + 3)(x^2 - 3x - 12)$

$\therefore (x + 3)$ is the Highest Common Factor

9) $x^3 + 2x^2 - 8x - 16, x^3 + 3x^2 - 8x - 24$

Current Method by Factorization

$x^3 + 2x^2 - 8x - 16$
If $x = -2$

$(-2)^3 + 2(-2)^2 - 8(-2) - 16 = 0$
$\therefore (x + 2)$ is one factor.

$$\begin{array}{r l}
x+2)\ x^3 + 2x^2 - 8x - 16\ (x^2 - 8 & \\
\underline{x^3 + 2x^2\qquad\qquad} & \\
-8x - 16 & \\
\underline{-8x - 16} & \\
0 &
\end{array}$$

$\therefore x^3 + 2x^2 - 8x - 16 = (x^2 - 8)(x + 2)$

$x^3 + 3x^2 - 8x - 24$
If $x = -3$
$(-3)^3 + 3(-3)^2 - 8(-3) - 24 = 0$
$\therefore (x + 3)$ is one factor.

$$\begin{array}{r l}
x+3)\ x^3 + 3x^2 - 8x - 24\ (x^2 - 8 & \\
\underline{x^3 + 3x^2\qquad\qquad} & \\
-8x - 24 & \\
\underline{-8x - 24} & \\
0 &
\end{array}$$

$\therefore x^3 + 3x^2 - 8x - 24 = (x^2 - 8)(x + 3)$
$\therefore$ Highest Common Factor is $(x + 3)$

Vedic Method by Factorization

$x^3 + 2x^2 - 8x - 16$
$S_c = -21$
Factors are $\pm1, \pm3, \pm7$, and ±21
Last term is -16
Factors are $\pm1, \pm2, \pm4, \pm8$, and ±16
Possible factors are $\pm2, -4$, and -8
$\therefore (x + 2)$ is one factor.

By Adyamadyena Sutra the first and last terms of another factors are $x^2 - 8$.
By Gunita Samuccaye Sutra
$3(x^2+bx-8) = -21$
$3(-7 + b) = -21$
$-7 + b = -7$
$b = 0$
$\therefore$ Another factor is $x^2 - 8$.
$\therefore x^3 + 2x^2 - 8x - 16 = (x^2 - 8)(x + 2)$
$x^3 + 3x^2 - 8x - 24$
$S_c = -28$
Factors are $\pm1, \pm2, \pm4, \pm7, \pm14$, and ±28
Last term $= -24$
Factors are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12$, and ±24
Possible factors are $-2, \pm3, 6$, and -8.
$\therefore (x + 3)$ is one factor.
By Adyamadyena Sutra the first and last terms of another factor x^2 and -8, i.e., $x^2 + bx - 8$
By Gunita Samuccay Sutra
$4(1 - 8 + b) = -28$
$-7 + b = -7$
$b = 0$
$\therefore$ Middle term is 0.
Another factor is $x^2 - 8$
$\therefore x^3 + 3x^2 - 8x - 24 = (x^2 - 8)(x + 3)$
$\therefore$ Highest Common Factor $= x + 3$

$\therefore$ H.C.F $= x+3$

10) $x^3 - x^2 - 5x - 3, x^3 - 4x^2 - 11x - 6$

Current Method by Factorization

$x^3 - x^2 - 5x - 3$
If $x = -1$
$(-1)^3 - (-1)^2 - 5(-1) - 3 = -1 - 1 + 5 - 3 = 0$
$\therefore x + 1$ is one factor.

$$
\begin{array}{r|l}
x+1) & x^3 - x^2 - 5x - 3\ (x^2 - 2x - 3 \\
& \underline{x^3 + x^2} \\
& \quad -2x^2 - 5x - 3 \\
& \quad \underline{-2x^2 - 2x} \\
& \qquad -3x - 3 \\
& \qquad \underline{-3x - 3} \\
& \qquad\quad 0
\end{array}
$$

$x^3 - x^2 - 5x - 3 = (x + 1)(x^2 - 2x - 3)$
$= (x + 1)(x^2 - 3x + x - 3)$
$= (x + 1)([x(x - 3) + 1(x - 3)]$
$= (x + 1)(x + 1)(x - 3)$
$= (x + 1)^2(x - 3)$

$x^3 - 4x^2 - 11x - 6$
If $x = -1$
$(-1)^3 - 4(-1)^2 - 11(-1) - 6 = 0$
$\therefore (x + 1)$ is one factor.

$$
\begin{array}{r|l}
x+1) & x^3 - 4x^2 - 11x - 6\ (x^2 - 5x - 6 \\
& \underline{x^3 + x^2} \\
& \quad -5x^2 - 11x \\
& \quad \underline{-5x^2 - 5x} \\
& \qquad -6x - 6 \\
& \qquad \underline{-6x - 6} \\
& \qquad\quad 0
\end{array}
$$

$x^3 - 4x^2 - 11x - 6 = (x + 1)(x^2 - 5x - 6)$
$= (x + 1)(x^2 - 6x + x - 6)$
$= (x+ 1)[x(x - 6) + 1(x - 6)]$
$= (x + 1)(x + 1)(x - 6)$
$= (x + 1)^2(x - 6)$

$\therefore$ Highest Common Factor is $(x + 1)^2$.

Vedic Method by Factorization

$x^3 - x^2 - 5x - 3$
$S_c = -8$
Last term is -3
Whose factors are 1, 1, and 3
But their total should be -1,
$\therefore 1, 1, -3$

$ab + bc + ca = -5$
$\therefore x^3 - x^2 - 5x - 3 = (x + 1)^2(x - 3)$
$x^3 - 4x^2 - 11x - 6$
$S_c = -20$
Last term is -6.
Whose factors are 1, 2, 3 or 1, 1, 6
Total should be -4,
$\therefore 1, 1, -6$

$ab + bc + ca = -11$
$\therefore x^3 - 4x^2 - 11x - 6 = (x + 1)^2(x - 6)$

$\therefore$ Highest Common Factor is $(x + 1)^2$.

11) $x^4 - 2x^3 - 4x - 7,\ x^4 + x^3 - 3x^2 - x + 2$

Current Method by Factorization

$x^4 - 2x^3 - 4x - 7$

if $x = -1$

$(-1)^4 - 2(-1)^3 - 4(-1) - 7 = 0$

$\therefore (x + 1)$ is one factor.

$$
\begin{array}{l}
x+1)\,x^4 - 2x^3 - 4x - 7\,(x^3 - 3x^2 + 3x - 7 \\
\quad \underline{x^4 + x^3} \\
\qquad -3x^3 - 4x - 7 \\
\qquad \underline{-3x^3 - 3x^2} \\
\qquad\quad 3x^2 - 4x - 7 \\
\qquad\quad \underline{3x^2 + 3x} \\
\qquad\qquad -7x - 7 \\
\qquad\qquad \underline{-7x - 7} \\
\qquad\qquad\quad 0
\end{array}
$$

$\therefore x^4 - 2x^3 - 4x - 7 = (x^3 - 3x^2 + 3x - 7)(x + 1)$

$x^4 + x^3 - 3x^2 - x + 2$

If $x = -1$

$(-1)^4 + (-1)^3 - 3(-1)^2 - (-1) + 2 = 0$

$\therefore x + 1$ is one factor.

$$
\begin{array}{l}
x + 1)\,x^4 + x^3 - 3x^2 - x + 2\,(x^3 - 3x + 2 \\
\quad \underline{x^4 + x^3} \\
\qquad -3x^2 - x + 2 \\
\qquad \underline{-3x^2 - 3x} \\
\qquad\qquad 2x + 2 \\
\qquad\qquad \underline{2x + 2} \\
\qquad\qquad\quad 0
\end{array}
$$

$\therefore x^4 + x^3 - 3x^2 - x + 2 = (x + 1)(x^3 - 3x + 2)$

$x^3 - 3x + 2$

If $x = 1$ or $1^3 - 3 \times 1 + 2 = 0$

$\therefore (x - 1)$ is one factor.

$$
\begin{array}{l}
x - 1)\,x^3 \quad - 3x + 2\,(x^2 + x - 2 \\
\quad \underline{x^3 - x^2} \\
\qquad x^2 - 3x + 2 \\
\qquad \underline{x^2 - x} \\
\qquad\quad -2x + 2 \\
\qquad\quad \underline{-2x + 2} \\
\qquad\qquad 0
\end{array}
$$

$x^4 + x^3 - 3x^2 - x + 2 = (x + 1)(x - 1)(x^2 + x - 2)$

$= (x + 1)(x - 1)(x^2 + 2x - x - 2)$

$= (x + 1)(x - 1)[x(x + 2) - 1(x + 2)]$

$= (x + 1)(x - 1)(x - 1)(x + 2)$

$\therefore$ Highest Common Factor is $(x + 1)$.

Vedic Method by Factorization

$x^4 - 2x^3 - 4x - 7$

$S_o = S_e$

$\therefore (x + 1)$ is one factor.

By Paravartya division another factor is $x^3 - 3x^2 + 3x - 7$

$x^3 - 3x^2 + 3x - 7$

$S_c = -6.$

Factors are $\pm 1, \pm 2, \pm 3,$ and ± 6.

Last term $= -7$.

Factors are $\pm 1, \pm 7$

Possible factor is -7

But $x - 7$ is not a factor.

$\therefore$ It has no further factors

$x^4 - 2x^3 - 4x - 7$

$= (x^3 - 3x^2 + 3x - 7)(x + 1)$

$x^4 + x^3 - 3x^2 - x + 2$

$S_c = 0$

$\therefore (x - 1)$ is one factor.

$S_o = S_e$

$\therefore (x + 1)$ is another factor.

By Paravartya Division with $x - 1$ we get another factor as $x^3 + 2x^2 - x - 2$

Again by Paravartya division by $x + 1$ we get $x^2 + x - 2$

$x^2 + x - 2 = x^2 + 2x - x - 2$

$= (x + 2)(x - 1)$

$\therefore x^4 + x^3 - 3x^2 - x + 2$

$= (x + 1)(x - 1)^2(x + 2)$

$\therefore$ Highest Common Factor is $x + 1$.

12) $2x^3 - 5x^2 + 11x + 7,\ 4x^3 - 11x^2 + 25x + 7$

Current Method by Factorization

Let $E = 2x^3 - 5x^2 + 11x + 7$
$4E = 8x^3 - 20x^2 + 44x + 28$
$(2x + 1)^3 = 8x^3 + 12x^2 + 6x + 1$
$4E = (2x + 1)^3 - 32x^2 + 38x + 27$
$= (2x + 1)^3 - (32x^2 + 16x - 54x - 27)$
$= (2x + 1)^3 - [16x(2x + 1) - 27(2x + 1)]$
$= (2x + 1)^3 - (2x + 1)(16x - 27)$
$= (2x + 1)[(2x + 1)^2 - 16x + 27]$
$= (2x + 1)(4x^2 + 4x + 1 - 16x + 27)$
$= (2x + 1)(4x^2 - 12x + 28)$

$\therefore 4E = (2x + 1)(x^2 - 3x + 7)$

Let $F = 4x^3 - 11x^2 + 25x + 7$
$16F = 64x^3 - 176x^2 + 400x + 112$
$(4x + 1)^3 = 64x^3 + 48x^2 + 12x + 1$
$16F = (4x + 1)^3 - 224x^2 + 388x + 111$
$= (4x + 1)^3 - (224x^2 - 388x - 111)$
$= (4x + 1)^3 - (224x^2 - 444x + 56x - 111)$
$= (4x + 1)^3 - [4x(56x - 111) + 1(56x - 111)]$
$= (4x + 1)^3 - (4x + 1)(56x - 111)$
$= (4x + 1)[(4x + 1)^2 - 56x + 111]$
$= (4x + 1)[16x^2 + 8x + 1 - 56x + 111]$
$= (4x + 1)[16x^2 - 48x + 112]$

$\therefore F = (4x + 1)(x^2 - 3x + 7)$

$\therefore$ Highest Common Factor is $x^2 - 3x + 7$.

Vedic Method by Factorization

$2x^3 - 5x^2 + 11x + 7$

First term = 2
Factors are 1, 2
$\therefore$ Factors are $(x + a)(2x^2 + bx + c)$
or $(2x + a)(x^2 + bx + c)$

$S_c = 15$
Factors are $\pm1, \pm3, \pm5$, and ±15
Last term is 7
Factors are $\pm1, \pm7$

By Gunita Samuccay, Sum of **Coefficients of factors is a factor of sum of coefficients of the product.**
$\therefore$ Possible factors are only
$(2x + 1), (2x - 1), (2x - 7)$
is one factor
If $(2x - 7)$ then another factor is
$(x^2 + bx - 1)$
By Gunita Samuccaye
$(-5)(1 + b - 1) = 15$
$-5b = 15$
$b = -3$
x^2 coefficient in the given expression is -5
By these factors x^2 coefficient is -13.
$\therefore$ This combination is not correct.
Is one factor
If $(2x - 1)$ then another factor is $(x^2 + bx - 7)$
By Gunita Samuccay $(1)(1 + b - 7) = 15$
$-6 + b = 15$
$b = -21$
x^2 coefficient is -5
By these factors x^2 coefficient is $2b - 1$
$= 2(-21) - 1 = -43$
$-5 \neq -43$
$\therefore$ This combination is not correct.
Is one factor
If $(2x + 1)$ then another factor is $(x^2 + bx + 7)$
By Gunita Samuccay Sutra
$(3)(1 + b + 7) = 15$
$8 + b = 5;\ b = -3$

x^2 coefficient is $2b + 1 = -6 + 1 = -5$
x coefficient is $b + 14 = -3 + 14 = 11$
$2x^3 - 5x^2 + 11x + 7 = (2x + 1)(x^2 - 3x + 7)$

$4x^3 - 11x^2 + 25x + 7$
First term = 4
Factors are 1, 2, and 4

$\therefore$ Factors are $(x + a)(4x^2 + bx + c)$
or $(4x + a)(x^2 + bx + c)$
or $(2x + a)(2x^2 + bx + c)$
$S_c = 25$
Factors are 1, 5, and 25.

Last term is 7. Factors are $\pm 1, \pm 7$
$\therefore$ Possible factors are $(4x+1)(2x - 1)(2x - 7)$

If $2x - 7$ is a factor then another factor is
$(2x^2 + bx - 1)$
By Gunita Samuccaye Sutra
$(-5)(2 + b - 1) = 25$
$1 + b = -5$
$b = -6$
x^2 coefficient in the given expression: -11.
By these factors x^2 coefficient $= -14 + 2b$
$= -14 + 2(-6) = -26$.
$\therefore$ This combination is not correct.

If $2x - 1$ then another factor is $(2x^2 + bx - 7)$
By Gunita Samuccaye Sutra
$(1)(2 + b - 7) = 25$
$-5 + b = 25$
$b = 30$
x^2 coefficient $= -2 + 2b = -2 + 60 = 58$
$\therefore$ This combination is not correct.

If $(4x + 1)$ then another factor is $(x^2 + bx + 7)$
By Gunita Samuccay Sutra
$(5)(1 + b + 7) = 25$
$8 + b = 5$
$b = -3$.
By these factors x^2 coefficient $= 1 + 4b$
$= 1 + 4(-3) = 1 - 12 = -11$
x coefficient $= 28 + b$
$= 28 - 3 = 25$
$\therefore 4x^3 - 11x^2 + 25x + 7 = (4x + 1)(x^2 - 3x + 7)$

$\therefore$ Highest Common Factor is $(x^2 - 3x + 7)$

13) $24x^4y + 72x^3y^2 - 6x^2y^3 - 90xy^4,\ 6x^4y^2 + 13x^3y^3 - 4x^2y^4 - 15xy^5$

Current Method by Factorization

$24x^4y + 72x^3y^2 - 6x^2y^3 - 90xy^4$
$6xy\,(4x^3 + 12x^2y - xy^2 - 15y^3)$
If $x = y$ then
$4y^3 + 12y^3 - y^3 - 15y^3 = 0$
$\therefore x - y$ is one factor

$$
\begin{array}{l}
x - y)\ 4x^3 + 12x^2y - \quad xy^2 - 15y^3\ (4x^2 + 16xy + 15y^2 \\
\qquad \underline{4x^3 - \ 4x^2y \qquad\qquad\qquad} \\
\qquad\qquad 16x^2y - \quad xy^2 \\
\qquad\qquad \underline{16x^2y - 16xy^2 \qquad} \\
\qquad\qquad\qquad 15xy^2 - 15y^3 \\
\qquad\qquad\qquad \underline{15xy^2 - 15y^3} \\
\qquad\qquad\qquad\qquad 0
\end{array}
$$

$4x^2 + 16xy + 15y^2$
$= 4x^2 + 6xy + 10xy + 15y^2$
$= 2x(2x + 3y) + 5y(2x + 3y)$
$= (2x + 3y)(2x + 5y)$

$\therefore 24x^4y + 72x^3y^2 - 6x^2y^3 - 90xy^4$
$= 6xy(x - y)(2x + 3y)(2x + 5y)$
$6x^4y^2 + 13x^3y^3 - 4x^2y^4 - 15xy^5$
$= xy^2(6x^3 + 13x^2y - 4xy^2 - 15y^3)$
If $x = y$ then
$6y^3 + 13y^3 - 4y^3 - 15y^3 = 0$
$\therefore x - y$ is one factor.

$$
\begin{array}{l}
x - y)\ 6x^3 + 13x^2y - \quad 4xy^2 - 15y^3\ (6x^2 + 19xy + 15y^2 \\
\qquad \underline{6x^3 - 6x^2y \qquad\qquad\qquad} \\
\qquad\qquad 19x^2y - \quad 4xy^2 \\
\qquad\qquad \underline{19x^2y - 19xy^2 \qquad} \\
\qquad\qquad\qquad 15xy^2 - 15y^3 \\
\qquad\qquad\qquad \underline{15xy^2 - 15y^3} \\
\qquad\qquad\qquad\qquad 0
\end{array}
$$

$6x^2 + 19xy + 15y^2$
$= 6x^2 + 9xy + 10xy + 15y^2$
$= 3x(2x + 3y) + 5y(2x + 3y)$
$= (2x + 3y)(3x + 5y)$
$\therefore 6x^4y^2 + 13x^3y^3 - 4x^2y^4 - 15xy^5$
$= xy^2(x - y)(2x + 3y)(3x + 5y)$

$\therefore$Highest Common Factor is $xy(x - y)(2x + 3y)$

Vedic Method by Factorization

$24x^4y + 72x^3y^2 - 6x^2y^3 - 90xy^4$
$= 6xy(4x^3 + 12x^2y - xy^2 - 15y^3)$
$4x^3 + 12x^2y - xy^2 - 15y^3$
$S_c = 0 \quad \therefore x = y$
$\therefore x - y$ is one factor.

By Paravartya division, we get,
$4x^2 + 16xy + 15y^2$ as another factor.
$4x^2 + 16xy + 15y^2$
$= 4x^2 + 6xy + 10xy + 15y^2$
By Adyamadyena and Anurupyena Sutras
$= (2x + 3y)(2x + 5y)$

$\therefore 24x^4y + 72x^3y^2 - 6x^2y^3 - 90xy^4$
$= 6xy(x - y)(2x + 3y)(2x + 5y)$
$6x^4y^2 + 13x^3y^3 - 4x^2y^4 - 15xy^5$
$= xy^2(6x^3 + 13x^2y - 4xy^2 - 15y^3)$
$6x^3 + 13x^2y - 4xy^2 - 15y^3$
$S_c = 0$
$x - y$ is one factor.

By Paravartya division, we get,
$6x^2 + 19xy + 15y^2$ as another factor.
$6x^2 + 19xy + 15y^2$
$= 6x^2 + 9xy + 10xy + 15y^2$
By Adyamadyena and Anurupyena Sutras
$= (2x + 3y)(3x + 5y)$

$\therefore 6x^4y^2 + 13x^3y^3 - 4x^2y^4 - 15xy^5$
$= xy^2(x - y)(2x + 3y)(3x + 5y)$

$\therefore$Highest Common Factor is
$xy(x - y)(2x + 3y)$

14) $x^5 - x^3 - x + 1,\ x^7 + x^6 + x^4 - 1$

Current Method by Factorization

$x^5 - x^3 - x + 1$

If $x = 1$

$1 - 1 - 1 + 1 = 0$

$\therefore x - 1$ is one factor.

$$
\begin{array}{r l}
x - 1)\ x^5 \qquad - x^3 - x + 1 & (x^4 + x^3 - 1 \\
\underline{x^5 - x^4} \qquad\qquad\quad & \\
x^4 - x^3 - x + 1 & \\
\underline{x^4 - x^3} \qquad\quad & \\
-x + 1 & \\
\underline{-x + 1} & \\
0 &
\end{array}
$$

$\therefore x^5 - x^3 - x + 1 = (x - 1)(x^4 + x^3 - 1)$

$x^7 + x^6 + x^4 - 1 = x^4(x^3 + 1) + x^6 - 1$

$= x^4(x^3 + 1) + (x^3 - 1)(x^3 + 1)$

$= (x^3 + 1)(x^4 + x^3 - 1)$

$\therefore$ Highest Common Factor is $x^4 + x^3 - 1$.

Vedic Method by Factorization

$x^5 - x^3 - x + 1$

$S_c = 0$

$\therefore x - 1$ is one factor.

By Paravartya division, we get $x^4 + x^3 - 1$ as another factor.

$\therefore x^5 - x^3 - x + 1 = (x - 1)(x^4 + x^3 - 1)$

$x^7 + x^6 + x^4 - 1$

$= x^4(x^3 + 1) + x^6 - 1$

$= x^4(x^3 + 1) + (x^3 + 1)(x^3 - 1)$

$= (x^3 + 1)(x^4 + x^3 - 1)$

$\therefore$ Highest Common Factor is $x^4 + x^3 - 1$.

15) $1 + x + x^3 - x^5,\ 1 - x^4 - x^6 + x^7$

Current Method by Factorization

$1 + x + x^3 - x^5$
If $x = -1$
$1 - 1 + (-1) - (-1) = 0$
$\therefore x + 1$ is one factor.

$$
\begin{array}{l}
x+1)\ -x^5 \qquad + x^3 + x + 1\ (-x^4 + x^3 + 1 \\
\qquad \underline{-x^5 - x^4 \qquad\qquad\quad} \\
\qquad\qquad x^4 + x^3 + x + 1 \\
\qquad\qquad \underline{x^4 + x^3 \qquad\qquad} \\
\qquad\qquad\qquad\quad x + 1 \\
\qquad\qquad\qquad\quad \underline{x + 1} \\
\qquad\qquad\qquad\qquad 0
\end{array}
$$

$1 + x + x^3 - x^5 = (x + 1)(1 + x^3 - x^4)$
$1 - x^4 - x^6 + x^7$
$= 1 - x^6 + x^7 - x^4$
$= (1 - x^3)(1 + x^3) + x^4(x^3 - 1)$
$= (1 - x^3)(1 + x^3) - x^4(1 - x^3)$
$= (1 - x^3)(1 + x^3 - x^4)$

$\therefore$ Highest Common Factor is $1 + x^3 - x^4$.

Vedic Method by Factorization

$1 + x + x^3 - x^5$
$S_o = S_e$

$\therefore (x + 1)$ is one factor.
By Paravartya division, we get,
$1 + x^3 - x^4$ as another factor.
$1 - x^4 - x^6 + x^7$
$= 1 - x^6 - x^4 + x^7$
$= (1 - x^3)(1 + x^3) - x^4(1 - x^3)$
$= (1 - x^3)(1 + x^3 - x^4)$

$\therefore$ Highest Common Factor is $1 + x^3 - x^4$.

16) $6 - 8a - 32a^2 - 18a^3,\ 20 - 35a - 95a^2 - 40a^3$

Current Method by Factorization

$6 - 8a - 32a^2 - 18a^3$
$= 2(3 - 4a - 16a^2 - 9a^3)$
$= 3 - 4a - 16a^2 - 9a^3$
If $a = -1$
$3 - 4(-1) - 16(1) - 9(-1)$
$= 3 + 4 - 16 + 9 = 0$
$\therefore 1 + a$ is one factor.

$$
\begin{array}{l}
a + 1)\ -9a^3 - 16a^2 - 4a + 3\ (\ -9a^2 - 7a + 3 \\
\quad \underline{-9a^3 - \ 9a^2} \\
\qquad -\ 7a^2 - 4a + 3 \\
\qquad \underline{-\ 7a^2 - 7a} \\
\qquad\qquad 3a + 3 \\
\qquad\qquad \underline{3a + 3} \\
\qquad\qquad\quad 0
\end{array}
$$

Therefore,
$6 - 8a - 32a^2 - 18a^3 = 3(a + 1)(3 - 7a - 9a^2)$

$20 - 35a - 95a^2 - 40a^3$
$= 5(4 - 7a - 19a^2 - 8a^3)$
$4 - 7a - 19a^2 - 8a^3$

If $a = -1$
$4 - 7(-1) - 19(1) - 8(-1)$
$= 4 + 7 - 19 + 8 = 0$
$\therefore a + 1$ is one factor.

$$
\begin{array}{l}
a + 1)\ -8a^3 - 19a^2 - 7a + 4\ (\ -8a^2 - 11a + 4 \\
\quad \underline{-8a^3 - \ 8a^2} \\
\qquad -11a^2 - 7a + 4 \\
\qquad \underline{-11a^2 - 11a} \\
\qquad\qquad 4a + 4 \\
\qquad\qquad \underline{4a + 4} \\
\qquad\qquad\quad 0
\end{array}
$$

Therefore,
$20 - 35a - 95a^2 - 40a^3 = 5(a + 1)(4 - 11a - 8a^2)$

$\therefore$ Highest Common Factor is $a + 1$.

Vedic Method by Factorization

$6 - 8a - 32a^2 - 18a^3 = 2(3 - 4a - 16a^2 - 9a^3)$
$3 - 4a - 16a^2 - 9a^3$
$S_o = S_e$
$\therefore a + 1$ is one factor.

By Paravartya division, we get,
$(3 - 7a - 9a^2)$ as another factor.

$6 - 8a - 32a^2 - 18a^3 = 2(a + 1)(3 - 7a - 9a^2)$
$20 - 35a - 95a^2 - 40a^3 = 5(4 - 7a - 19a^2 - 8a^3)$
$4 - 7a - 19a^2 - 8a^3$
$S_o = S_e$

$\therefore a + 1$ is one factor.
By Paravartya division, we get,
$(4 - 11a - 8a^2)$ as another factor.

Therefore,
$20 - 35a - 95a^2 - 40a^3 = 5(a + 1)(4 - 11a - 8a^2)$

$\therefore$ Highest Common Factor is $a + 1$.

17) $9x^2 - 15x^3 - 45x^4 - 12x^5$, $42x - 49x^2 - 203x^3 - 84x^4$

Current Method by Factorization

$9x^2 - 15x^3 - 45x^4 - 12x^5$
$= 3x^2(3 - 5x - 15x^2 - 4x^3)$
Let $E = 3 - 5x - 15x^2 - 4x^3$
$9E = 27 - 45x - 135x^2 - 36x^3$
$(3 + 4x)^3 = 27 + 108x + 144x^2 + 64x^3$
$9E = (3 + 4x)^3 - 153x - 279x^2 - 100x^3$
$= (3 + 4x)^3 - x(153 + 279x + 100x^2)$
$= (3 + 4x)^3 - x(153 + 204x + 75x + 100x^2)$
$= (3 + 4x)^3 - x[51(3 + 4x) + 25x(3 + 4x)]$
$= (3 + 4x)^3 - x(3 + 4x)(51 + 25x)$
$= (3 + 4x)[(3 + 4x)^2 - x(51 + 25x)]$
$= (3 + 4x)(9 + 24x + 16x^2 - 51x - 25x^2)$
$= (3 + 4x)(9 - 27x - 9x^2)$
$\therefore E = (3 + 4x)(1 - 3x - x^2)$
$42x - 49x^2 - 203x^3 - 84x^4$
$= 7x(6 - 7x - 29x^2 - 12x^3)$
$6 - 7x - 29x^2 - 12x^3$
If $x = -2$
$6 - 7(-2) - 29(4) - 12(-8) = 0$
$\therefore x + 2$ is one factor.

$$
\begin{array}{l}
x + 2)\ -12x^3 - 29x^2 - 7x + 6\ (-12x^2 - 5x + 3 \\
\quad\ \underline{-12x^3 - 24x^2} \\
\qquad\quad -5x^2 - 7x + 6 \\
\qquad\quad \underline{-5x^2 - 10x} \\
\qquad\qquad\qquad 3x + 6 \\
\qquad\qquad\qquad \underline{3x + 6} \\
\qquad\qquad\qquad\quad 0
\end{array}
$$

$12x^2 + 5x - 3$
$= 12x^2 + 9x - 4x - 3$
$= 3x(4x + 3) - 1(4x + 3)$
$= (3x - 1)(4x + 3)$

$\therefore 42x - 49x^2 - 203x^3 - 84x^4$
$= 7x(x + 2)(3 + 4x)(1 - 3x)$
$\therefore$ Highest Common Factor is $x(3 + 4x)$

Vedic Method by Factorization

$9x^2 - 15x^3 - 45x^4 - 12x^5 \longrightarrow P$
$= 3x^2(3 - 5x - 15x^2 - 4x^3)$
$- 4x^3 - 15x^2 - 5x + 3$
$S_c = -21$
Factors are $\pm1, \pm3, \pm7, \pm21$
First Term is -4
Factors are $\pm1, \pm2, \pm4$
$\therefore$ Factors are
$(x + a)(-4)(a^2 + bx + c)$
or $(4x + a)(-x^2 + bx + c)$
or $(2x + a)(-2x^2 + bx + c)$
Last term is 3. Factors are $\pm1, \pm3$.
Possible factors are $(4x - 1)(4x + 3)(4x - 3)(2x + 1)(2x - 1)(2x - 3)$
If $(2x-3)$ is a factor then another factor is $(-2x^2+bx-1)$
By Gunita Samuccay Sutra
$(-1)(-2 + b - 1) = -21$
$-3 + b = 21$
$b = 24.$
x^2 coefficient in the given expression is -15. By this factor x^2 coefficient is
$6 + 2b$
$= 6 + 48 = 54.$
$\therefore$ This combination is not correct.

If $(2x - 1)$ is a factor then another factor is $(-2x^2 + bx - 3)$
By Gunita Samuccaye Sutra
$(1)(-2 + b - 3) = -21$
$-5 + b = -21$
$b = -16$
x^2 coefficient $= 2 + 2b = 2 - 32 = -30$
$\therefore$ This combination is also not correct.
If $(2x + 1)$ is a factor then another factor is $(-2x^2 + bx + 3)$
By Gunita Samuccaye Sutra
$(3)(-2 + b + 3) = -21$
$1 + b = -7$
$b = -8$
x^2 coefficient $= -2 + 2b = -2 - 16 = -18$
$\therefore$ This is not correct.
If $(4x - 3)$ is a factor the another factor is

$(-x^2 + bx - 1)$
By Gunita Samuccaye Sutra
$(1)(-1 + b - 1) = -21$
$-2 + b = -21$
$b = -19$
x^2 coefficient $= 3 + 4b = 3 - 76 = -73 \neq -15$

If $(4x - 1)$ is a factor then another factor is
$(-x^2 + bx - 3)$
By Gunita Samuccaye Sutra
$(3)(-1 + b - 3) = -21$
$-4 + b = -7 \Rightarrow b = -3.$
x^2 coefficient $= 1 + 4b$
$= 1 - 12 = -11 \neq -15.$

If $(4x + 3)$ is a factor then another factor is
$(-x^2 + bx + 1)$
By Gunita Samuccaye Sutra
$(7)(-1 + b + 1) = -21$
$b = -3$
x^2 coefficient $= -3 + 4b = -3 - 12 = -15$
x coefficient $= 3b + 4 = -9 + 4 = -5$

$\therefore 9x^2 - 15x^3 - 45x^4 - 12x^5$
$= 3x^2(4x + 3)(1 - 3x - x^2)$
$42x - 49x^2 - 203x^3 - 84x^4$
$= 7x(6 - 7x - 29x^2 - 12x^3)$
$-12x^3 - 29x^2 - 7x + 6$
First term is -12

Factors are $\pm 1, \pm 12, \pm 2, \pm 6, \pm 3, \pm 4$
Possible factors are
$(x + a)(-12x^2 + bx + c)$
$(12x + a)(-x^2 + bx + c)$
$(2x + a)(-6x^2 + bx + c)$
$(6x + a)(-2x^2 + bx + c)$
$(3x + a)(-4x^2 + bx + c)$
$(4x + a)(-3x^2 + bx + c)$

$S_c = -42$
Factors are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 7, \pm 14, \pm 21,$
± 42
Last term is 6
Factors are $\pm 1, \pm 6, \pm 3, \pm 2$
Possible factors are $(x+6)(x-3)(x+2)(x-2)$
$(2x + 1)(2x - 1)(2x - 3)$
$(6x + 1)(3x + 1)(3x - 1)(3x - 2)$
$(4x - 1)(4x + 3)(4x - 3)$

On verification we get
$(x+2)(3x-1)(4x+3)$ as factors

$\therefore 42x - 49x^2 - 203x^3 - 84x^4$
$= 7x(x+2)(4x+3)(1-3x)$

$\therefore$ Highest Common Factor is $x(4x+3)$

18) $3x^5 - 5x^3 + 2$, $2x^5 - 5x^2 + 3$

Current Method by Factorization

$3x^5 - 5x^3 + 2$ —— 1

If $x = 1$ or $3 - 5 + 2 = 0$

$\therefore x - 1$ is one factor.

$x - 1)\ 3x^5 \quad - 5x^3 + 2\ (3x^4 + 3x^3 - 2x^2 - 2x - 2$

$\underline{3x^5 - 3x^4}$

$3x^4 - 5x^3 + 2$

$\underline{3x^4 - 3x^3}$

$- 2x^3 + 2$

$\underline{- 2x^3 + 2x^2}$

$- 2x^2 + 2$

$\underline{- 2x^2 + 2x}$

$- 2x + 2$

$\underline{- 2x + 2}$

0

$3x^4 + 3x^3 - 2x^2 - 2x - 2$

If $x = 1$

$3 + 3 - 2 - 2 - 2 = 0$

$\therefore (x - 1)$ is one factor.

$x - 1)\ 3x^4 + 3x^3 - 2x^2 - 2x - 2\ (3x^3 + 6x^2 + 4x + 2$

$\underline{3x^4 - 3x^3}$

$6x^3 - 2x^2$

$\underline{6x^3 - 6x^2}$

$4x^2 - 2x$

$\underline{4x^2 - 4x}$

$2x - 2$

$\underline{2x - 2}$

0

$\therefore 3x^5 - 5x^3 + 2 = (x - 1)^2(3x^3 + 6x^2 + 4x + 2)$

$2x^5 - 5x^2 + 3$

If $x = 1$

$2 - 5 + 3 = 0$

$\therefore (x - 1)$ is one factor.

$x - 1)\ 2x^5 \quad - 5x^2 + 3 (2x^4 + 2x^3 + 2x^2 - 3x - 3$

$\underline{2x^5 - 2x^4}$

$2x^4 \quad - 5x^2 + 3$

$\underline{2x^4 - 2x^3}$

$2x^3 - 5x^2 + 3$

$\underline{2x^3 - 2x^2}$

$- 3x^2 + 3$

$\underline{- 3x^2 + 3x}$

$- 3x + 3$

$\underline{- 3x + 3}$

0

Vedic Method by Factorization

$3x^5 - 5x^3 + 2$

$S_c = 0$

$\therefore x - 1$ is one factor

By Paravartya division, we get,

$3x^4 + 3x^3 - 2x^2 - 2x - 2$ as another factor.

$3x^4 + 3x^3 - 2x^2 - 2x - 2$

$S_c = 0$

$\therefore x - 1$ is one factor

By Paravartya division, we get

$3x^3 + 6x^2 + 4x + 2$ as another factor.

$3x^4 + 3x^3 - 2x^2 - 2x - 2$

$S_c = 0$

$\therefore x - 1$ is one factor

By Paravartya division, we get

$3x^3 + 6x^2 + 4x + 2$ as another factor

$\therefore 3x^5 - 5x^3 + 2$

$= (x - 1)^2(3x^3 + 6x^2 + 4x + 2)$

$2x^5 - 5x^2 + 3$

$S_c = 0$ $\therefore$ (x-1) is one factor.

By Paravartya division, we get,

$2x^4 + 2x^3 + 2x^2 - 3x - 3$ as another factor.

$2x^4 + 2x^3 + 2x^2 - 3x - 3$

$S_c = 0$

$\therefore (x - 1)$ is one factor.

By Paravartya division, we get,

$2x^3 + 4x^2 + 6x + 3$ as another factor.

$\therefore 2x^5 - 5x^2 + 3 = (x - 1)^2(2x^3 + 4x^2 + 6x + 3)$

$\therefore$ Highest Common Factor is $(x - 1)^2$

$2x^4 + 2x^3 + 2x^2 - 3x - 3$
If x = 1 or $2 + 2 + 2 - 3 - 3 = 0$

$\therefore$ (x – 1) is one factor.

$$
\begin{array}{l}
x - 1)\ 2x^4 + 2x^3 + 2x^2 - 3x - 3\ (2x^3 + 4x^2 + 6x + 3 \\
\qquad \underline{2x^4 - 2x^3} \\
\qquad\qquad 4x^3 + 2x^2 \\
\qquad\qquad \underline{4x^3 - 4x^2} \\
\qquad\qquad\qquad 6x^2 - 3x \\
\qquad\qquad\qquad \underline{6x^2 - 6x} \\
\qquad\qquad\qquad\qquad 3x - 3 \\
\qquad\qquad\qquad\qquad \underline{3x - 3} \\
\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

$\therefore 2x^5 - 5x^2 + 3 = (x - 1)^2\,(2x^3 + 4x^2 + 6x + 3)$
$\therefore$ Highest Common Factor is $(x - 1)^2$.

19) $2 - 3a + 5a^2 - 2a^3,\ 2 - 5a + 8a^2 - 3a^3$

Current Method by Factorization

$2 - 3a + 5a^2 - 2a^3$
if $a = 2$
$2 - 3(2) + 5(4) - 2(8) = 2 - 6 + 20 - 16 = 0$
$\therefore$ $(a - 2)$ is one factor

$$
\begin{array}{l}
a-2)\ -2a^3 + 5a^2 - 3a + 2\ (-2a^2 + a - 1 \\
\quad\ \ \underline{-2a^3 + 4a^2} \\
\qquad\qquad a^2 - 3a + 2 \\
\qquad\qquad \underline{a^2 - 2a} \\
\qquad\qquad\quad -a + 2 \\
\qquad\qquad\quad \underline{-a + 2} \\
\qquad\qquad\qquad 0
\end{array}
$$

$2 - 3a + 5a^2 - 2a^3 = (a - 2)(-2a^2 + a - 1)$
$2 - 5a + 8a^2 - 3a^3$
If $a = 2$
$2 - 5(2) + 8(4) - 3(8) = 2 - 10 + 32 - 24 = 0$
$\therefore$ $(a - 2)$ is one factor

$$
\begin{array}{l}
a-2)\ -3a^3 + 8a^2 - 5a + 2\ (-3a^2 + 2a - 1 \\
\quad\ \ \underline{-3a^3 + 6a^2} \\
\qquad\qquad 2a^2 - 5a \\
\qquad\qquad \underline{2a^2 - 4a} \\
\qquad\qquad\quad -a + 2 \\
\qquad\qquad\quad \underline{-a + 2} \\
\qquad\qquad\qquad 0
\end{array}
$$

$\therefore$ $2 - 5a + 8a^2 - 3a^3 = (a - 2)(-3a^2 + 2a - 1)$
$\therefore$ $(a - 2)$ is the Highest Common Factor

Vedic Method by Factorization

$2 - 3a + 5a^2 - 2a^3$
$S_c = 2$ factors are 1, 2
Absolute term = 2 factors are 1, 2
$\therefore$ Possible factor is $a - 2$
By Paravartya division
we get $-2a^2 + a - 1$ as another factor
$2 - 3a + 5a^2 - 2a^3 = (a - 2)(-2a^2 + a - 1)$
$2 - 5a + 8a^2 - 3a^3$
$S_c = 2$ factors are 1, 2
Absolute term is 2 factors are 1, 2
$\therefore$ Possible factor is $a - 2$
$\therefore$ By Paravartya division
we get $-3a^2 + 2a - 1$ as another factor
$\therefore$ $2 - 5a + 8a^2 - 3a^3 = (a - 2)(-3a^2 + 2a - 1)$
$\therefore$ $a - 2$ is the Highest Common Factor

Conclusion

Request: Attention

Preparation of this Lecture notes-V has taken unusually longer time due to many personal inconveniences during the course of the work. But we are glad that it could be successfully completed. The readers are requested to kindly go through the details. If they notice any mistakes or errors which might have occurred due to oversight, they are requested to bring them to the notice of the Project Director (Prof. C. Santhamma) so that they can be got rectified if necessary .The Director feels that the chapter on Differentiation and Integration needs a more comprehensive and systematic work to cover all types of equations including various degrees and orders, which feature the director feels, that it is worthy of an attempt and will be taken up later as a separate work making use of Vedic principles.

--- Prof. C. Santhamma

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