Vedic Mathematics - Methods

Preface

The Sanskrit word Veda is derived from the root Vid, meaning to know without limit. The word Veda covers all Veda-sakhas known to humanity. The Veda is a repository of all knowledge, fathomless, ever revealing as it is delved deeper.

Swami Bharati Krishna Tirtha (1884-1960), former Jagadguru Sankaracharya of Puri culled a set of 16 Sutras (aphorisms) and 13 Sub - Sutras (corollaries) from the Atharva Veda. He developed methods and techniques for amplifying the principles contained in the aphorisms and their corollaries, and called it Vedic Mathematics.

According to him, there has been considerable literature on Mathematics in the Veda-sakhas. Unfortunately most of it has been lost to humanity as of now. This is evident from the fact that while, by the time of Patanjali, about 25 centuries ago, 1131 Veda-sakhas were known to the Vedic scholars, only about ten Veda-sakhas are presently in the knowledge of the Vedic scholars in the country.

The Sutras apply to and cover almost every branch of Mathematics. They apply even to complex problems involving a large number of mathematical operations. Application of the Sutras saves a lot of time and effort in solving the problems, compared to the formal methods presently in vogue. Though the solutions appear like magic, the application of the Sutras is perfectly logical and rational. The computation made on the computers follows, in a way, the principles underlying the Sutras. The Sutras provide not only methods of calculation, but also ways of thinking for their application.

This book on Vedic Mathematics seeks to present an integrated approach to learning Mathematics with keenness of observation and inquisitiveness, avoiding the monotony of accepting theories and working from them mechanically. The explanations offered make the processes clear to the learners. The logical proof of the Sutras is detailed in algebra, which eliminates the misconception that the Sutras are a jugglery.

Application of the Sutras improves the computational skills of the learners in a wide area of problems, ensuring both speed and accuracy, strictly based on rational and logical reasoning. The knowledge of such methods enables the teachers to be more resourceful to mould the students and improve their talent and creativity. Application of the Sutras to specific problems involves rational thinking, which, in the process, helps improve intuition that is the bottom - line of the mastery of the mathematical geniuses of the past and the present such as Aryabhatta, Bhaskaracharya, Srinivasa Ramanujan, etc.

This book makes use of the Sutras and Sub-Sutras stated above for presentation of their application for learning Mathematics at the secondary school level in a way different from what is taught at present, but strictly embodying the principles of algebra for empirical accuracy. The innovation in the presentation is the algebraic proof for every elucidation of the Sutra or the Sub-Sutra concerned.

Sri Sathya Sai Veda Pratishtan

I. Why Vedic Mathematics?

Many Indian Secondary School students consider Mathematics a very difficult subject. Some students encounter difficulty with basic arithmetical operations. Some students feel it difficult to manipulate symbols and balance equations. In other words, abstract and logical reasoning is their hurdle.

Many such difficulties in learning Mathematics enter into a long list if prepared by an experienced teacher of Mathematics. Volumes have been written on the diagnosis of 'learning difficulties' related to Mathematics and remedial techniques. Learning Mathematics is an unpleasant experience to some students mainly because it involves mental exercise.

Of late, a few teachers and scholars have revived interest in Vedic Mathematics which was developed, as a system derived from Vedic principles, by Swami Bharati Krishna Tirthaji in the early decades of the 20th century.

Dr. Narinder Puri of the Roorke University prepared teaching materials based on Vedic Mathematics during 1986 - 89. A few of his opinions are stated hereunder:

i) Mathematics, derived from the Veda, provides one line, mental and super-fast methods along with quick cross checking systems.

ii) Vedic Mathematics converts a tedious subject into a playful and blissful one which students learn with smiles.

iii) Vedic Mathematics offers a new and entirely different approach to the study of Mathematics based on pattern recognition. It allows for constant expression of a student's creativity, and is found to be easier to learn.

iv) In this system, for any problem, there is always one general technique applicable to all cases and also a number of special pattern problems. The element of choice and flexibility at each stage keeps the mind lively and alert to develop clarity of thought and intuition, and thereby a holistic development of the human brain automatically takes place.

v) Vedic Mathematics with its special features has the inbuilt potential to solve the psychological problem of Mathematics - anxiety.

J.T.Glover (London, 1995) says that the experience of teaching Vedic Mathematics' methods to children has shown that a high degree of mathematical ability can be attained from an early stage while the subject is enjoyed for its own merits.

A.P. Nicholas (1984) puts the Vedic Mathematics system as 'one of the most delightful chapters of the 20th century mathematical history'.

Prof. R.C. Gupta (1994) says 'the system has great educational value because the Sutras contain techniques for performing some elementary mathematical operations in simple ways, and results are obtained quickly'.

Prof. J.N. Kapur says 'Vedic Mathematics can be used to remove math-phobia, and can be taught to (school) children as enrichment material along with other high speed methods'.

Dr. Michael Weinless, Chairman of the Department of Mathematics at the M.I.U, Iowa says thus: 'Vedic Mathematics is easier to learn, faster to use and less prone to error than conventional methods. Furthermore, the techniques of Vedic Mathematics not only enable the students to solve specific mathematical problems; they also develop creativity, logical thinking and intuition.'

Keeping the above observations in view, let us enter Vedic Mathematics as given by Sri Bharati Krishna Tirthaji (1884 - 1960), Sankaracharya of Govardhana Math, Puri. Entering into the methods and procedures, one can realize the importance and applicability of the different formulae (Sutras) and methods.

II. Vedic Mathematical Formulae

What we call VEDIC MATHEMATICS is a mathematical elaboration of '**Sixteen Simple Mathematical formulae from theVedas**' as brought out by **Sri Bharati Krishna Tirthaji**. In the text authored by the Swamiji, nowhere has the list of the Mathematical formulae (Sutras) been given. But the Editor of the text has compiled the list of the formulae from stray references in the text. The list so compiled contains Sixteen Sutras and Thirteen Sub - Sutras as stated hereunder.

SIXTEEN SUTRAS

1. एकाधिकेन पूर्वेण
   Ekādhikena Pūrveṇa (also a corollary)
2. निखिलं नवतश्चरमं दशतः
   Nikhilaṃ Navataścaramaṃ Daśataḥ
3. ऊर्ध्वतिर्यग्भ्याम्
   Ūrdhva-tiryagbhyām
4. परावर्त्य योजयेत्
   Parāvartya Yojayet
5. शून्यं साम्यसमुच्चये
   Śūnyaṃ Sāmyasamuccaye
6. (आनुरूप्ये) शून्यमन्यत्
   (Ānurūpye) Śūnyamanyat
7. संकलनव्यवकलनाभ्याम्
   Saṅkalana-vyavakalanābhyam (also a corollary)
8. पूरणापूरणाभ्याम्
   Pūraṇāpūraṇābhyām
9. चलनकलनाभ्याम्
   Calana-Kalanābhyām
10. यावदूनम्
    Yāvadūnam
11. व्यष्टिसमष्टिः
    Vyaṣṭisamaṣṭiḥ
12. शेषाण्यङ्केन चरमेण
    Śeṣāṇyaṅkena Carameṇa
13. सोपान्त्यद्वयमन्त्यम्
    Sopāntyadvayamantyam
14. एकन्यूनेन पूर्वेण
    Ekanyūnena Pūrveṇa
15. गुणितसमुच्चयः
    Guṇitasamuccayaḥ
16. गुणकसमुच्चयः
    Guṇakasamuccayaḥ

THIRTEEN SUB – SUTRAS

1. आनुरूप्येण
   Ānurūpyeṇa
2. शिष्यते शेषसंज्ञः
   Śiṣyate Śeṣasaṃjñaḥ
3. आद्यमाद्येनान्त्यमन्त्येन
   Ādyamādyenāntya-mantyena
4. केवलैः सप्तकं गुण्यात्
   Kevalaiḥ Saptakaṃ Guṇyāt
5. वेष्टनम्
   Veṣṭanaṃ
6. यावदूनं तावदूनम्
   Yāvadūnaṃ Tāvadūnaṃ
7. यावदूनं तावदूनीकृत्य वर्गं च योजयेत्
   Yāvadunaṃ Tāvadūnikṛtya Vargañca Yojayet
8. अन्त्ययोर्दशकेऽपि
   Antyayordaśake'pi
9. अन्त्ययोरेव
   Antyayoreva
10. समुच्चयगुणितः
    Samuccayaguṇitaḥ
11. लोपस्थापनाभ्याम्
    Lopanasthāpanābhyām
12. विलोकनम्
    Vilokanam
13. गुणितसमुच्चयः समुच्चयगुणितः
    Gunitasamuccayaḥ
    Samuccayaguṇitaḥ

In the text, the words Sutra, aphorism, formula are used synonymously. So are also the words Upa-sutra, Sub-sutra, Sub-formula, corollary used.

Now we shall have the literal meaning, contextual meaning, process, different methods of application along with examples for the Sutras. Explanation, methods, further short-cuts, algebraic proof, etc follow. What follows relates to a single formula or a group of formulae related to the methods of Vedic Mathematics.

1. Ekadhikena Purvena

The Sutra (formula) Ekādhikena Pūrvena means: "By one more than the previous one".

i) Squares of numbers ending in 5 :

Now we relate the sutra to the 'squaring of numbers ending in 5'. Consider the example 25^2.

Here the number is 25. We have to find out the square of the number. For the number 25, the last digit is 5 and the 'previous' digit is 2. Hence, 'one more than the previous one', that is, 2+1=3. The Sutra, in this context, gives the procedure'to multiply the previous digit 2 by one more than itself, that is, by 3'. It becomes the L.H.S (left hand side) of the result, that is, 2 X 3 = 6. The R.H.S (right hand side) of the result is5^2, that is, 25.

Thus $25^2 = 2 \times 3 / 25 = 625$.

In the same way,

$35^2 = 3 \times (3+1) /25 = 3 \times 4/ 25 = 1225$;

$65^2 = 6 \times 7 / 25 = 4225$;

$105^2 = 10 \times 11/25 = 11025$;

$135^2 = 13 \times 14/25 = 18225$;

Apply the formula to find the squares of the numbers 15, 45, 85, 125, 175 and verify the answers.

Algebraic proof:

a) Consider $(ax + b)^2 \equiv a^2. x^2 + 2abx + b^2$.

Thisidentity for $x = 10$ and $b = 5$ becomes

$$(10a + 5)^2 = a^2 . 10^2 + 2. 10a . 5 + 5^2$$

$$= a^2 . 10^2 + a.10^2 + 5^2$$

$$= (a^2 + a) . 10^2 + 5^2$$

$= a\,(a + 1)\,.\,10^2 + 25.$

Clearly 10a + 5 represents two-digit numbers 15, 25, 35, -------,95 for the values a = 1, 2, 3, -------,9 respectively. In such a case the number $(10a + 5)^2$ is of the form whose L.H.S is **a (a + 1)** and R.H.S is 25, that is, **a (a + 1) / 25**.

Thus any such two digit number gives the result in the same fashion.

Example: $45 = (40 + 5)^2$, It is of the form $(ax+b)^2$ for a = 4, x=10 and b = 5. giving the answer a (a+1) / 25 that is, 4 (4+1) / 25 + 4 X 5 / 25 = 2025.

b) Any three digit number is of the form ax^2+bx+c for x =10, $a \neq 0$, $a, b, c \in W$.

Now $(ax^2+bx+ c)^2 = a^2 x^4 + b^2x^2 + c^2 + 2abx^3 + 2bcx + 2cax^2$

$$= a^2 x^4+2ab.x^3+(b^2+ 2ca)x^2+2bc\,.\,x+ c^2.$$

This identity for x = 10, c = 5becomes $(a\,.\,10^2 + b\,.10 + 5)^2$

$$= a^2.10^4+ 2.a.b.10^3 + (b^2+ 2.5.a)10^2+ 2.b.5.10 + 5^2$$

$$= a^2.10^4+ 2.a.b.10^3 + (b^2 + 10\,a)10^2 + b.10^2+ 5^2$$

$$= a^2.10^4+ 2ab.10^3+ b^2.10^2+ a\,.\,10^3 + b\,10^2+ 5^2$$

$$= a^2.10^4 + (2ab + a).10^3 + (b^2+ b)10^2 +5^2$$

$$= [\,a^2.10^2 +2ab.10 + a.10 + b^2 + b]\,10^2+ 5^2$$

$$= (10a + b)\,(\,10a+b+1).10^2 + 25$$

$= P\ (P+1)\ 10^2 + 25$, where $P = 10a+b$.

Hence any three digit number whose last digit is 5 gives the same result as in **(a)** for P=10a + b, the 'previous' of 5.

Example : $165^2 = (1\ .\ 10^2 + 6\ .\ 10 + 5)^2$.

It is of the form $(ax^2+bx+c)^2$ for a = 1, b = 6, c = 5 and x = 10. It gives the answer P(P+1) / 25, where P = 10a + b = 10 X 1 + 6 = 16, the 'previous'. The answer is 16 (16+1) / 25 = 16 X 17 / 25 = 27225.

Apply *Ekadhikena purvena* to find the squares of the numbers 95, 225, 375, 635, 745, 915, 1105, 2545.

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ii) Vulgar fractions whose denominators are numbers ending in NINE :

We now take examples of 1 / a9, where a = 1, 2, -----, 9. In the conversion of such vulgar fractions into recurring decimals, Ekadhika process can be effectively used both in division and multiplication.

a) Division Method : Value of 1 / 19.

The numbers of decimal places before repetition is the difference of numerator and denominator, i.e.,, 19 -1=18 places.

For the denominator 19, the *purva* (previous) is 1.

Hence *Ekadhikena purva* (one more than the previous) is 1 + 1 = 2.

The sutra is applied in a different context. Now the method of division is as follows:

Step. 1 : Divide numerator 1 by 20.

i.e.,, 1 / 20 = 0.1 / 2 = $._{1}0$ (0 times, 1 remainder)

Step. 2 : Divide 10 by 2

i.e.,, 0.0_05(5 times, 0 remainder)

Step. 3 : Divide 5 by 2

i.e.,, 0.05_12 (2 times, 1 remainder)

Step. 4 : Divide $_12$ i.e.,, 12 by 2

i.e.,, 0.0526 (6 times, No remainder)

Step. 5 : Divide 6 by 2

i.e.,, 0.05263 (3 times, No remainder)

Step. 6 : Divide 3 by 2

i.e.,, 0.05263_11(1 time, 1 remainder)

Step. 7 : Divide $_11$ i.e.,, 11 by 2

i.e.,, 0.052631_15 (5 times, 1 remainder)

Step. 8 : Divide $_15$ i.e.,, 15 by 2

i.e.,, 0.0526315_17 (7 times, 1 remainder)

Step. 9 : Divide $_17$ i.e.,, 17 by 2

i.e.,, $0.05263157\ _18$ (8 times, 1 remainder)

Step. 10 : Divide $_18$ i.e.,, 18 by 2

i.e.,, 0.0526315789 (9 times, No remainder)

Step. 11 : Divide 9 by 2

i.e.,, $0.0526315789\ _14$ (4 times, 1 remainder)

Step. 12 : Divide $_14$ i.e.,, 14 by 2

i.e.,, 0.052631578947 (7 times, No remainder)

Step. 13 : Divide 7 by 2

i.e.,, $0.052631578947{}_{1}3$ (3 times, 1 remainder)

Step. 14 : Divide ${}_{1}3$ i.e.,, 13 by 2

i.e.,, $0.0526315789473{}_{1}6$ (6 times, 1 remainder)

Step. 15 : Divide ${}_{1}6$ i.e.,, 16 by 2

i.e.,, 0.052631578947368 (8 times, No remainder)

Step. 16 : Divide 8 by 2

i.e.,, 0.0526315789473684 (4 times, No remainder)

Step. 17 : Divide 4 by 2

i.e.,, 0.05263157894736842 (2 times, No remainder)

Step. 18 : Divide 2 by 2

i.e.,, 0.052631578947368421 (1 time, No remainder)

Now from step 19, i.e.,, dividing 1 by 2, Step 2 to Step. 18 repeats thus giving

$$1 / 19 = \overline{0.052631578947368421} \text{ or } 0.\dot{0}5263157894736842\dot{1}$$

Note that we have completed the process of division only by using '2'. Nowhere the division by 19 occurs.

b) Multiplication Method: Value of 1 / 19

First we recognize the last digit of the denominator of the type 1 / a9. Here the last digit is 9.

For a fraction of the form in whose denominator 9 is the last digit, we take the case of 1 / 19 as follows:

For 1 / 19, 'previous' of 19 is 1. And one more than of it is 1 + 1 = 2.

Therefore 2 is the multiplier for the conversion. We write the last digit in the numerator as 1 and follow the steps leftwards.

Step. 1 : 1

Step. 2 : 21(multiply 1 by 2, put to left)

Step. 3 : 421(multiply 2 by 2, put to left)

Step. 4 : 8421(multiply 4 by 2, put to left)

Step. 5 : $_1$68421 (multiply 8 by 2 =16, 1 carried over, 6 put to left)

Step. 6 : $_1$368421 (6 X 2 =12,+1 [carry over]

= 13, 1 carried over, 3 put to left)

Step. 7 : 7368421 (3 X 2, = 6 +1 [Carryover]

= 7, put to left)

Step. 8 : $_1$47368421 (as in the same process)

Step. 9 : 947368421 (Do – continue to step 18)

Step. 10 : $_1$8947368421

Step. 11 : $_1$78947368421

Step. 12 : $_1$578947368421

Step. 13 : $_1$1578947368421

Step. 14 : 31578947368421

Step. 15 : 631578947368421

Step. 16 : $_1$2631578947368421

Step. 17 : 52631578947368421

Step. 18 : 1052631578947368421

Now from step 18 onwards the same numbers and order towards left continue.

Thus **1 / 19 = 0.052631578947368421**

It is interesting to note that we have

i) not at all used division process

ii) instead of dividing 1 by 19 continuously, just multiplied 1 by 2 and continued to multiply the resultant successively by 2.

Observations :

a) For any fraction of the form 1 / a9 i.e.,, in whose denominator 9 is the digit in the units place and a is the set of remaining digits, the value of the fraction is in recurring decimal form and the repeating block's right most digit is 1.

b) Whatever may be a9, and the numerator, it is enough to follow the said process with (a+1) either in division or in multiplication.

c) Starting from right most digit and counting from the right, we see (in the given example 1 / 19)

Sum of 1st digit + 10th digit = 1 + 8 = 9

Sum of 2nd digit + 11th digit = 2 + 7 = 9

\- - - - - - - - -- - - - - - - - - - - - - - - - - -

Sum of 9th digit + 18th digit = 9+ 0 = 9

From the above observations, we conclude that if we find first 9 digits, further digits can be derived as complements of 9.

i) Thus at the step 8 in division process we have 0.0526315_17 and next step. 9 gives 0.052631578

Now the complements of the numbers

0, 5, 2, 6, 3, 1, 5, 7, 8 from 9

9, 4, 7, 3, 6, 8, 4, 2, 1 follow the right order

i.e.,, 0.052631578947368421

Now taking the multiplication process we have

Step. 8 : $_147368421$

Step. 9 : 947368421

Now the complements of 1, 2, 4, 8, 6, 3, 7, 4, 9 from 9
i.e.,, 8, 7, 5, 1, 3, 6, 2, 5, 0 precede in successive steps, giving the answer.

0.052631578947368421.

d) When we get (Denominator – Numerator) as the product in the multiplicative process, half the work is done. We stop the multiplication there and mechanically write the remaining half of the answer by merely taking down complements from 9.

e) Either division or multiplication process of giving the answer can be put in a single line form.

Algebraic proof :

Any vulgar fraction of the form 1 / a9 can be written as

$1 / a9 = 1 / ((a + 1) x - 1)$ where $x = 10$

$$= \frac{1}{(a+1)x\left[1 - 1/(a+1)x\right]}$$

$$= \frac{1}{(a+1)x}\left[1 - 1/(a+1)x\right]^{-1}$$

$$= \frac{1}{(a+1)x}\left[1 + 1/(a+1)x + 1/(a+1)x^2 + \text{----------}\right]$$

$$= 1/(a+1)x + 1/(a+1)^2x^2 + 1/(a+1)^3x^3 + \text{----}\textit{ad infinitum}$$

$$= 10^{-1}(1/(a+1)) + 10^{-2}(1/(a+1)^2) + 10^{-3}(1/(a+1)^3) + \text{---}\textit{ad infinitum}$$

This series explains the process of *ekadhik*.

Now consider the problem of 1 / 19. From above we get

$$1 / 19 = 10^{-1} (1/(1+1)) + 10^{-2} (1/(1+1)^2) + 10^{-3} (1/(1+1)^3) + ----$$

(since a=1)

$= 10^{-1} (1/2) + 10^{-2} (1/2)^2 + 10^{-3} (1/3)^3 +$ ----------

$= 10^{-1} (0.5) + 10^{-2} (0.25) + 10^{-3} (0.125)+$ ----------

= 0.05 + 0.0025 + 0.000125 + 0.00000625+ - - - - - - -

= 0.052631 - - - - - - -

Example1 :

1. Find 1 / 49 by *ekadhikena* process.

Now 'previous' is 4. 'One more than the previous' is 4 + 1 = 5.

Now by division right ward from the left by '5'.

1 / 49 = $._1 0$ - - - - - - - - - - - - -(divide 1 by 50)

= .02 - - - - - - - - - (divide 2 by 5, 0 times, 2 remainder)

= $.02_2 0$ - - - - - - --(divide 20 by 5, 4 times)

= .0204 - - - - - - -(divide 4 by 5, 0 times, 4 remainder)

= $.0204_4 0$ -- - -- - (divide 40 by 5, 8 times)

= .020408 - - - - - (divide 8 by 5, 1 time, 3 remainder)

= $.020408_3 1$ - - - -(divide 31 by 5, 6 times, 1 remainder)

= $.0204081_1 6$ - - - - - - - continue

$= .02040816_{1}3_{3}2_{2}6_{1}5306_{1}1_{1}2_{2}2_{2}4_{4}4_{4}8$ - -- - - - -

On completing 21 digits, we get 48

i.e.,,Denominator - Numerator = 49 – 1 = 48 stands.

i.e, half of the process stops here. The remaining half can be obtained as complements from 9.

$$\text{Thus } 1 / 49 = 0.\dot{0}2040816326530612244 8$$

$$97959183673469387755\dot{1}$$

Now finding 1 / 49 by process of multiplication left ward from right by 5, we get

1 / 49 = ---1

= --51

= --$_{2}551$

= ---$_{2}7551$

= ---- ${}_{4}8{}_{3}9{}_{4}7{}_{2}9{}_{4}59{}_{4}1{}_{1}8{}_{3}3{}_{3}6{}_{1}7{}_{2}3{}_{3}4{}_{4}69{}_{4}3{}_{3}8{}_{3}7{}_{2}7551$

i.e.,,Denominator – Numerator = 49 – 1 = 48 is obtained as 5X9+3

(Carry over) = 45 + 3 = 48. Hence half of the process is over. The remaining half is automatically obtained as complements of 9.

Thus 1 / 49 = ---------------979591836734693877551

$$= 0.\dot{0}2040816326530612244 8$$

$$97959183673469387755\dot{1}$$

Example 2: Find 1 / 39 by Ekadhika process.

Now by multiplication method, Ekadhikena purva is 3 + 1 = 4

1 / 39 = -------------------------------------1

= -------------------------------------41

= ----------------------------------$_1$641

= ---------------------------------$_2$5641

= --------------------------------$_2$25641

= -------------------------------$_1$025641

Here the repeating block happens to be block of 6 digits. Now the rule predicting the completion of half of the computation does not hold. The complete block has to be computed by *ekadhika* process.

Now continue and obtain the result. Find reasons for the non–applicability of the said 'rule'.

Find the recurring decimal form of the fractions 1 / 29, 1 / 59, 1 / 69, 1 / 79, 1 / 89 using Ekadhika process if possible. Judge whether the rule of completion of half the computation holds good in such cases.

Note : The Ekadhikena Purvena sutra can also be used for conversion of vulgar fractions ending in 1, 3, 7 such as 1 / 11, 1 / 21, 1 / 31 - - -- ,1 / 13, 1 / 23, - - - -, 1 / 7, 1 / 17, - - - - - by writing them in the following way and solving them.

2. Nikhilam navatascaramam Dasatah

The formula simply means : "all from 9 and the last from 10"

The formula can be very effectively applied in multiplication of numbers, which are nearer to bases like 10, 100, 1000i.e., to the powers of 10 . The procedure of multiplication using the Nikhilam involves minimum number of steps, space, time saving and only mental calculation. The numbers taken can be either less or more than the base considered.

The difference between the number and the base is termed as deviation. Deviation may be positive or negative. Positive deviation is written without the positive sign and the negative deviation, is written using Rekhank (a bar on the number). Now observe the following table.

Number	Base	Number – Base	Deviation
14	10	14 - 10	4
8	10	8 - 10	-2 or $\overline{2}$
97	100	97 - 100	-03 or $\overline{03}$
112	100	112 - 100	12
993	1000	993 - 1000	-007 or $\overline{007}$
1011	1000	1011 - 1000	011

Some rules of the method (near to the base) in Multiplication

a) Since deviation is obtained by Nikhilam sutra we call the method as Nikhilam multiplication.

Eg : 94. Now deviation can be obtained by 'all from 9 and the last from 10' sutrai.e., the last digit 4 is from 10 and remaining digit 9 from 9 gives 06.

b) The two numbers under consideration are written one below the other. The deviations are written on the right hand side.

Eg : Multiply 7 by 8.

Now the base is 10. Since it is near to both the numbers,

7

we write the numbers one below the other. 8

Take the deviations of both the numbers from the base and represent

$7 \quad \bar{3}$

Rekhank or the minus sign before the deviations $\qquad 8 \quad \bar{2}$

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or $7 \quad -3$

$8 \quad -2$

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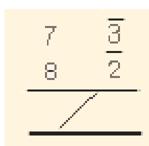
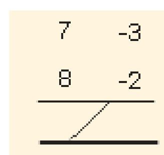
or remainders 3 and 2 implies that the numbers to be multiplied are both less than 10

c) The product or answer will have two parts, one on the left side and the other on the right. A vertical or a slant linei.e., a slash may be drawn for the demarcation of the two parts i.e.,



(or)



d) The R.H.S. of the answer is the product of the deviations of the numbers. It shall contain the number of digits equal to number of zeroes in the base.

i.e., $7 \quad \bar{3}$

$8 \quad \bar{2}$

____

/ (3x2) = 6

Since base is 10, 6 can be taken as it is.

e) L.H.S of the answer is the sum of one number with the deviation of the other. It can be arrived at in any one of the four ways.

i) Cross-subtract deviation 2 on the second row from the original number7 in the first row i.e., 7-2 = 5.

ii) Cross–subtract deviation 3 on the first row from the original number8 in the

second row (converse way of(i))
i.e., 8 - 3 = 5

iii) Subtract the base 10 from the sum of the given numbers.
i.e., (7 + 8) – 10 = 5

iv) Subtract the sum of the two deviations from the base.
i.e., 10 – (3 + 2) = 5

Hence 5 is left hand side of the answer.

$$\begin{array}{lcc} \text{Thus} & 7 & \bar{3} \\ & 8 & \bar{2} \\ \hline & 5\ / & \end{array}$$

Now (d) and (e) together give the solution

$$\begin{array}{cc} 7 & \bar{3} \\ 8 & \bar{2} \\ \hline 5\ / & 6 \end{array} \quad \text{i.e.,} \quad \begin{array}{r} 7 \\ \times\ 8 \\ \hline 56 \end{array}$$

f) If R.H.S. contains less number of digits than the number of zeros in the base, the remaining digits are filled up by giving zero or zeroes on the left side of the R.H.S. If the number of digits are more than the number of zeroes in the base, the excess digit or digits are to be added to L.H.S of the answer.

The general form of the multiplication under Nikhilam can be shown as follows :

Let N_1 and N_2 be two numbers near to a given base in powers of 10, andD_1 and D_2 are their respective deviations from the base. ThenN_1 X N_2 can be represented as

$$\begin{array}{c|c} N_1 & D_1 \\ N_2 & D_2 \\ \hline (N_1 + D_2) \text{ or } & D_1 \times D_2 \\ (N_2 + D_1) & \\ \hline \end{array}$$

Case (i) : Both the numbers are lower than the base.We have already considered the example 7 x 8 , with base 10.

Now let us solve some more examples by taking bases 100 and 1000 respectively.

Ex. 1: Find 97 X 94. Here base is 100. Now following the rules, the working is as follows:

$$\begin{array}{l|l}
97 & \overline{03} \\
94 & \overline{06} \\
\hline
(97 - 06) \text{ or } & 3 \times 6 \\
(94 - 03) & \\
\hline
\end{array}$$

Ex. 2: 98 X 97 Base is 100.

$$\begin{array}{l|l}
98 & \overline{02} \\
97 & \overline{03} \\
\hline
(98 - 03) \text{ or } & 2 \times 3 \\
(97 - 02) & \\
\hline
\end{array}$$

Ex. 3: 75X95. Base is 100.

$$\begin{array}{l|l}
75 & \overline{25} \\
95 & \overline{05} \\
\hline
(75 - 05) \text{ or } & 25 \times 5 \quad = 70/_{1}25 \text{(observe rule-f)} \\
(95 - 25) & \quad\quad\quad\;\; = (70+1)/25 = 7125 \\
\hline
\end{array}$$

Ex. 4: 986 X 989. Base is 1000.

$$\begin{array}{l|l}
986 & \overline{014} \\
989 & \overline{011} \\
\hline
(986 - 14) \text{ or } & 14 \times 11 \quad = 975/154 = 975154 \\
(989 - 14) & \\
\hline
\end{array}$$

Ex. 5: 994X988. Base is 1000.

994 $\overline{006}$

988 $\overline{012}$

(994 - 12) or / 6 X 12 = 982/072 (rule-f)

(988 - 06) = 982072

Ex. 6: 750X995.

750 $\overline{250}$

995 $\overline{005}$

(750 - 005) or / 250 X 5 = $745/_{1}250$ (rule-f)

(995 - 250) = 746250

Case (ii) : Both the numbers are higher than the base.

The method and rules follow as they are. The only difference is the positive deviation. Instead of cross – subtract, we follow cross – add.

Ex. 7: 13X12. Base is 10

13 03

12 02

(13 + 02) or / 3 X 2 = 15/6 = 156

(12 + 03)

Ex. 8: 18X14. Base is 10.

18 08

14 04

(18 + 06) or / 8 X 4 = $22/_{3}2$

(14 + 03) (rule - f , 3 carry over)

= 252

Ex. 9: 104X102. Base is 100.

104 04

102 02

106 / 4x2 = 10608 (rule -f)

Ex. 10: 1275X1004. Base is 1000.

$$
\begin{array}{ll}
1275 & 275 \\
1004 & 004 \\
\hline
\end{array}
$$

$$1279 / 275 \times 4 \quad = \quad 1279 / {}_{1}100 \qquad (\text{ rule -f })$$

$$= \quad 1280100$$

Case (iii): One number is more and the other is less than the base.

In this situation one deviation is positive and the other is negative. So the product of deviations becomes negative. So the right hand side of the answer obtained will therefore have to be subtracted. To have a clear representation and understanding a vinculum is used. It proceeds into normalization.

Ex.11: 13X7. Base is 10

$$
\begin{array}{ll}
13 & 03 \\
7 & \overline{03} \\
\hline
(13 - 03) \text{ or } & 3 \times \overline{3} \quad = \quad 10 / \overline{9} = 100 - 9 = 91 \\
(7 + 03) & \\
\hline
\end{array}
$$

Note : Conversion of common number into vinculum number and vice versa.

Eg :

$9 = 10 - 1 = \overline{11}$

$98 = 100 - 2 = 10\overline{2}$

$196 = 200 - 4 = 20\overline{4}$

$3\overline{2} = 30 - 2 = 28$

$14\overline{5} = 140 - 5 = 135$

$32\overline{2} = 300 - 22 = 278$. etc

The procedure can be explained in detail using Nikhilam Navatascaram Dasatah, Ekadhikenapurvena, Ekanyunena purvena in the foregoing pages of this book.]

Ex. 12: 108 X 94. Base is 100.

108	08	
94	$\overline{06}$	
(108 - 06) or (94 + 08)	$8 \times \bar{6}$	$= 102 / \overline{48} = 10152$ (Since the complement of 48 is 52 - base 10)

Ex. 13: 998 X 1025. Base is 1000.

998	$\overline{002}$	
1025	$\overline{025}$	
(998 - 25) or (1025 + 2)	$\bar{2} \times 25$	$= 1023 / \overline{050} = 1022950$ (Since the complement of 50 is 950 for the base 1000

Algebraic Proof:

Case (i):

Let the two numbers N1 and N2 be less than the selected base say x.

$N_1 = (x-a)$, $N_2 = (x-b)$. Here a and b are the corresponding deviations of the numbersN_1 and N_2 from the base x. Observe that x is a multiple of 10.

Now $N_1 \times N_2 = (x-a)(x-b) = x.x - x.b - a.x + ab$

$= x(x - a - b) + ab$. [rule – e(iv), d]

$= x[(x - a) - b] + ab = x(N_1 - b) + ab$[rule–e(i),d]

or $= x[(x - b) - a] = x(N_2 - a) + ab$. [rule –e (ii),d]

$x(x - a - b) + ab$ can also be written as

$x[(x - a) + (x - b) - x] + ab = x[N_1 + N_2 - x] + ab$ [rule –e(iii),d].

A difficult can be faced, if the vertical multiplication of the deficit digits or deviationsi.e., a.b yields a product consisting of more than the required digits. Then rule-f will enable us to surmount the difficulty.

Case (ii) :

When both the numbers exceed the selected base, we have $N_1 = x + a, N_2 = x + b$, x being the base. Now the identity $(x+a)(x+b) = x(x+a+b) + a.b$ holds good, of course with relevant details mentioned in case(i).

Case (iii) :

When one number is less and another is more than the base, we can use $(x-a)(x+b) = x(x-a+b)-ab$. and the procedure is evident from the examples given.

Find the following products by Nikhilam formula.

1) 7 X 4 2) 93 X 85 3) 875 X 994

4) 1234 X 1002 5) 1003 X 997 6) 11112 X 9998

7) 1234 X 1002 8) 118 X 105

Nikhilam in Division

Consider some two digit numbers (dividends) and same divisor 9. Observe the following example.

i) 13 ÷ 9 The quotient (Q) is 1, Remainder (R) is 4.

$$\text{since } 9 \;)\; 13 \;(\; 1 \\ \;\;\;\;\;\; 9 \\ \overline{\;\;\;4\;\;\;}$$

ii) 34 ÷ 9, Q is 3, R is 7.

iii) 60 ÷ 9, Q is 6, R is 6.

iv) 80 ÷ 9, Q is 8, R is 8.

Now we have another type of representation for the above examples as given hereunder:

i) Split each dividend into a left hand part for the Quotient and right - hand part for the remainder by a slant line or slash.

Eg. 13 as 1 / 3, 34 as 3 / 4 , 80 as 8 / 0.

ii) Leave some space below such representation, draw a horizontal line.

Eg. 1 / 3 3 / 4 8 / 0

_______ , _______ , _______

iii) Put the first digit of the dividend as it is under the horizontal line. Put the same digit under the right hand part for the remainder, add the two and place the sumi.e., sum of the digits of the numbers as the remainder.

Eg.

```
 1 / 3       3 / 4      8 / 0
     1           3          8
_______ ,  _______ ,  _______
 1 / 4       3 / 7      8 / 8
```

Now the problem is over. i.e.,

13 ÷ 9 gives Q = 1, R = 4

34 ÷ 9 gives Q = 3, R = 7

80 ÷ 9 gives Q = 8, R = 8

Proceeding for some more of the two digit number division by 9, we get

a) 21 ÷ 9 as

```
9)  2 /  1     i.e     Q=2, R=3
         2
    ______
    2  /  3
```

b) 43 ÷ 9 as

```
9)  4  /  3     i.e    Q = 4, R = 7.
          4
    _______
    4  /  7
```

The examples given so far convey that in the division of two digit numbers by 9, we canmechanically take the first digit down for the quotient – column and that, by adding the quotient to the second digit, we can get the remainder.

Now in the case of 3 digit numbers, let us proceed as follows.

i)

```
9  )  104  ( 11              9 ) 10   /  4
       99                          1  /  1
```

$$\begin{array}{ccc} \overline{\quad 5 \quad} & \text{as} & \overline{11 \,/\, 5} \end{array}$$

ii)

$$\begin{array}{ccc} 9\ )\ 212\ (\ 23 & & 9\ )\ 21\ /\ 2 \\ \underline{\ 207\ } & \text{as} & \underline{\ \ 2\ /\ 3\ } \\ 5 & & 23\ /\ 5 \end{array}$$

iii)

$$\begin{array}{ccc} 9\ )\ 401\ (44 & & 9\ )\ 40\ /\ 1 \\ \underline{\ 396\ } & \text{as} & \underline{\ \ 4\ /\ 4\ } \\ 5 & & 44\ /\ 5 \end{array}$$

Note that the remainder is the sum of the digits of the dividend. The first digit of the dividend from left is added mechanically to the second digit of the dividend to obtain the second digit of the quotient. This digit added to the third digit sets the remainder. The first digit of the dividend remains as the first digit of the quotient.

Consider 511 ÷ 9

Add the first digit 5 to second digit 1 getting 5 + 1 = 6. Hence Quotient is 56. Now second digit of 56 i.e., 6 is added to third digit 1 of dividend to get the remainder i.e., 1 + 6 = 7

Thus

$$\begin{array}{ll} 9\ ) & 51\ /\ 1 \\ & \underline{\ \ 5\ /\ 6\ } \\ & 56\ /\ 7 \end{array}$$

Q is 56, R is 7.

Extending the same principle even to bigger numbers of still more digits, we can get the results.

Eg : 1204 ÷ 9

i) Add first digit 1 to the second digit 2. 1 + 2 = 3

ii) Add the second digit of quotient 13. i.e., 3 to third digit ‘0’ and obtain the Quotient. 3 + 0 = 3, 133

iii) Add the third digit of Quotient 133 i.e.,3 to last digit '4' of the dividend and write the final Quotient and Remainder. R = 3 + 4 = 7, Q = 133

```
In symbolic form        9 )  120  /  4
                               13  /  3
                             ----------
                             133  /  7
```

Another example.

```
                       9 ) 13210   / 1         132101 ÷ 9
        gives
                            1467   /  7        Q = 14677, R = 8
                           -------------
                            14677  /  8
```

In all the cases mentioned above, the remainder is less than the divisor. What about the case when the remainder is equal or greater than the divisor?

Eg.

```
        9 )  3  /  6           9)   24  /  6
                   3                 2  /   6
             --------               ----------
                          or
             3  /  9 (equal)        26 / 12 (greater).
```

We proceed by re-dividing the remainder by 9, carrying over this Quotient to the quotient side and retaining the final remainder in the remainder side.

```
     9 )  3  /  6             9 ) 24  /  6
             /  3                  2  /  6
          --------                ---------
          3  /  9                 26  /  12
          --------                ---------
          4  /  0                 27  /   3

     Q = 4,       R = 0        Q = 27,       R = 3.
```

When the remainder is greater than divisor, it can also be represented as

```
          9 )  24  /  6
                2  /  6
               ---------
               26 /1 / 2
                     / 1
               ---------
                1    / 3
```

```
      ________
        27   / 3
```

Now consider the divisors of two or more digits whose last digit is 9,when divisor is 89.

We Know 113 = 1 X 89 + 24, Q =1, R = 24

10015 = 112 X 89 + 47, Q = 112, R = 47.

Representing in the previous form of procedure, we have

```
89 )  1  /  13            89 ) 100  / 15
         /   11                 12  /  32
      _________             ____________

      1  /  24                 112  /  47
```

But how to get these? What is the procedure?

Now Nikhilam rule comes to rescue us. The nikhilam states "all from 9 and the last from 10". Now if you want to find 113 ÷ 89, 10015 ÷ 89, you have to apply nikhilam formula on 89 and get the complement 11.Further while carrying the added numbers to the place below the next digit, we have to multiply by this 11.

```
89 )  1  /  13        89 )  100  /15

     ___/  11___      11    11  /         first digit 1 x 11

      1  /  24                1 /  1     total second is 1x11
                          _________22    total of 3rd digit is 2 x 11

                            112  /  47
```

What is 10015 ÷ 98 ? Apply Nikhilam and get 100 – 98 = 02. Set off the 2 digits from the right as the remainder consists of 2 digits. While carrying the added numbers to the place below the next digit, multiply by 02.

Thus

```
98 )   100   /  15

02       02  /            i.e., 10015 ÷ 98 gives
          0  /   0          Q = 102,     R = 19
      _______/   04_

      102   /  19
```

In the same way

```
897 )  11  /  422
---
103     1  /  03
           /   206
       ___________
       12 /  658
```

gives   11,422 ÷ 897,   Q = 12, R=658.

In this way we have to multiply the quotient by 2 in the case of 8, by 3 in the case of 7, by 4 in the case of 6 and so on. i.e., multiply the Quotient digit by the divisors complement from 10. In case of more digited numbers we apply Nikhilam and proceed. Any how, this method is highly useful and effective for division when the numbers are near to bases of 10.

*** Guess the logic in the process of division by 9.**

*** Obtain the Quotient and Remainder for the following problems.**

1) 311 ÷ 9   **2) 120012 ÷ 9**   **3) 1135 ÷ 97**

4) 2342 ÷ 98   **5) 113401 ÷ 997**

6) 11199171 ÷ 99979

Observe that by nikhilam process of division, even lengthier divisions involve no division or no subtraction but only a few multiplications of single digits with small numbers and a simple addition. But we know fairly well that only a special type of cases are being dealt and hence many questions about various other types of problems arise. The answer lies in Vedic Methods.

3. Urdhva - tiryagbhyam

Urdhva – tiryagbhyam is the general formula applicable to all cases of multiplication and also in the division of a large number by another large number. It means

(a) Multiplication of two 2 digit numbers.

Ex.1: Find the product 14 X 12

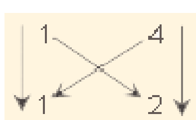
i) The right hand most digit of the multiplicand, the first number (14) i.e.,4 is multiplied by the right hand most digit of the multiplier, the second number (12)i.e., 2. The product 4 X 2 = 8 forms the right hand most part of the answer.

ii) Now, diagonally multiply the first digit of the multiplicand (14) i.e., 4 and second digit of the multiplier (12)i.e., 1 (answer 4 X 1=4); then multiply the second digit of the multiplicand i.e.,1 and first digit of the multiplier i.e., 2 (answer 1 X 2 = 2); add these two i.e.,4 + 2 = 6. It gives the next, i.e., second digit of the answer. Hence second digit of the answer is 6.

iii) Now, multiply the second digit of the multiplicand i.e., 1 and second digit of the multiplieri.e., 1 vertically, i.e., 1 X 1 = 1. It gives the left hand most part of the answer.

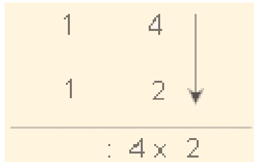
Thus the answer is 16 8.

Symbolically we can represent the process as follows :

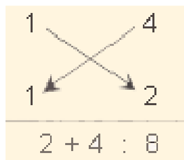


The symbols are operated from right to left .

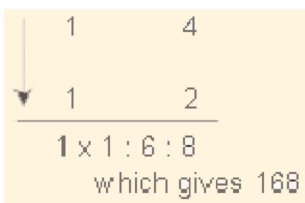
Step i) :



Step ii) :



Step iii) :



Now in the same process, answer can be written as

$$\begin{array}{r} 23 \\ 13 \\ \hline \end{array}$$

2 : 6 + 3 : 9 = 299 (Recall the 3 steps)

Ex.3

$$\begin{array}{r} 41 \\ \text{X } 41 \\ \hline \end{array}$$

16 : 4 + 4 : 1 = 1681.

What happens when one of the results i.e., either in the last digit or in the middle digit of the result, contains more than 1 digit ? Answer is simple. The right – hand – most digit there of is to be put down there and the preceding,i.e., left –hand –side digit or digits should be carried over to the left and placed under the previous digit or digits of the upper row. The digits carried over may be written as in Ex. 4.

Ex.4: 32 X 24

Step (i) : 2 X 4 = 8

Step (ii) : 3 X 4 = 12; 2 X 2 = 4; 12 + 4 = 16.

Here 6 is to be retained. 1 is to be carried out to left side.

Step (iii) : 3 X 2 = 6. Now the carried over digit 1 of 16 is to be added.
i.e., 6 + 1 = 7.

Thus 32 X 24 = 768

We can write it as follows

$$\begin{array}{r} 32 \\ 24 \\ \hline 668 \\ 1 \\ \hline 768. \end{array}$$

Note that the carried over digit from the result (3X4) + (2X2) = 12+4 = 16 i.e.,1 is placed under the previous digit 3 X 2 = 6 and added.

After sufficient practice, you feel no necessity of writing in this way and simply operate or perform mentally.

Ex.5 28 X 35.

Step (i) : 8 X 5 = 40. 0 is retained as the first digit of the answer and 4 is carried over.

Step (ii) : 2 X 5 = 10; 8 X 3 = 24; 10 + 24 = 34; add the carried over 4 to 34. Now the result is 34 + 4 = 38. Now 8 is retained as the second digit of the answer and3 is carried over.

Step (iii) : 2 X 3 = 6; add the carried over 3 to 6. The result 6 + 3 = 9 is the third or final digit from right to left of the answer.

Thus 28 X 35 = 980.

Ex.6

```
  48
  47
______
 1606
  65
_______
 2256
```

Step (i): 8 X 7 = 56; 5, the carried over digit is placed below the second digit.

Step (ii): (4 X 7) + (8 X 4) = 28 + 32 = 60; 6, the carried over digit is placed below the third digit.

Step (iii): Respective digits are added.

Algebraic proof :

a) Let the two 2 digit numbers be (ax+b) and (cx+d). Note that x = 10. Now consider the product

$(ax + b)(cx + d) = ac.x^2 + adx + bcx + b.d$

$= ac.x^2 + (ad + bc)x + b.d$

Observe that

i) The first term i.e., the coefficient of x^2 (i.e., 100, hence the digit in the100th place) is obtained by vertical multiplication of a and c i.e.,the digits in10th place (coefficient of x) of both the numbers;

ii) The middle term, i.e., the coefficient of x (i.e., digit in the 10th place) is obtained by cross wise multiplication of a and d; and of b and c; and the addition of the two products;

iii) The last (independent of x) term is obtained by vertical multiplication of the independent terms b and d.

b) Consider the multiplication of two 3 digit numbers.

Let the two numbers be($ax^2 + bx + c$) and ($dx^2 + ex + f$). Note that x=10

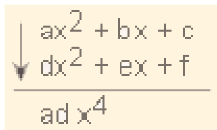
Now the product is

$$
\begin{array}{r}
ax^2 + bx + c \\
\times\ dx^2 + ex + f \\
\hline
ad.x^4+bd.x^3+cd.x^2+ae.x^3+be.x^2+ce.x+af.x^2+bf.x+cf
\end{array}
$$

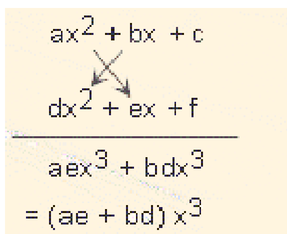
$= ad.x^4 + (bd + ae).\ x^3 + (cd + be + af).x^2 + (ce + bf)x + cf$

Note the following points :

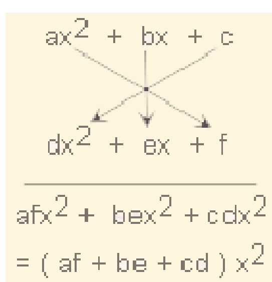
i) The coefficient of x^4 , i.e., ad is obtained by the vertical multiplication of the firstcoefficient from the left side :



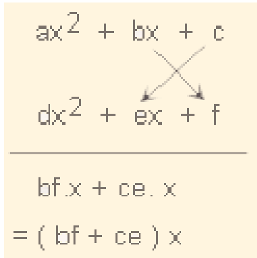
ii)The coefficient of x^3 , i.e., (ae + bd) is obtained by the cross –wise multiplication of the first two coefficients and by the addition of the two products;



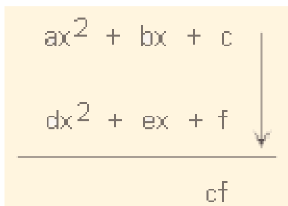
iii) The coefficient of x^2 is obtained by the multiplication of the first coefficient of the multiplicand(ax^2+bx +c) i.e., a; by the last coefficient of the multiplier (dx^2 +ex +f)i.e.,f ; of the middle one i.e., b of the multiplicand by the middle one i.e., e of the multiplier and of the last onei.e., c of the multiplicand by the first one i.e., d of the multiplier and by the addition of all the three productsi.e., af + be +cd :



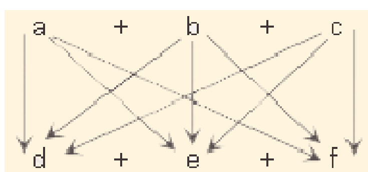
iv) The coefficient of x is obtained by the cross wise multiplication of the second coefficienti.e., b of the multiplicand by the third one i.e., f of the multiplier, and conversely the third coefficienti.e., c of the multiplicand by the second coefficient i.e., e of the multiplier and by addition of the two products,i.e., bf + ce ;



v) And finally the last (independent of x) term is obtained by the vertical multiplication of the last coefficients c and f i.e., cf



Thus the process can be put symbolically as (from left to right)



Consider the following example

124 X 132.

Proceeding from right to left

i) 4 X 2 = 8. First digit = 8

ii) (2 X 2) + (3 X 4) = 4 + 12 = 16. The digit 6 is retained and 1 is carried over to left side. Second digit = 6.

iii) (1 X 2) + (2 X 3) + (1 X 4) = 2 + 6 + 4 =12. The carried over 1 of above step is added i.e., 12 + 1 = 13. Now 3 is retained and 1 is carried over to left side. Thus third digit = 3.

iv) (1X 3) + (2 X 1) = 3 + 2 = 5. the carried over 1 of above step is added i.e., 5 + 1 = 6 . It is retained. Thus fourth digit = 6

v) (1 X 1) = 1. As there is no carried over number from the previous step it is retained. Thus fifth digit = 1

124 X 132 = 16368.

Let us work another problem by placing the carried over digits under the first row and proceed.

```
 234
 x 316
 ______
  61724
   1222
 ______
  73944
```

i) 4 X 6 = 24 : 2, the carried over digit is placed below the second digit.

ii) (3 X 6) + (4 x 1) = 18 + 4 = 22 ; 2, the carried over digit is placed below third digit.

iii) (2 X 6) + (3 X 1) + (4 X 3) = 12 + 3 + 12 = 27 ; 2, the carried over digit is placed below fourth digit.

iv) (2 X 1) + (3 X 3) = 2 + 9 = 11; 1, the carried over digit is placed below fifth digit.

v) (2 X 3) = 6.

vi) Respective digits are added.

Note :

1. We can carry out the multiplication in urdhva - tiryak process from left to right or right to left.

2. The same process can be applied even for numbers having more digits.

3. urdhva –tiryak process of multiplication can be effectively used in multiplication regarding algebraic expressions.

Example 1 : Find the product of (a+2b) and (3a+b).

$$\begin{array}{ccc} a & + & 2b \\ 3a & + & b \\ \hline \multicolumn{3}{c}{3a^2 + 7ab + 2b^2} \end{array}$$

Example 2 :

$$\begin{array}{r} 3a^2 + 2a + 4 \\ \times\ 2a^2 + 5a + 3 \\ \hline \end{array}$$

i) 4 X 3 = 12

ii) (2 X 3) + (4 X 5) = 6 + 20 =26 i.e., 26a

iii) (3 X 3) + (2 X 5) + (4 X 2) = 9 + 10 + 8 = 27 i.e., $27a^2$

iv) (3 X 5) + (2 X 2) = 15 + 4 = 19 i.e., $19\,a^3$

v) 3 X 2 = 6 i.e.,$6a^4$

Hence the product is $6a^4 + 19a^3 + 27a^2 + 26a + 12$

Example 3 : Find $(3x^2 + 4x + 7)\ (5x + 6)$

Now
$$\begin{array}{r} 3.x^2 + 4x + 7 \\ 0.x^2 + 5x + 6 \\ \hline \end{array}$$

i) 7 X 6 = 42

ii) (4 X 6) + (7 X 5) = 24 + 35 = 59 i.e., 59x

iii) (3 X 6) + (4 X 5) + (7 X 0) = 18 + 20 + 0 = 38 i.e., $38x^2$

iv) (3 X 5) + (0 X 4) = 15 + 0 = 15 i.e., $15x^3$

v) 3 X 0 = 0

Hence the product is $15x^3 + 38x^2 + 59x + 42$

Find the products using urdhva tiryagbhyam process.

1) 25 X 16 **2) 32 X 48** **3) 56 X 56**

4) 137 X 214 **5) 321 X 213** **6) 452 X 348**

7) (2x + 3y) (4x + 5y) **8) $(5a^2 + 1)(3a^2 + 4)$**

9) $(6x^2 + 5x + 2)(3x^2 + 4x + 7)$ **10) $(4x^2 + 3)(5x + 6)$**

Urdhva – tiryak in converse for division process:

As per the statement it an used as a simple argumentation for division process particularly in algebra.

Consider the division of $(x^3 + 5x^2 + 3x + 7)$ by $(x - 2)$process by converse of urdhva – tiryak :

i) x^3 divided by x gives x^2 . It is the first term of the Quotient.

$$\frac{x^3 + 5x^2 + 3x + 7}{x - 2}$$

$Q = x^2 + - - - - - - - - - - -$

ii) $x^2 \times -2 = -2x^2$. But $5x^2$ in the dividend hints$7x^2$ more since $7x^2 - 2x^2 = 5x^2$. This 'more' can be obtained from the multiplication of x by 7x. Hence second term of Q is 7x.

$$\frac{x^3 + 5x^2 + 3x + 7}{x - 2}$$ gives $Q = x^2 + 7x + - - - - - - - -$

iii)We now have $-2 \times 7x = -14x$. But the 3rd term in the dividend is 3x for which '17x more' is required since $17x - 14x = 3x$.

Now multiplication of x by 17 gives 17x. Hence third term of quotient is 17

Thus

$$\frac{x^3 + 5x^2 + 3x + 7}{x - 2} \text{ gives } Q = x^2 + 7x + 17$$

iv) Now last term of Q, i.e., 17 multiplied by –2 gives 17X–2 = -34 but the relevant term in dividend is 7. So 7 + 34 = 41 'more' is required. As there no more terms left in dividend, 41 remains as the remainder.

$$\frac{x^3 + 5x^2 + 3x + 7}{x - 2} \text{ gives } Q = x^2 + 7x + 17 \text{ and } R = 41.$$

Find the Q and R in the following divisions by using the converse process of urdhva – tiryagbhyam method :

1) $\dfrac{3x^2 - x - 6}{3x - 7}$ **2)** $\dfrac{16x^2 + 24x + 9}{4x + 3}$

3) $\dfrac{x^3 + 2x^2 + 3x + 5}{x - 3}$ **4)** $\dfrac{12x^4 - 3x^2 - 3x + 12}{x^2 + 1}$

4. Paravartya Yojayet

'Paravartya – Yojayet' means **'transpose and apply'**

(i) Consider the division by divisors of more than one digit, and when the divisors are slightly greater than powers of 10.

Example 1 : Divide 1225 by 12.

Step 1 : (From left to right) write the Divisor leaving the first digit, write the other digit or digits using negative (-) sign and place them below the divisor as shown.

$$\begin{array}{l} 12 \\ \underline{-2} \end{array}$$

Step 2 : Write down the dividend to the right. Set apart the last digit for the remainder.

i.e.,, 12 122 5
- 2

Step 3 : Write the 1st digit below the horizontal line drawn under thedividend. Multiply the digit by –2, write the product below the 2nd digit and add.

i.e.,,

```
 12      122   5
 -2      -2
______   _____

          10
```

Since 1 x –2 = -2and 2 + (-2) = 0

Step 4 : We get second digits' sum as '0'. Multiply the second digits' sum thus obtained by –2 and writes the product under 3rd digit and add.

```
 12        122    5
- 2        -20
____     __________

           102     5
```

Step 5 : Continue the process to the last digit.

i.e.,

```
 12        122     5
 - 2        -20    -4
______    ___________

            102      1
```

Step 6: The sum of the last digit is the Remainder and the result to its left is Quotient.
Thus Q = 102 andR = 1

Example 2 : Divide 1697 by 14.

```
 14       1 6 9  7
- 4       -4–8–4
_____     _________

         1 2 1  3
```

Q = 121, R = 3.

Example 3 : Divide 2598 by 123.

Note that the divisor has 3 digits. So we have to set up the last two digits of the dividend for the remainder.

```
1 2 3        25   98    Step ( 1 ) & Step ( 2 )
 -2-3
______       ________
```

Now proceed the sequence of steps write –2 and –3 as follows :

```
1 2 3        25   98
 -2-3         -4   -6
______              -2–3
             __________

              21    1 5
```

Since 2 X (-2, -3)= -4 , -6;5 – 4 = 1
and (1 X (-2,-3); 9 – 6 – 2 = 1; 8 – 3 = 5.
Hence Q = 21 and R = 15.

Example 4 : Divide 239479 by 11213. The divisor has 5 digits. So the last 4digits of the dividend are to be set up for Remainder.

```
 1 1213       2 3   9 4 7 9
-1-2-1-3        -2   -4-2-6        with 2
________               -1-2-1-3     with 1
              ______________

              2 1      40 0 6
```

Hence Q = 21, R = 4006.

Example 5 : Divide 13456 by 1123

```
1 12  3        1 3 4  5        6
 -1–2–3          -1-2-3
_______            -2-4       –6
                ______________

                1 2 0–2        0
```

Note that the remainder portion contains –20, i.e.,, a negative quantity. To over come this situation, take 1 over from the quotient column, i.e.,, 1123 over to the right side, subtract the remainder portion 20 to get the actual remainder.

Thus Q = 12 – 1 = 11, and R = 1123 - 20 = 1103.

Find the Quotient and Remainder for the problems using paravartya – yojayet method.

1) 1234 ÷ 112 2) 11329 ÷ 1132

3) 12349÷ 133 4) 239479÷1203

Now let us consider the application of paravartya – yojayet in algebra.

Example 1 : Divide $6x^2 + 5x + 4$ by $x - 1$

$$\begin{array}{l|l} \underline{X-1} & 6x^2 + 5x + 4 \\ 1 & \underline{\quad 6 + 11 \quad} \\ & 6x + 11 + 15 \end{array}$$

Thus $Q = 6x+11, R=15$.

Example 2 : Divide $x^3 - 3x^2 + 10x - 4$ by $x - 5$

$$\begin{array}{l|l} \underline{X-5} & x^3 - 3x^2 + 10x - 4 \\ 5 & \underline{\quad 5 \quad + 10 \quad 100} \\ & x^2 + 2x + 20, + 96 \end{array}$$

Thus $Q= x^2 + 2x + 20$, $R = 96$.

The procedure as a mental exercise comes as follows :

i) x^3 / xgives x^2 i.e.,, 1 the first coefficient in the Quotient.

ii) Multiply 1 by + 5,(obtained after reversing the sign of second term in the Quotient) and add to the next coefficient in the dividend. It gives 1 X(+5) = +5, adding to the next coefficient, i.e.,, –3 + 5 = 2. This is next coefficient in Quotient.

iii) Continue the process : multiply 2 by +5, i.e.,, 2 X +5 =10, add to the next coefficient 10 + 10 = 20. This is next coefficient in Quotient. Thus Quotient is $x^2 + 2x + 20$

iv) Nowmultiply 20 by + 5i.e.,, 20 x 5 = 100. Add to the next (last) term, 100 + (-4) = 96, which becomesR,i.e.,, R =9.

Example 3:

$$\frac{x^4 - 3x^3 + 7x^2 + 5x + 7}{x + 4}$$

Now thinking the method as in example (1), we proceed as follows.

$$\begin{array}{l|l}
\underline{x + 4} & x^4 \ - 3x^3 \ + 7x^2 \ + 5x \ + \ 7 \\
-4 & \underline{\quad - 4 \ + 28 \ - \ 140 \ + \ 540} \\
 & x^3 - 7x^2 + 35x - 135 \quad 547
\end{array}$$

Thus Q = $x^3 - 7x^2 + 35x - 135$ and R = 547.

or we proceed orally as follows:

x^4 / x gives 1 as first coefficient.

i) -4 X 1 = - 4 : add to next coefficient – 4 + (-3) = - 7 which gives next coefficient in Q.

ii) – 7 X - 4 = 28 : then 28 + 7 = 35, the next coefficient in Q.

iii) 35 X - 4 = - 140 : then – 140 + 5 = - 135, the next coefficient in Q.

iv) - 135 X - 4 = 540 : then 540 + 7 = 547 becomes R.

Thus Q = $x^3 - 7x^2 + 35x - 135$, R = 547.

Note :

1. We can follow the same procedure even the number of terms is more.
2. If any term is missing, we have to take the coefficient of the term as zero and proceed.

Now consider the divisors of second degree or more as in the following example.

Example :4 $2x^4 - 3x^3 - 3x + 2$ by $x^2 + 1$.

Here x^2 term is missing in the dividend. Hence treat it as 0 .x^2 or 0 . And the x term in divisor is also absent we treat it as 0 . x. Now

$x^2 \quad + 1 \qquad 2x^4 - 3x^3 + 0 . x^2 - 3x + 2$

$\underline{x^2 + 0 . x + 1} \qquad 0 \quad - 2$

$0 \quad - 1 \qquad\qquad 0 \quad + 3$

$\qquad\qquad\qquad 0 \quad + 2$

$2 \quad - 3 \quad - 2 \qquad 0 \quad 4$

Thus Q = $2x^2 - 3x - 2$andR = $0 . x + 4 = 4$.

Example 5 : $2x^5 - 5x^4 + 3x^2 - 4x + 7$by $x^3 - 2x^2 + 3$.

We treat the dividend as $2x^5 - 5x^4 + 0. x^3 + 3x^2 - 4x + 7$ and divisor as $x^3 - 2x^2 + 0 . x + 3$ and proceed as follows :

$\underline{x^3 - 2x^2 + 0 . x + 3} \qquad 2x^5 - 5x^4 + 0.x^3 + 3x^2 - 4x + 7$

$2 \quad 0 \quad - 3 \qquad 4 \quad 0 \quad - 6$

$\qquad\qquad -2 \quad 0 \quad + 3$

$\qquad\qquad\qquad - 4 \quad 0 \quad + 6$

$2 \quad - \quad 1 \quad - 2 \qquad - 7 \quad - 1 \quad +13$

Thus Q = $2x^2 - x - 2$, R = $- 7x^2 - x + 13$.

You may observe a very close relation of the method paravartya in this aspect with regard to REMAINDER THEOREM and HORNER PROCESS of Synthetic division. And yet paravartya goes much farther and is capable of numerous applications in other directions also.

Apply paravartya – yojayet to find out the Quotient and Remainder in each of the following problems.

1) **$(4x^2 + 3x + 5) \div (x+1)$**
2) **$(x^3 - 4x^2 + 7x + 6) \div (x - 2)$**
3) **$(x^4 - x^3 + x^2 + 2x + 4) \div (x^2 - x - 1)$**
4) **$(2x^5 + x^3 - 3x + 7) \div (x^3 + 2x - 3)$**
5) **$(7x^6 + 6x^5 - 5x^4 + 4x^3 - 3x^2 + 2x - 1) \div (x-1)$**

Paravartya in solving simple equations :

Recall that 'paravartya yojayet' means 'transpose and apply'. The rule relating to transposition enjoins invariable change of sign with every change of side. i.e., + becomes - and conversely ; and X becomes ÷ and conversely. Further it can be extended to the transposition of terms from left to right and conversely and from numerator to denominator and conversely in the concerned problems.

Type (i) :

Consider the problem $7x - 5 = 5x + 1$

$$7x - 5x = 1 + 5$$

i.e.,, $2x = 6$ $x = 6 \div 2 = 3.$

Observe that the problem is of the type $ax + b = cx + d$ from which we get by 'transpose' $(d - b)$, $(a - c)$ and

$$x = \frac{d - b.}{a - c}$$

In this example $a = 7$, $b = - 5$, $c = 5$, $d = 1$

Hence

$$x = \frac{1 - (- 5)}{7 - 5} = \frac{1+5}{7-5} = \frac{6}{2} = 3$$

Example 2: Solve for x,$3x + 4 = 2x + 6$

$$x = \frac{d - b}{a - c} = \frac{6 - 4}{3 - 2} = \frac{2}{1} = 2$$

Type (ii) : Consider problems of the type $(x + a)(x+b) = (x+c)(x+d)$. By paravartya, we get

$$x = \frac{cd - ab}{(a + b) - (c + d)}$$

It is trivial form the following steps

$$(x + a)(x + b) = (x + c)(x + d)$$
$$x^2 + bx + ax + ab = x^2 + dx + cx + cd$$
$$bx + ax - dx - cx = cd - ab$$
$$x(a + b - c - d) = cd - ab$$

$$x = \frac{cd - ab}{a + b - c - d} \qquad x = \frac{cd - ab}{(a + b) - (c + d.)}$$

Example 1 : $(x - 3)(x - 2) = (x + 1)(x + 2)$.

By paravartya

$$x = \frac{cd - ab}{a + b - c - d} = \frac{1(2) - (-3)(-2)}{-3 - 2 - 1 - 2}$$

$$= \frac{2 - 6}{-8} = \frac{-4}{-8} = \frac{1}{2}$$

Example 2 : $(x + 7)(x - 6) = (x + 3)(x - 4)$.

Now $$x = \frac{cd - ab}{a + b - c - d} = \frac{(3)(-4) - (7)(-6)}{7 + (-6) - 3 - (-4)}$$

$$= \frac{-12 + 42}{7 - 6 - 3 + 4} = \frac{30}{2} = 15$$

Note that if cd - ab = 0 i.e.,, cd = ab, i.e.,, if the product of the absolute terms be the same on both sides, the numerator becomes zero giving x = 0.

For the problem $(x + 4)(x + 3) = (x - 2)(x - 6)$

Solution is x = 0 since 4 X 3 = - 2 X - 6. = 12

Type (iii) :

Consider the problems of the type $$\frac{ax + b}{cx + d} = \frac{m}{n}$$

By cross – multiplication,

$$n\,(ax + b) = m\,(cx + d)$$
$$nax + nb = mcx + md$$
$$nax - mcx = md - nb$$
$$x(na - mc) = md - nb$$

$$x = \frac{md - nb}{na - mc.}$$

Now look at the problem once again

$$\frac{ax + b}{cx + d} = \frac{m}{n}$$

paravartya gives md - nb, na - mc and

$$x = \frac{md - nb}{na - mc}$$

Example 1: $\dfrac{3x + 1}{4x + 3} = \dfrac{13}{19}$

$$x = \frac{md - nb}{na - mc} = \frac{13\,(3) - 19(1)}{19\,(3) - 13(4)} = \frac{39 - 19}{57 - 52} = \frac{20}{5}$$

$$= 4$$

Example 2: $\dfrac{4x + 5}{3x + 13/2} = \dfrac{7}{8}$

$$x = \frac{(7)\,(13/2) - (8)(5)}{(8)(4) - (7)(3)}$$

$$= \frac{(91/2) - 40}{32 - 21} = \frac{(91 - 80)/2}{32 - 21} = \frac{11}{2 \text{ X } 11} = \frac{1}{2}$$

Type (iv) : Consider the problems of the type $\dfrac{m}{x + a} + \dfrac{n}{x + b} = 0$

Take L.C.M and proceed.

$$\frac{m(x+b) + n(x+a)}{(x + a)(x + b)} = 0$$

$$\frac{mx + mb + nx + na}{x + a)(x + b)} = 0$$

$$(m + n)x + mb + na = 0 \quad \therefore (m + n)x = - mb - na$$

$$x = \frac{-mb - na}{(m + n)}$$

Thus the problem $\dfrac{m}{x + a} + \dfrac{n}{x + b} = 0$, by paravartya process

gives directly

$$x = \frac{-mb - na}{(m + n)}$$

Example 1 : $\dfrac{3}{x + 4} + \dfrac{4}{x - 6} = 0$

gives $x = \dfrac{-mb - na}{(m + n)}$ Note that m = 3, n = 4, a = 4, b = - 6

$$= \frac{-(3)(-6) - (4)(4)}{(3 + 4)} = \frac{18 - 16}{7} = \frac{2}{7}$$

Example 2 :

$$\frac{5}{x + 1} + \frac{6}{x - 21} = 0$$

gives

$$x = \frac{-(5)\,(-21) - (6)\,(1)}{5 + 6} = \frac{105 - 6}{11} = \frac{99}{11} = 9$$

I . Solve the following problems using the sutra Paravartya – yojayet.

1) $3x + 5 = 5x - 3$

2) $(2x/3) + 1 = x - 1$

3) $\dfrac{7x + 2}{3x - 5} = \dfrac{5}{8}$

4) $\dfrac{x + 1/3}{3x - 1} = 1$

5) $\dfrac{5}{x + 3} + \dfrac{2}{x - 4} = 0$

6) $(x + 1)\,(x + 2) = (x - 3)\,(x - 4)$

7) $(x - 7)\,(x - 9) = (x - 3)\,(x - 22)$

8) $(x + 7)\,(x + 9) = (x + 3)\,(x + 21)$

II)

1.Show that for the type of equations

$$\frac{m}{x + a} + \frac{n}{x + b} + \frac{p}{x + c} = 0, \quad \text{the solution is}$$

$$x = \frac{- mbc - nca - pab}{m(b + c) + n(c+a) + p(a + b)}, \text{ if } m + n + p = 0.$$

2. Apply the above formula to set the solution for the problem

Problem $$\frac{3}{x+4} + \frac{2}{x+6} - \frac{5}{x+5} = 0$$

---

some more simple solutions :

$$\frac{m}{x+a} + \frac{n}{x+b} = \frac{m+n}{x+c}$$

Now this can be written as,

$$\frac{m}{x+a} + \frac{n}{x+b} = \frac{m}{x+c} + \frac{n}{x+c}$$

$$\frac{m}{x+a} - \frac{m}{x+c} = \frac{n}{x+c} - \frac{n}{x+b}$$

$$\frac{m(x+c) - m(x+a)}{(x+a)(x+c)} = \frac{n(x+b) - n(x+c)}{(x+c)(x+b)}$$

$$\frac{mx + mc - mx - ma}{(x+a)(x+c)} = \frac{nx + nb - nx - nc}{(x+c)(x+b)}$$

$$\frac{m(c-a)}{x+a} = \frac{n(b-c)}{x+b}$$

$$m(c-a).x + m(c-a).b = n(b-c).x + n(b-c).a$$

$$x[m(c-a) - n(b-c)] = na(b-c) - mb(c-a)$$

$$\text{or } x[m(c-a) + n(c-b)] = na(b-c) + mb(a-c)$$

Thus

$$x = \frac{mb(a - c) + na\,(b - c)}{m(c-a) + n(c-b).}$$

By paravartya rule we can easily remember the formula.

Example 1 : solve 3

5. Sunyam Samya Samuccaye

The Sutra 'Sunyam Samyasamuccaye' says the 'Samuccaya is the same, that Samuccaya is Zero.' i.e., it should be equated to zero. The term 'Samuccaya' has several meanings under different contexts.

i) We interpret, 'Samuccaya' as a term which occurs as a common factor in all the terms concerned and proceed as follows.

Example 1: The equation $7x + 3x = 4x + 5x$ has the same factor ' x ' in all its terms. Hence by the sutra it is zero,i.e., $x = 0$.

Otherwise we have to work like this:

$$7x + 3x = 4x + 5x$$
$$10x = 9x$$
$$10x - 9x = 0$$
$$x = 0$$

This is applicable not only for 'x' but also any such unknown quantity as follows.

Example 2: $5(x+1) = 3(x+1)$

No need to proceed in the usual procedure like

$$5x + 5 = 3x + 3$$
$$5x - 3x = 3 - 5$$
$$2x = -2 \quad \text{or} \quad x = -2 \div 2 = -1$$

Simply think of the contextual meaning of 'Samuccaya'

Now Samuccaya is (x + 1)

$x + 1 = 0$ gives $x = -1$

ii) Now we interpret 'Samuccaya' as product of independent terms in expressions like (x+a) (x+b)

Example 3: $(x + 3)(x + 4) = (x - 2)(x - 6)$

Here Samuccaya is $3 \times 4 = 12 = -2 \times -6$

Since it is same , we derive $x = 0$

This example, we have already dealt in type (ii) of Paravartya in solving simple equations.

iii) We interpret ' Samuccaya 'as the sum of the denominators of two fractions having the same numerical numerator.

Consider the example.

$$\frac{1}{3x-2} + \frac{1}{2x-1} = 0$$

for this we proceed by takingL.C.M.

$$\frac{(2x-1)+(3x-2)}{(3x-2)(2x-1)} = 0$$

$$\frac{5x-3}{(3x-2)(2x-1)} = 0$$

$$5x - 3 = 0 \qquad 5x = 3$$

$$x = \frac{3}{5}$$

Instead of this, we can directly put the Samuccaya i.e., sum of the denominators

i.e., $3x - 2 + 2x - 1 = 5x - 3 = 0$

giving $5x = 3$ $\quad x = 3 / 5$

It is true and applicable for all problems of the type

$$\frac{m}{ax+b} + \frac{m}{cx+d} = 0$$

Samuccaya is ax+b+cx+d and solution is ($m \neq 0$)

$$x = \frac{-(b+d)}{(a+c)}$$

iii) We now interpret 'Samuccaya' as combination or total.

If the sum of the numerators and the sum of the denominators be the same, then that sum = 0.

Consider examples of type

$$\frac{ax+b}{ax+c} = \frac{ax+c}{ax+b}$$

In this case, $(ax+b)(ax+b) = (ax+c)(ax+c)$

$$a^2x^2 + 2abx + b^2 = a^2x^2 + 2acx + c^2$$
$$2abx - 2acx = c^2 - b^2$$
$$x(2ab - 2ac) = c^2 - b^2$$

$$x = \frac{c^2-b^2}{2a(b-c)} = \frac{(c+b)(c-b)}{2a(b-c)} = \frac{-(c+b)}{2a}$$

As per Samuccaya $(ax+b) + (ax+c) = 0$

$$2ax+b+c = 0$$
$$2ax = -b-c$$

$$x = \frac{-(c+b)}{2a}$$ Hence the statement.

Example 4:

$$\frac{3x+4}{3x+5} = \frac{3x+5}{3x+4}$$

Since $N_1 + N_2 = 3x + 4 + 3x + 5 = 6x + 9$,

And $D_1 + D_2 = 3x + 4 + 3x + 5 = 6x + 9$

We have$N_1 + N_2 = D_1 + D_2 = 6x + 9$

Hence from Sunya Samuccaya we get $6x + 9 = 0$

$6x = -9$

$$x = \frac{-9}{6} = \frac{-3}{2}$$

Example 5:

$$\frac{5x+7}{5x+12} = \frac{5x+12}{5x+7}$$

Hence $N_1 + N_2 = 5x + 7 + 5x + 12 = 10x + 19$

And $D_1 + D_2 = 5x + 12 + 5x + 7 = 10x + 19$

$N_1 + N_2 = D_1 + D_2$ gives $10x + 19 = 0$

$10x = -19$

$$x = \frac{-19}{10}$$

Consider the examples of the type, where $N_1 + N_2 = K (D_1 + D_2)$, where K is a numerical constant, then also by removing the numerical constant K, we can proceed as above.

Example 6:

$$\frac{2x+3}{4x+5} = \frac{x+1}{2x+3}$$

Here $N_1 + N_2 = 2x + 3 + x + 1 = 3x + 4$

$$D_1 + D_2 = 4x + 5 + 2x + 3 = 6x + 8$$
$$= 2 (3x + 4)$$

Removing the numerical factor 2, we get 3x + 4 on both sides.

$3x + 4 = 0 \quad 3x = -4 \quad x = - 4 / 3.$

v) 'Samuccaya' with the same meaning as above, i.e., case (iv), we solve the problems leading to quadratic equations. In this context, we take the problems as follows;

If $N_1 + N_2 = D_1 + D_2$ and also the differences
$N_1 \sim D_1 = N_2 \sim D_2$ then both the things are equated to zero, the solution gives the two values for x.

Example 7:

$$\frac{3x + 2}{2x + 5} = \frac{2x + 5}{3x + 2}$$

In the conventional text book method, we work as follows :

$$\frac{3x + 2}{2x + 5} = \frac{2x + 5}{3x + 2}$$

$$(3x + 2) (3x + 2) = (2x + 5) (2x + 5)$$
$$9x^2 + 12x + 4 = 4x^2 + 20x + 25$$
$$9x^2 + 12x + 4 - 4x^2 - 20x - 25 = 0$$
$$5x^2 - 8x - 21 = 0$$
$$5x^2 - 15x + 7x - 21 = 0$$
$$5x (x - 3) + 7 (x - 3) = 0$$
$$(x - 3) (5x + 7) = 0$$
$$x - 3 = 0 \text{ or } 5x + 7 = 0$$
$$x = 3 \text{ or } - 7 / 5$$

Now ‘Samuccaya’ sutra comes to help us in a beautiful way as follows :

Observe $N_1 + N_2 = 3x + 2 + 2x + 5 = 5x + 7$
$D_1 + D_2 = 2x + 5 + 3x + 2 = 5x + 7$

Further $N_1 \sim D_1 = (3x + 2) - (2x + 5) = x - 3$
$N_2 \sim D_2 = (2x + 5) - (3x + 2) = - x + 3 = - (x - 3)$

Hence $5x + 7 = 0$, $x - 3 = 0$

$5x = -7$, $x = 3$

i.e., $x = -7 / 5$, $x = 3$

Note that all these can be easily calculated by mere observation.

Example 8:

$$\frac{3x + 4}{6x + 7} = \frac{5x + 6}{2x + 3}$$

Observe that

$N_1 + N_2 = 3x + 4 + 5x + 6 = 8x + 10$

and $D_1 + D_2 = 6x + 7 + 2x + 3 = 8x + 10$

Further $N_1 \sim D_1 = (3x + 4) - (6x + 7)$

$= 3x + 4 - 6x - 7$

$= -3x - 3 = -3 (x + 1)$

$N_2 \sim D_2 = (5x + 6) - (2x + 3) = 3x + 3 = 3(x + 1)$

By 'Sunyam Samuccaye' we have

$8x + 10 = 0$ $\quad$ $3(x + 1) = 0$

$8x = -10$ $\quad$ $x + 1 = 0$

$x = - 10 / 8$ $\quad$ $x = -1$

$= - 5 / 4$

vi)'Samuccaya' with the same sense but with a different context and application .

Example 9:

$$\frac{1}{x - 4} + \frac{1}{x - 6} = \frac{1}{x - 2} + \frac{1}{x - 8}$$

Usually we proceed as follows.

$$\frac{x-6+x-4}{(x-4)\ (x-6)} = \frac{x-8+x-2}{(x-2)\ (x-8)}$$

$$\frac{2x-10}{x^2-10x+24} = \frac{2x-10}{x^2-10x+16}$$

$$(2x - 10) (x^2 - 10x + 16) = (2x - 10) (x^2 - 10x + 24)$$
$$2x^3-20x^2+32x-10x^2+100x-160 = 2x^3-20x^2+48x-10x^2+100x-240$$
$$2x^3 - 30x^2 + 132x - 160 = 2x^3 - 30x^2 + 148x - 240$$
$$132x - 160 = 148x - 240$$
$$132x - 148x = 160 - 240$$
$$- 16x = - 80$$
$$x = - 80 / - 16 = 5$$

Now 'Samuccaya' sutra, tell us that, if other elements being equal, the sum-total of the denominators on the L.H.S. and their total on the R.H.S. be the same, that total is zero.

Now $D_1 + D_2 = x - 4 + x - 6 = 2x - 10$, and
$D_3 + D_4 = x - 2 + x - 8 = 2x - 10$

By Samuccaya, $2x - 10$ gives $2x = 10$

$$x = \frac{10}{2} = 5$$

Example 10:

$$\frac{1}{x-8} + \frac{1}{x-9} = \frac{1}{x-5} + \frac{1}{x-12}$$

$D_1 + D_2 = x - 8 + x - 9 = 2x - 17$, and
$D_3 + D_4 = x - 5 + x - 12 = 2x - 17$
Now $2x - 17 = 0$ gives $2x = 17$

$$x = \frac{17}{2} = 8\tfrac{1}{2}$$

Example 11:

$$\frac{1}{x+7} - \frac{1}{x+10} = \frac{1}{x+6} - \frac{1}{x+9}$$

This is not in the expected form. But a little work regarding transposition makes the above as follows.

$$\frac{1}{x+7} + \frac{1}{x+9} = \frac{1}{x+6} + \frac{1}{x+10}$$

Now 'Samuccaya' sutra applies

$D_1 + D_2 = x + 7 + x + 9 = 2x + 16$, and
$D_3 + D_4 = x + 6 + x + 10 = 2x + 16$

Solution is given by $2x + 16 = 0$ i.e., $2x = -16$.
$x = -16 / 2 = -8$.

Solve the following problems using Sunyam Samya-Samuccaye process.

1. $\mathbf{7(x+2) + 3(x+2) = 6(x+2) + 5(x+2)}$

2. $\mathbf{(x+6)(x+3) = (x-9)(x-2)}$

3. $\mathbf{(x-1)(x+14) = (x+2)(x-7)}$

4. $$\frac{1}{4x-3} + \frac{1}{x-2} = 0$$

5. $$\frac{4}{3x+1} + \frac{4}{5x+7} = 0$$

6. $$\frac{2x+11}{2x+5} = \frac{2x+5}{2x+11}$$

7. $$\frac{3x+4}{6x+7} = \frac{x+1}{2x+3}$$

8. $\dfrac{4x - 3}{2x + 3} = \dfrac{x + 4}{3x - 2}$

9. $\dfrac{1}{x - 2} + \dfrac{1}{x - 5} = \dfrac{1}{x - 3} + \dfrac{1}{x - 4}$

10. $\dfrac{1}{x - 7} - \dfrac{1}{x - 6} = \dfrac{1}{x - 10} - \dfrac{1}{x - 9}$

Sunyam Samya Samuccaye in Certain Cubes:

Consider the problem $\mathbf{(x - 4)^3 + (x - 6)^3 = 2(x - 5)^3}$. For the solution by the traditional method we follow the steps as given below:

$$(x - 4)^3 + (x - 6)^3 = 2(x - 5)^3$$
$$x^3 - 12x^2 + 48x - 64 + x^3 - 18x^2 + 108x - 216$$
$$= 2(x^3 - 15x^2 + 75x - 125)$$
$$2x^3 - 30x^2 + 156x - 280 = 2x^3 - 30x^2 + 150x - 250$$
$$156x - 280 = 150x - 250$$
$$156x - 150x = 280 - 250$$
$$6x = 30$$
$$x = 30 / 6 = 5$$

But once again observe the problem in the vedic sense

We have $(x - 4) + (x - 6) = 2x - 10$. Taking out the numerical factor 2, we have $(x - 5) = 0$, which is the factor under the cube on R.H.S. In such a case "Sunyam samya Samuccaye" formula gives that $x - 5 = 0$. Hence $x = 5$

Think of solving the problem $\mathbf{(x-249)^3 + (x+247)^3 = 2(x-1)^3}$

The traditional method will be horrible even to think of.

But $(x - 249) + (x + 247) = 2x - 2 = 2(x - 1)$. And $x - 1$. on R.H.S.

cube, it is enough to state that x – 1 = 0 by the 'sutra'.

x = 1 is the solution. No cubing or any other mathematical operations.

Algebraic Proof :

Consider $(x - 2a)^3 + (x - 2b)^3 = 2(x - a - b)^3$ it is clear that

$$x - 2a + x - 2b = 2x - 2a - 2b$$
$$= 2(x - a - b)$$

Now the expression,

$$x^3 - 6x^2a + 12xa^2 - 8a^3 + x^3 - 6x^2b + 12xb^2 - 8b^3 =$$
$$2(x^3 - 3x^2a - 3x^2b + 3xa^2 + 3xb^2 + 6axb - a^3 - 3a^2b - 3ab^2 - b^3)$$

$$= 2x^3 - 6x^2a - 6x^2b + 6xa^2 + 6xb^2 + 12xab - 2a^3 - 6a^2b - 6ab^2 - 2b^3$$

cancel the common terms on both sides

$$12xa^2 + 12xb^2 - 8a^3 - 8b^3 = 6xa^2 + 6xb^2 + 12xab - 2a^3 - 6a^2b - 6ab^2 - 2b^3$$
$$6xa^2 + 6xb^2 - 12xab = 6a^3 + 6b^3 - 6a^2b - 6ab^2$$
$$6x(a^2 + b^2 - 2ab) = 6[a^3 + b^3 - ab(a + b)]$$
$$x(a - b)^2 = [(a + b)(a^2 + b^2 - ab) - (a + b)ab]$$
$$= (a + b)(a^2 + b^2 - 2ab)$$
$$= (a + b)(a - b)^2$$

$$\therefore \quad x = a + b$$

Solve the following using "Sunyam Samuccaye" process :

1. $(x - 3)^3 + (x - 9)^3 = 2(x - 6)^3$

2. $(x + 4)^3 + (x - 10)^3 = 2(x - 3)^3$

3. $(x + a + b - c)^3 + (x + b + c - a)^3 = 2(x + b)^3$

Example :

$$\frac{(x + 2)^3}{(x + 3)^3} = \frac{x + 1}{x + 4}$$

with the text book procedures we proceed as follows

$$\frac{x^3 + 6x^2 + 12x + 8}{x^3 + 9x^2 + 27x + 27} = \frac{x + 1}{x + 4}$$

Now by cross multiplication,

$$(x + 4) (x^3 + 6x^2 + 12x + 8) = (x + 1) (x^3 + 9x^2 + 27x + 27)$$
$$x^4 + 6x^3 + 12x^2 + 8x + 4x^3 + 24x^2 + 48x + 32 =$$
$$x^4 + 9x^3 + 27x^2 + 27x + x^3 + 9x^2 + 27x + 27$$
$$x^4 + 10x^3 + 36x^2 + 56x + 32 = x^4 + 10x^3 + 36x^2 + 54x + 27$$
$$56x + 32 = 54x + 27$$
$$56x - 54x = 27 - 32$$
$$2x = - 5$$
$$x = - 5 / 2$$

Observe that ($N_1 + D_1$) with in the cubes on
L.H.S. is $x + 2 + x + 3 = 2x + 5$ and

$N_2 + D_2$ on the right hand side
is $x + 1 + x + 4 = 2x + 5$.

By vedic formula we have $2x + 5 = 0$ $\quad x = - 5 / 2$.

Solve the following by using vedic method :

1. $$\frac{(x + 3)^3}{(x + 5)^3} = \frac{x+1}{x+7}$$

2.

$$\frac{(x - 5)^3}{(x - 7)^3} = \frac{x - 3}{x - 9}$$

6. Anurupye - Sunyamanyat

The Sutra Anurupye Sunyamanyat says : 'If one is in ratio, the other one is zero'.

We use this Sutra in solving a special type of simultaneous simple equations in which the coefficients of 'one' variable are in the same ratio to each other as the independent terms are to each other. In such a context the Sutra says the 'other' variable is zero from which we get two simple equations in the first variable (already considered) and of course give the same value for the variable.

Example 1:

$$3x + 7y = 2$$
$$4x + 21y = 6$$

Observe that the y-coefficients are in the ratio 7 : 21 i.e., 1 : 3, which is same as the ratio of independent terms i.e., 2 : 6 i.e., 1 : 3. Hence the other variable x = 0 and 7y = 2 or 21y = 6 gives y = 2 / 7

Example 2:

$$323x + 147y = 1615$$
$$969x + 321y = 4845$$

The very appearance of the problem is frightening. But just an observation and anurupye sunyamanyat give the solution x = 5, because coefficient of x ratio is

323 : 969 = 1 : 3 and constant terms ratio is 1615 : 4845 = 1 : 3.

y = 0 and 323 x = 1615 or 969 x = 4845 gives x = 5.

Solve the following by anurupye sunyamanyat.

1. $12x + 78y = 12$ **2.** $3x + 7y = 24$

$16x + 96y = 16$ **$12x + 5y = 96$**

3. **$4x - 6y = 24$**
$7x - 9y = 36$

4. **$ax + by = bm$**
$cx + dy = dm$

In solving simultaneous quadratic equations, also we can take the help of the 'sutra' in the following way:

Example 3 :

Solve for x and y

$$x + 4y = 10$$
$$x^2 + 5xy + 4y^2 + 4x - 2y = 20$$

$x^2 + 5xy + 4y^2 + 4x - 2y = 20$ can be written as

$(x + y) (x + 4y) + 4x - 2y = 20$

$10 (x + y) + 4x - 2y = 20$ (Since $x + 4y = 10$)

$10x + 10y + 4x - 2y = 20$

$14x + 8y = 20$

Now $x + 4y = 10$

$14x + 8y = 20$ and $4 : 8 :: 10 : 20$

from the Sutra, $x = 0$ and $4y = 10$, i.e.,, $8y= 20$ $y = 10/4 = 2½$

Thus **x = 0** and **y = 2½** is the solution.

7. Sankalana - Vyavakalanabhyam

This Sutra means 'by addition and by subtraction'. It can be applied in solving a special type of simultaneous equations where the x - coefficients and the y - coefficients are found interchanged.

Example 1:

$$45x - 23y = 113$$
$$23x - 45y = 91$$

In the conventional method we have to make equal either the coefficient of x or coefficient of y in both the equations. For that we have to multiply equation (1) by 45 and equation (2) by 23 and subtract to get the value of x and then substitute the value of x in one of the equations to get the value of y or we have to multiply equation (1) by 23 and equation (2) by 45 and then subtract to get value of y and then substitute the value of y in one of the equations, to get the value of x. It is difficult process to think of.

From Sankalana – vyavakalanabhyam

add them,

i.e., $(45x - 23y) + (23x - 45y) = 113 + 91$

i.e., $68x - 68y = 204 \quad \therefore x - y = 3$

subtract one from other,

i.e., $(45x - 23y) - (23x - 45y) = 113 - 91$

i.e., $22x + 22y = 22 \quad \therefore x + y = 1$

and repeat the same sutra, we get $x = 2$ and $y = - 1$

Very simple addition and subtraction are enough, however big the coefficients may be.

Example 2:

$$1955x - 476y = 2482$$
$$476x - 1955y = -4913$$

Oh ! what a problem ! And still

just add, $2431(x - y) = - 2431 \quad \therefore x - y = -1$

subtract, $1479 (x + y) = 7395 \quad \therefore x + y = 5$

once again add, $2x = 4 \quad \therefore x = 2$

subtract $- 2y = - 6 \quad \therefore y = 3$

Solve the following problems usingSankalana – Vyavakalanabhyam.

1. **$3x + 2y = 18$**
   $2x + 3y = 17$

2. **$5x - 21y = 26$**
   $21x - 5y = 26$

3. **$659x + 956y = 4186$**
   $956x + 659y = 3889$

8. Puranapuranabhyam

The Sutra can be taken as Purana - Apuranabhyam which means by the completion or non - completion. Purana is well known in the present system. We can see its application in solving the roots for general form of quadratic equation.

We have : $ax^2 + bx + c = 0$

$$x^2 + (b/a)x + c/a = 0 \quad (\text{ dividing by a })$$

$$x^2 + (b/a)x = - c/a$$

completing the square (i.e.,, purana) on the L.H.S.

$$x^2 + (b/a)x + (b^2/4a^2) = -c/a + (b^2/4a^2)$$

$$[x + (b/2a)]^2 = (b^2 - 4ac) / 4a^2$$

Proceeding in this way we finally get $x = \dfrac{- b \pm \sqrt{b^2 - 4ac}}{2a}$

Now we apply purana to solve problems.

Example 1. $x^3 + 6x^2 + 11 x + 6 = 0.$

Since $(x + 2)^3 = x^3 + 6x^2 + 12x + 8$
Add (x + 2) to both sides
We get $x^3 + 6x^2 + 11x + 6 + x + 2 = x + 2$
i.e.,, $x^3 + 6x^2 + 12x + 8 = x + 2$
i.e.,, $(x + 2)^3 = (x + 2)$
this is of the form $y^3 = y$ for $y = x + 2$
solution $y = 0, y = 1, y = - 1$
i.e.,, $x + 2 = 0,1,-1$
which gives $x = -2,-1,-3$

Example 2: $x^3 + 8x^2 + 17x + 10 = 0$

We know $(x + 3)^3 = x^3 + 9x^2 + 27x + 27$
So adding on the both sides, the term $(x^2 + 10x + 17)$, we get
$x^3 + 8x^2 + 17x + x^2 + 10x + 17 = x^2 + 10x + 17$
i.e.,, $x^3 + 9x^2 + 27x + 27 = x^2 + 6x + 9 + 4x + 8$
i.e.,, $(x + 3)^3 = (x + 3)^2 + 4 (x + 3) – 4$

$y^3 = y^2 + 4y - 4$ for $y = x + 3$

$y = 1, 2, -2.$

Hence $x = -2, -1, -5$

Thus purana is helpful in factorization.

Further purana can be applied in solving Biquadratic equations also.

Solve the following using purana – apuranabhyam.

1. $\mathbf{x^3 - 6x^2 + 11x - 6 = 0}$
2. $\mathbf{x^3 + 9x^2 + 23x + 15 = 0}$
3. $\mathbf{x^2 + 2x - 3 = 0}$
4. $\mathbf{x^4 + 4x^3 + 6x^2 + 4x - 15 = 0}$

9. Calana - Kalanabhyam

In the book on **Vedic Mathematics** Sri Bharati Krishna Tirthaji mentioned the Sutra 'Calana - Kalanabhyam' at only two places. The Sutra means 'Sequential motion'.

i) In the first instance it is used to find the roots of a quadratic equation $7x^2 - 11x - 7 = 0$. Swamiji called the sutra as calculus formula. Its application at that point is as follows.Now by calculus formula we say: $14x-11 = \pm\sqrt{317}$

A Note follows saying every Quadratic can thus be broken down into two binomial factors. An explanation in terms of first differential, discriminant with sufficient number of examples are given under the chapter 'Quadratic Equations'.

ii) At the Second instance under the chapter 'Factorization and Differential Calculus' for factorizing expressions of 3rd, 4th and 5th degree, the procedure is mentioned as'Vedic Sutras relating to Calana – Kalana – Differential Calculus'.

Further other Sutras 10 to 16 mentioned below are also used to get the required results. Hence the sutra and its various applications will be taken up at a later stage for discussion.

But sutra – 14 is discussed immediately after this item.

Now the remaining sutras :

10. YĀVADŨNAM (The deficiency)
11. VYAŞŢISAMAŞŢIH (Whole as one and one as whole)
12. ŚEŞĀNYAŃ KENA CARAMEŅA (Remainder by the last digit)
13. SOPĀNTYADVAYAMANTYAM (Ultimate and twice the penultimate)
15. GUŅITASAMUCCAYAH (The whole product is the same)
16. GUŅAKA SAMUCCAYAH (Collectivity of multipliers)

The Sutras have their applications in solving different problems in different contexts. Further they are used along with other Sutras. So it is a bit of inconvenience to deal each Sutra under a separate heading exclusively and also independently. Of course they will be mentioned and also be applied in solving the problems in the forth coming chapter wherever necessary. This decision has been taken because up to now, we have treated each Sutra independently and have not continued with any other Sutra even if it is necessary. When the need for combining Sutras for filling the gaps in the process arises, we may opt for it. Now we shall deal the fourteenth Sutra, the Sutra left so far untouched.

10. Ekanyunena Purvena

The Sutra Ekanyunena purvena comes as a Sub-sutra to Nikhilam which gives the meaning 'One less than the previous' or 'One less than the one before'.

1) The use of this sutra in case of multiplication by 9,99,999.. is as follows .

Method :

a) The left hand side digit (digits) is (are) obtained by applying the ekanyunena purvena i.e. by deduction 1 from the left side digit (digits) .

e.g. (i) 7 x 9; 7 – 1 = 6 (L.H.S. digit)

b) The right hand side digit is the complement or difference between the multiplier and the left hand side digit (digits) . i.e. 7 X 9 R.H.S is 9 - 6 = 3.

c) The two numbers give the answer; i.e. 7 X 9 = 63.

Example 1: 8 x 9 **Step (a)** gives 8 – 1 = 7 (L.H.S. Digit)
Step (b) gives 9 – 7 = 2 (R.H.S. Digit)
Step (c) gives the answer 72

Example 2: 15 x 99 **Step (a) :** 15 – 1 = 14
Step (b) : 99 – 14 = 85 (or 100 – 15)
Step (c) : 15 x 99 = 1485

Example 3: 24 x 99
Answer :

$$\begin{array}{l|l} (24 - 1) & (99 - 23) \\ = 23 & = 76 \text{ (or } 100 - 24) \end{array} \quad = 2376$$

Example 4: 356 x 999
Answer :

$$\begin{array}{l|l} (356 - 1) & (999 - 335) \\ = 355 & = 644 \end{array} \quad = 355644$$

Example 5: 878 x 9999
Answer :

$$\begin{array}{l|l} (878 - 1) & (9999 - 877) \\ = 877 & = 9122 \text{ } (10000 - 876) \end{array} \quad = 8779122$$

Note the process : The multiplicand has to be reduced by 1 to obtain the LHS and the rightside is mechanically obtained by the subtraction of the L.H.S from the multiplier which is practically a direct application of Nikhilam Sutra.

Now by Nikhilam

$$\begin{array}{lll} 24 - 1 = 23 & \text{L.H.S.} & \\ \times\ 99 - 23 = 76 & \text{R.H.S.} & (100-24) \\ \hline 23 / 76 & & = 2376 \end{array}$$

Reconsider the **Example 4:**

$$\begin{array}{ll} 356 - 1 = 355 & \text{L.H.S.} \\ \times\ 999 - 355 = 644 & \text{R.H.S.} \\ \hline 355 / 644 & = 355644 \end{array}$$

and in **Example 5:** 878 x 9999 we write

$$\begin{array}{lll} 0878 - 1 = 877 & \text{L.H.S.} \\ \times\ 9999 - 877 = 9122 & \text{R.H.S.} \\ \hline 877 \,/\, 9122 & = 8779122 \end{array}$$

Algebraic proof :

As any two digit number is of the form (10x + y), we proceed

$$\begin{aligned} & (10x + y) \times 99 \\ &= (10x + y) \times (100 - 1) \\ &= 10x \,.\, 10^2 - 10x + 10^2 \,.y - y \\ &= x \,.\, 10^3 + y \,.\, 10^2 - (10x + y) \\ &= x \,.\, 10^3 + (y - 1) \,.\, 10^2 + [10^2 - (10x + y)] \end{aligned}$$

Thus the answer is a four digit number whose 1000^{th} place is x,100^{th} place is (y - 1) and the two digit number which makes up the 10^{th} and unit place is the number obtained by subtracting the multiplicand from 100.(or apply nikhilam).

Thus in 37 X 99. The 1000^{th} place is x i.e. 3

100^{th} place is (y - 1) i.e. (7 - 1) = 6

Number in the last two places 100-37=63.

Hence answer is 3663.

Apply Ekanyunena purvena to find out the products

1. 64 x 99 **2. 723 x 999** **3. 3251 x 9999**

4. 43 x 999 **5. 256 x 9999** **6. 1857 x 99999**

We have dealt the cases

i) When the multiplicand and multiplier both have the same number of digits
ii) When the multiplier has more number of digits than the multiplicand.

In both the cases the same rule applies. But what happens when the multiplier has lesser digits?

i.e. for problems like 42 X 9, 124 X 9, 26325 X 99 etc., For this let us have a re-look in to the process for proper understanding.

Multiplication table of 9.

	a	b
2 x 9 =	1	8
3 x 9 =	2	7
4 x 9 =	3	6
- - - - - - - - - -		
8 x 9 =	7	2
9 x 9 =	8	1
10 x 9 =	9	0

Observe the left hand side of the answer is always one less than the multiplicand (here multiplier is 9) as read from Column (a) and the right hand side of the answer is the complement of the left hand side digit from 9 as read from Column (b)

Multiplication table when both multiplicand and multiplier are of 2 digits.

	a	b	
11 x 99 =	10	89	= (11–1) / 99 – (11–1) = 1089
12 x 99 =	11	88	= (12–1) / 99 – (12–1) = 1188
13 x 99 =	12	87	= (13–1) / 99 – (13–1) = 1287
--			
18 x 99 =	17	82	----------------------------
19 x 99 =	18	81	
20 x 99 =	19	80	= (20–1) / 99 – (20–1) = 1980

The rule mentioned in the case of above table also holds good here

Further we can state that the rule applies to all cases, where the multiplicand and the multiplier have the same number of digits.

Consider the following **Tables**.

(i)

	a	b
11 x 9 =	9	9
12 x 9 =	10	8
13 x 9 =	11	7
---------------------		
18 x 9 =	16	2
19 x 9 =	17	1
20 x 9 =	18	0

(ii)

21 x 9 = 18   9
22 x 9 = 19   8
23 x 9 = 20   7
----
28 x 9 = 25   2
29 x 9 = 26   1
30 x 9 = 27   0

(iii)

35 x 9 = 31   5
46 x 9 = 41   4
53 x 9 = 47   7
67 x 9 = 60   3
-------------------------so on.

From the above tables the following points can be observed:

1) **Table (i)** has the multiplicands with 1 as first digit except the last one. Here L.H.S of products are uniformly 2 less than the multiplicands. So also with20 x 9

2) **Table (ii)** has the same pattern. Here L.H.S of products are uniformly 3 less than the multiplicands.

3) **Table (iii)** is of mixed example and yet the same result i.e. if 3 is first digit of the multiplicand then L.H.S of product is 4 less than the multiplicand; if 4 is first digit of the multiplicand then, L.H.S of the product is 5 less than the multiplicand and so on.

4) The right hand side of the product in all the tables and cases is obtained by subtracting the R.H.S. part of the multiplicand by Nikhilam.

Keeping these points in view we solve the problems:

Example1 : 42 X 9

i) Divide the multiplicand (42) of by a Vertical line or by theSign **:** into a right hand portion consisting of as many digits as the multiplier.

i.e. 42 has to be written as **4/2** or **4:2**

ii) Subtract from the multiplicand one more than the whole excess portion on the left. i.e. left portion of multiplicand is 4.

one more than it 4 + 1 = 5.

We have to subtract this from multiplicand
i.e. write it as

```
  4 : 2
    :-5
---------------
  3 : 7
```

This gives the L.H.S part of the product.

This step can be interpreted as "take the ekanyunena and sub tract from the previous" i.e. the excess portion on the left.

iii) Subtract the R.H.S. part of the multiplicand by nikhilam process.
i.e. R.H.S of multiplicand is 2

its nikhilam is 8

It gives the R.H.S of the product

i.e. answer is 3 : 7 : 8 = 378.

Thus 42 X 9 can be represented as

```
      4 : 2
        :-5 : 8
------------------
      3 : 7 : 8 = 378.
```

Example 2 : 124 X 9

Here Multiplier has one digit only .

We write 12 : 4

Now step (ii),12 + 1 = 13

```
i.e.    12 : 4
        -1 : 3
      ------------
```

Step (iii) R.H.S. of multiplicand is 4. Its Nikhilam is 6

```
124 x 9 is    12 : 4
              -1 : 3 : 6
          -----------------
              11 : 1 : 6    =    1116
```

The process can also be represented as
124 x 9 = [124 – (12 + 1)] : (10 – 4) = (124 – 13) : 6 = 1116

Example 3: 15639 x 99

Since the multiplier has 2 digits, the answer is
[15639 – (156 + 1)] : (100 – 39) = (15639 – 157) : 61 = 1548261

Find the products in the following cases.

1. 58 x 9 2. 62 x 9 3. 427 x 99

4. 832 x 9 5. 24821 x 999 6. 111011 x 99

Ekanyunena Sutra is also useful in Recurring Decimals. We can take up this under a separate treatment.

Thus we have a glimpse of majority of the Sutras. At some places some Sutras are mentioned as Sub-Sutras. Any how we now proceed into the use of Sub-Sutras. As already mentioned the book on Vedic Mathematics enlisted 13 Upa-Sutras.

But some approaches in the Vedic Mathematics book prompted some serious research workers in this field to mention some other Upa-Sutras. We can observe those approaches and developments also.

11. Anurupyena

The upa-Sutra 'anurupyena' means 'proportionality'. This Sutra is highly useful to find products of two numbers when both of them are near the Common bases i.e powers of base 10 . It is very clear that in such cases the expected 'Simplicity ' in doing problems is absent.

Example 1: 46 X 43

As per the previous methods, if we select 100 as base we get

46	-54	This is much more difficult and of no use.
43	-57	

Now by 'anurupyena' we consider a working base In three ways. We can solve the problem.

Method 1: Take the nearest higher multiple of 10. In this case it is 50.

Treat it as 100 / 2 = 50. Now the steps are as follows:

i) Choose the working base near to the numbers under consideration.
i.e., working base is 100 / 2 = 50

ii) Write the numbers one below the other

i.e.
```
   4  6
   4  3
  ______
```

iii) Write the differences of the two numbers respectively from 50 against each number on right side

i.e.
```
   46   -04
   43   -07
  __________
```

iv) Write cross-subtraction or cross- addition as the case may be under the line drawn.

```
      46          -04
      43          -07
  ____________________
  (46 - 7 )  or
  (43 - 4 )
  ____________________
   = 39
```

v) Multiply the differences and write the product in the left side of the answer.

```
     46   -04
     43   -07
  _____________
   39  / -4 x –7

        = 28
```

vi) Since base is 100 / 2 = 50 , 39 in the answer represents 39X50.

Hence divide 39 by 2 because 50 = 100 / 2

Thus 39 ÷ 2 gives 19½ where 19 is quotient and 1 is remainder . This 1 as Reminder gives one 50 making the L.H.S of the answer 28 + 50 = 78(or Remainder ½ x 100 + 28)

i.e. R.H.S 19 and L.H.S 78 together give the answer1978 We represent it as

```
      46   -04
      43   -07
     ___________
  2)  39  /  28
     ___________
     19½ /  28

   = 19 / 78 = 1978
```

Example 2: 42 X 48.

With 100 / 2 = 50 as working base, the problem is as follows:

```
      42   -08
      48   -02
     ___________
  2)  40  /  16
     ___________
      20  /  16

    42 x 48 = 2016
```

Method 2: For the **example 1:** 46X43. We take the same working base 50. We treat it as 50=5X10. i.e. we operate with 10 but not with 100 as in method

now

```
        46           -04
        43           -07
   _____________________________
   ( 46 - 7 )  or  / -4 x -7
   ( 43 - 4 )     /
     = 39        /     = 28
   _____________________________
     39 x 5     /      8
       +2      /
 (carried over)
```

(195 + 2) / 8 = 1978

[Since we operate with 10, the R.H.S portion shall have only unit place .Hence out of the product 28, 2 is carried over to left side. The L.H.S portion of the answer shall be multiplied by 5, since we have taken 50 = 5 X 10.]

Now in the **example 2:** 42 x 48 we can carry as follows by treating 50 = 5 x 10

42	-08	
48	-02	
40	$_{1}6$	
x 5		
200	$_{1}6$	= 2016

Method 3: We take the nearest lower multiple of 10 since the numbers are 46 and 43 as in the first example, We consider 40 as working base and treat it as 4 X 10.

46	06
43	03
(46 + 3) or	6 x 3
(43 + 6)	
= 49	= $_{1}8$

Since 10 is in operation 1 is carried out digit in 18.

Since 4 X 10 is working base we consider 49 X 4 on L.H.S of answer i.e. 196 and 1 carried over the left side, giving L.H.S. of answer as 1978. Hence the answer is 1978.

We proceed in the same method for 42 X 48

42	02	
48	08	working base 4 x10 = 40
50	$_{1}6$	
x 4		since 10 is in operation
200	$_{1}6$	= 2016

Let us see the all the three methods for a problem at a glance

Example 3: 24 X 23

Method - 1: Working base = 100 / 5 = 20

```
 24   04
 23   03
__________
5) 27 /  12
__________
5 2/5 / 12   =  5 / 52  = 552
```

[Since 2 / 5 of 100 is 2 / 5 x 100 = 40 and 40 + 12 = 52]

Method - 2: Working base 2 X 10 = 20

```
    24      04
    23      03
   ____________
    27   /  12     it is more convenient than
X   2              method-1 Guss the reason
   ____________
    54   /  12     =  552
```

Now as 20 itself is nearest lower multiple of 10 for the problem under consideration, the case of method – 3 shall not arise.

Let us take another example and try all the three methods.

Example 4: 492 X 404

Method - 1 : working base = 1000 / 2 = 500

```
   492   -008
   404   -096
   ____________
2) 396  /  768      since 1000 is in operation
   ____________
   198  /  768      =  198768
```

Method 2: working base = 5 x 100 = 500

492	-008
404	-096
396 /	$_7$68
X 5	
1980 /	$_7$68 = 198768

Method - 3.

Since 400 can also be taken as working base, treat 400 = 4 X 100 as working base.

Thus

492	092	
404	004	since 100 is in operation
496 /	$_3$68	
X 4		
1984 /	$_3$68	= 198768

No need to repeat that practice in these methods finally takes us to work out all these mentally and getting the answers straight away in a single line.

Example 5: 3998 X 4998

Working base = 10000 / 2 = 5000

3998	-1002
4998	-0002

2) 3996 / 2004 since 10,000 is in operation

1998 / 2004 = 19982004

or taking working base = 5 x 1000 = 5,000 and

3998	-1002	
4998	-0002	since 2 is carried out
3996 /	$_2$004	
X 5		
19980 /	$_2$004	= 19982004

What happens if we take 4000 i.e. 4 X 1000 as working base?

$$\begin{array}{ll} 3998 & \overline{0002} \\ 4998 & 0998 \\ \hline \\ 4996 \ / & 1996 \end{array} \qquad \text{Since 1000 is an operation}$$

As 1000 is in operation, $1\overline{996}$ has to be written as ${}_1\overline{996}$ and 4000 as base, the L.H.S portion 5000 has to be multiplied by 4. i. e. the answer is

$$\begin{array}{rl} 4996 \ / & {}_1\overline{996} \\ \times \quad 4 & \\ \hline 19982 \ / & 004 \end{array} \quad = 19982004$$

A simpler example for better understanding.

Example 6: 58 x 48

Working base 50 = 5 x 10 gives

$$\begin{array}{rl} 50 & +8 \\ 48 & \overline{2} \\ \hline 56 \ / & {}_1\overline{6} \\ \times \ 5 & \\ \hline 280 \ / & {}_1\overline{6} \end{array} \quad = 280 / 4 = 2784$$

Since 10 is in operation.

Use anurupyena by selecting appropriate working base and method.

Find the following product.

1. 46 x 46 **2. 57 x 57** **3. 54 x 45**

4. 18 x 18 **5. 62 x 48** **6. 229 x 230**

7. 47 x 96 **8. 87965 x 99996** **9. 49x499**

10. 389 x 512

12. Adyamadyenantya – mantyena

The Sutra ' adyamadyenantya-mantyena' means 'the first by the first and the last by the last'.

Suppose we are asked to find out the area of a rectangular card board whose length and breadth are respectively 6ft . 4 inches and 5 ft. 8 inches. Generally we continue the problem like this.

Area = Length X Breath

= 6′ 4" X 5′ 8" Since 1′ = 12", conversion

= (6 X 12 + 4) (5 X 12 + 8) in to single unit

= 76" 68" = 5168 Sq. inches.

Since 1 sq. ft. =12 X 12 = 144sq.inches we have area

$$\frac{5168}{144} = 144)\ 5168\ (35$$

```
432
----
 848
 720   i.e., 35 Sq. ft 128 Sq. inches
----
 128
```

By Vedic principles we proceed in the way "the first by first and the last by last"

i.e. 6′ 4" can be treated as 6x + 4 and 5′ 8" as 5x + 8,

Where x= 1ft. = 12 in;x^2 is sq. ft.

Now (6x + 4)(5x + 8)

$= 30x^2 + 6.8.x + 4.5.x + 32$
$= 30x^2 + 48x + 20x + 32$
$= 30x^2 + 68. x + 32$
$= 30x^2 + (5x + 8). x + 32$ Writing 68 = 5 x 12 + 8
$= 35x^2 + 8. x + 32$
= 35 Sq. ft. + 8 x 12 Sq. in + 32 Sq. in
= 35 Sq. ft. + 96 Sq. in + 32 Sq. in
= 35 Sq. ft. + 128 Sq. in

It is interesting to know that a mathematically untrained and even uneducated carpenter simply works in this way by mental argumentation. It goes in his mind like this

6′ 4"

5′ 8"

First by first i.e. 6′ X 5′ = 30 sq. ft.

Last by last i.e. 4" X 8" = 32 sq. in.

Now cross wise 6 X 8 + 5 x 4 = 48 +20 = 68.

Adjust as many '12' s as possible towards left as 'units' i.e. 68 = 5 X 12 +8 , 5 twelve's as 5 square feet make the first 30+5 = 35 sq. ft ; 8 left becomes 8 x 12 square inches and go towards right i.e. 8 x 12 = 96 sq. in. towards right ives 96+32 = 128sq.in.

Thus he got area in some sort of 35 squints and another sort of 128 sq. units. i.e. 35 sq. ft 128 sq. in

Another Example:

	4' 6"	
	3' 4"	
4x3	4x4+6x3	6x4
	=16+18	
=12	=34	=24
	=2x12+10	

Now 12 + 2 = 14, 10 x 12 + 24 = 120 + 24 = 144

Thus 4′ 6″ x 3′ 4″ = 14 Sq. ft. 144 Sq. inches.

Since 144 sq. in = 12 X 12 = 1 sq. ft The answer is 15 sq. ft.

We can extend the same principle to find volumes of parallelepiped also.

I. Find the area of the rectangles in each of the following situations.

1). l = 3′ 8" , b = 2′ 4 " **2). l = 12′ 5" , b = 5′ 7"**

3). l = 4 yard 3 ft. b = 2 yards 5 ft.(1yard =3ft)

4). l = 6 yard 6 ft. b = 5 yards 2 ft.

II. Find the area of the trapezium in each of the following cases. Recall area =½ h (a + b) where a, b are parallel sides and h is the distance between them.

1). a = 3′ 7", b = 2′ 4", h = 1′ 5"

2). a = 5′ 6", b = 4′ 4", h = 3′ 2"

3). a = 8′ 4", b = 4′ 6", h = 5′ 1".

Factorization of quadratics:

The usual procedure of factorizing a quadratic is as follows:

$$\begin{aligned} & 3x^2 + 8x + 4 \\ &= 3x^2 + 6x + 2x + 4 \\ &= 3x (x + 2) + 2 (x + 2) \\ &= (x + 2) (3x + 2) \end{aligned}$$

But by mental process, we can get the result immediately. The steps are as follows.

i). Split the middle coefficient in to two such parts that the ratio of the first coefficient to the first part is the same as the ratio of the second part to the last coefficient. Thus we split the coefficient of middle term of $3x^2 + 8x + 4$ i.e. 8 in to two such parts 6 and 2 such that the ratio of the first coefficient to the first part of the middle coefficient i.e. 3:6 and the ratio of the second pat to the last coefficient, i.e. 2: 4 are the same. It is clear that 3:6 = 2:4. Hence such split is valid. Now the ratio 3: 6 = 2: 4 = 1:2 gives one factor x+2.

ii). Second factor is obtained by dividing the first coefficient of the quadratic by the fist coefficient of the factor already found and the last coefficient of the quadratic by the last coefficient of the factor.

i.e. the second factor is

$$\frac{3x^2}{x} + \frac{4}{2} = 3x + 2$$

Hence $3x^2 + 8x + 4 = (x + 2)(3x + 2)$

Eg.1: $4x^2 + 12x + 5$

i) Split 12 into 2 and 10 so that as per rule 4 : 2 = 10 : 5 = 2 : 1i.e.,, $2x + 1$ is first factor.

ii) Now

$$\frac{4x^2}{2x} + \frac{5}{1} = 2x + 5 \quad \text{is second factor.}$$

Eg.2: $15x^2 - 14xy - 8y^2$

i) Split –14 into –20, 6 so that 15 : - 20 = 3 : - 4 and 6 : - 8 = 3 : - 4. Both are same.i.e., $(3x - 4y)$ is one factor.

ii) Now

$$\frac{15x^2}{3x} + \frac{8y^2}{-4y} = 5x + 2y \text{ is second factor.}$$

Thus $15x^2 - 14xy - 8y^2 = (3x - 4y)(5x + 2y)$.

It is evident that we have applied two sub-sutras **'anurupyena'** i.e.'proportionality' and **'adyamadyenantyamantyena'** i.e. 'the first by the first and the last by the last' to obtain the above results.

Factorise the following quadratics applying appropriate vedic maths sutras:

1). $3x^2 + 14x + 15$

2). $6x^2 - 23x + 7$

3). $8x^2 - 22x + 5$

4). $12x^2 - 23xy + 10y^2$

13. Yavadunam Tavadunikrtya Varganca Yojayet

The meaning of the Sutra is 'what ever the deficiency subtract that deficit from the number and write along side the square of that deficit'.

This Sutra can be applicable to obtain squares of numbers close to bases of powers of 10.

Method-1 : Numbers near and less than the bases of powers of 10.

Eg 1: 9^2 Here base is 10.

The answer is separated in to two parts by a'/'

Note that deficit is 10 - 9 = 1

Multiply the deficit by itself or square it

$1^2 = 1$. As the deficiency is 1, subtract it from the number i.e., 9–1 = 8.

Now put 8 on the left and 1 on the right side of the vertical line or slash i.e., 8/1.

Hence 81 is answer.

Eg. 2: 96^2 Here base is 100.

Since deficit is 100-96=4 and square of it is 16 and the deficiency subtracted from the number 96 gives 96-4 = 92, we get the answer 92 / 16 Thus $96^2 = 9216$.

Eg. 3: 994^2 Base is 1000

Deficit is 1000 - 994 = 6. Square of it is 36.

Deficiency subtracted from 994 gives 994 - 6 = 988

Answer is 988 / 036 [since base is 1000]

Eg. 4: 9988^2 Base is 10,000.

Deficit = 10000 - 9988 = 12.

Square of deficit = $12^2 = 144$.

Deficiency subtracted from number = 9988 - 12 = 9976.

Answer is 9976 / 0144   [since base is 10,000].

Eg. 5: 88^2 Base is 100.

Deficit = 100 - 88 = 12.

Square of deficit = 12^2 = 144.

Deficiency subtracted from number =88 - 12 = 76.

Now answer is 76 / $_1$44 =7744   [since base is 100]

Algebraic proof:

The numbers near and less than the bases of power of 10 can be treated as (x-y), where x is the base and y, the deficit.

Thus

(1) 9 = (10 -1) (2) 96 = (100-4) (3) 994 = (1000-6)

(4) 9988 = (10000-12) (v) 88 = (100-12)

$$\begin{aligned}(x - y)^2 &= x^2 - 2xy + y^2 \\ &= x (x - 2y) + y^2 \\ &= x (x - y - y) + y^2 \\ &= \text{Base (number – deficiency) + (deficit)}^2\end{aligned}$$

Thus

$$\begin{aligned}985^2 &= (1000 - 15)^2 \\ &= 1000 (985 - 15) + (15)^2 \\ &= 1000 (970) + 225 \\ &= 970000 + 225 \\ &= 970225.\end{aligned}$$

or we can take the identity $a^2 - b^2 = (a + b) (a - b)$ and proceed as

$$a^2 - b^2 = (a + b) (a - b).$$

gives $a^2 = (a + b) (a - b) + b^2$

Thus for a = 985 and b = 15;

$$a^2 = (a + b) (a - b) + b^2$$

$$985^2 = (985 + 15) (985 - 15) + (15)^2$$

$= 1000 (970) + 225$

$= 970225.$

Method. 2 : Numbers near and greater than the bases of powers of 10.

Eg.(1): 13^2 .

Instead of subtracting the deficiency from the number we add and proceed as in Method-1.

for 13^2 , base is 10, surplus is 3.

Surplus added to the number = 13 + 3 = 16.

Square of surplus = $3^2 = 9$

Answer is 16 / 9 = 169.

Eg.(2): 112^2

Base = 100, Surplus = 12,

Square of surplus = $12^2 = 144$

add surplus to number = 112 + 12 = 124.

Answer is 124 / $_1$44 = 12544

Or think of identity $a^2 = (a + b) (a - b) + b^2$ for a = 112, b = 12:

$$\begin{aligned} 112^2 &= (112 + 12) (112 - 12) + 12^2 \\ &= 124 (100) + 144 \\ &= 12400 + 144 \\ &= 12544. \end{aligned}$$

$$\begin{aligned} (x + y)^2 &= x^2 + 2xy + y^2 \\ &= x (x + 2y) + y^2 \\ &= x (x + y + y) + y^2 \end{aligned}$$

= Base (Number + surplus) + (surplus)^2

gives

$$\begin{aligned} 112^2 &= 100 (112 + 12) + 12^2 \\ &= 100 (124) + 144 \end{aligned}$$

= 12400 + 144
= 12544.

Eg. 3: 10025^2

= (10025 + 25) / 25^2

= 10050 / 0625 [since base is 10,000]

= 100500625.

Method - 3: This is applicable to numbers which are near to multiples of 10, 100, 1000 .... etc. For this we combine the upa-Sutra 'anurupyena' and 'yavadunam tavadunikritya varganca yojayet' together.

Example 1: 388^2 Nearest base = 400.

We treat 400 as 4 x 100. As the number is less than the base we proceed as follows

Number 388, deficit = 400 - 388 = 12

Since it is less than base, deduct the deficit

i.e. 388 - 12 = 376.

multiply this result by 4 since base is 4 X 100 = 400.

376 x 4 = 1504

Square of deficit = 12^2 = 144.

Hence answer is 1504 / $_1$44 = 150544 [since we have taken multiples of 100].

Example 2: 485^2 Nearest base = 500.

Treat 500 as 5 x 100 and proceed

485^2 = (485 - 15) / 15^2 [since deficit is 15]
= 470 / $_2$25 since 500 base is taken as 5 x 100
X 5 and 2 of 225 is carried over
= 2350 / $_2$25
= 235225

Example 3: 67^2 Nearest base = 70

$67^2 = (67 - 3) / 3^2$ deficit is 3

$= 64 / 9$

$\times 7$ [since 7 x 10 = 70]

$= 448 / 9$

$= 4489$

Example 4: 416^2 Nearest (lower) base = 400

Here surplus = 16 and 400 = 4 x 100

$416^2 = (416 + 16) / 16^2$

$= 432 / {}_{2}56$ since base is multiple 100;

$\times \mathbf{4}$ and 2 of 256 is carried over

$= 1728 / {}_{2}56$

$= 173056$

Example 5: 5012^2 Nearest lower base is 5000 = 5 x 1000

Surplus = 12

$5012^2 = (5012 + 12) / 12^2$

$= 5024 / 144$

$\times 5$

$= 25120 / 144$

$= 25120144$

Apply yavadunam to find the following squares.

1. 7^2 **2. 98^2** **3. 987^2** **4. 14^2**

5. 116^2 **6. 1012^2** **7. 19^2** **8. 475^2**

9. 796^2 **10. 108^2** **11. 9988^2** **12. 6014^2.**

So far we have observed the application of yavadunam in finding the squares of number. Now with a slight modification yavadunam can also be applied for finding the cubes of numbers.

Cubing of Numbers:

Example : Find the cube of the number 106.

We proceed as follows:

i) For 106, Base is 100. The surplus is 6.

Here we add double of the surplus i.e. 106+12 = 118.

(Recall in squaring, we directly add the surplus)

This makes the left-hand -most part of the answer.

i.e. answer proceeds like 118 / - - - - -

ii) Put down the new surplus i.e. 118-100=18 multiplied by the initial surplus

i.e. 6=108.

Since base is 100, we write 108 in carried over form 108 i.e. .

As this is middle portion of the answer, the answer proceeds like 118 / $_{1}08$ /....

iii) Write down the cube of initial surplus i.e. 6^3 = 216 as the last portion

i.e. right hand side last portion of the answer.

Since base is 100, write 216 as $_{2}16$ as 2 is to be carried over.

Answer is 118 / $_{1}08$ / $_{2}16$

Now proceeding from right to left and adjusting the carried over, we get the answer

119 / 10 / 16 = 1191016.

Eg.(1): 102^3 = (102 + 4) / 6 X 2 / 2^3

= 106 = 12 = 08

= 1061208.

Observe initial surplus = 2, next surplus =6 and base = 100.

Eg.(2): 94^3

Observe that the nearest base = 100. Here it is deficit contrary to the above examples.

i) Deficit = -6. Twice of it -6 X 2 = -12
add it to the number = 94 -12 =82.

ii) New deficit is -18.
Product of new deficit x initial deficit = -18 x -6 = 108

iii) $\text{deficit}^3 = (-6)^3 = -216$.

Hence the answer is $82 \,/\, {}_{1}08 \,/\, -\overline{{}_{2}16}$

Since 100 is base 1 and -2 are the carried over. Adjusting the carried over in order, we get the answer

(82 + 1) /(08 – 03) / (100 – 16)

= 83 / = 05 / = 84 = 830584

$\overline{16}$ becomes 84 after taking1 from middle most portion i.e. 100. (100-16=84).

Now 08 - 01 = 07 remains in the middle portion, and2 $\overline{\text{o}}$r 2 carried to it makes the middle as 07 - 02 = 05. Thus we get the above result.

Eg.(3):

998^3 Base = 1000; initial deficit = - 2.

$998^3 = (998 - 2 \times 2) \;/\; (-6 \times -2) \;/\; (-2)^3$

= 994 / = 012 / = -008

= 994 / 011 / 1000 - 008

= 994 / 011 / 992

= 994011992.

Find the cubes of the following numbers using yavadunam sutra.

1. 105 2. 114 3. 1003 4. 10007 5. 92
6. 96 7. 993 8. 9991 9. 1000008 10. 999992.

14. Antyayor Dasakepi

The Sutra signifies numbers of which the last digits added up give 10. i.e. the Sutra works in multiplication of numbers for example: 25 and 25, 47 and 43, 62 and 68, 116 and 114. Note that in each case the sum of the last digit of first number to the last digit of second number is 10. Further the portion of digits or numbers left wards to the last digits remain the same. At that instant use Ekadhikena on left hand side digits. Multiplication of the last digits gives the right hand part of the answer.

Example 1 : 47 X 43

See the end digits sum 7 + 3 = 10 ; then by the sutras antyayor dasakepi and ekadhikena we have the answer.

47 x 43 = (4 + 1) x 4/ 7 x 3
= 20 / 21
= 2021.

Example 2: 62 x 68

2 + 8 = 10, L.H.S. portion remains the same i.e.,, 6.

Ekadhikena of 6 gives 7

62 x 68 = (6 x 7)/ (2 x 8)
= 42 / 16
= 4216.

Example 3: 127 x 123

As antyayor dasakepi works, we apply ekadhikena

127 x 123 = 12 x 13/ 7 x 3
= 156 / 21
= 15621.

Example 4: 65 x 65

We have already worked on this type. As the present sutra is applicable.

We have 65 x 65 = 6 x 7/ 5 x 5
= 4225.

Example 5: 395^2

$395^2 = 395 \times 395$
$= 39 \times 40 / 5 \times 5$
$= 1560 / 25$
$= 156025.$

Use Vedic sutras to find the products

1. 125 x 125 **2. 34 x 36** **3. 98 x 92**

4. 401 x 409 **5. 693 x 697** **6. 1404 x 1406**

It is further interesting to note that the same rule works when the sum of the last 2, last 3, last 4 - - - digits added respectively equal to 100, 1000, 10000 -- - - . The simple point to remember is to multiply each product by 10, 100, 1000, - - as the case may be . Your can observe that this is more convenient while working with the product of 3 digit numbers.

Eg. 1: 292 x 208

Here 92 + 08 = 100, L.H.S portion is same i.e. 2

292 x 208 = (2 x 3)/ 92 x 8

$\longrightarrow$ 60 / =736 (for 100 raise the L.H.S. product by 0)

= 60736.

Eg. 2: 848 X 852

Here 48 + 52 = 100, L.H.S portion is 8 and its 'ekhadhikena' is 9.

Now R.H.S product 48 X 52 can be obtained by 'anurupyena' mentally.

$$\begin{array}{ll} 48 & \bar{2} \\ 52 & 2 \\ \hline 2)\ \underline{50} & \bar{4} \\ 25 & \end{array} \quad = 24 / (100 - 4) = 96$$

= 2496

and write 848 x 852 = 8 x 9 / 48 x 52

$\longrightarrow$ 720 = $_2$496

= 722496.

[Since L.H.S product is to be multiplied by 10 and 2 to be carried over as the base is 100].

Eg. 3: 693 x 607

693 x 607 = 6 x 7/ 93 x 7
= 420 / 651
= 420651.

Find the following products using 'antyayordasakepi'

1. 318 x 312 **2. 425 x 475** **3. 796 x 744**

4. 902 x 998 **5. 397 x 393** **6. 551 x 549**

15. Antyayoreva

'Atyayoreva' means 'only the last terms'. This is useful in solving simple equations of the following type.

The type of equations are those whose numerator and denominator on the L.H.S. bearing the independent terms stand in the same ratio to each other as the entire numerator and the entire denominator of the R.H.S. stand to each other.

Let us have a look at the following example.

Example 1:

$$\frac{x^2 + 2x + 7}{x^2 + 3x + 5} = \frac{x + 2}{x + 3}$$

In the conventional method we proceed as

$$\frac{x^2 + 2x + 7}{x^2 + 3x + 5} = \frac{x + 2}{x + 3}$$

$$(x + 3)(x^2 + 2x + 7) = (x + 2)(x^2 + 3x + 5)$$
$$x^3 + 2x^2 + 7x + 3x^2 + 6x + 21 = x^3 + 3x^2 + 5x + 2x^2 + 6x + 10$$
$$x^3 + 5x^2 + 13x + 21 = x^3 + 5x^2 + 11x + 10$$

Canceling like terms on both sides

$$13x + 21 = 11x + 10$$
$$13x - 11x = 10 - 21$$
$$2x = -11$$
$$x = -11 / 2$$

Now we solve the problem using anatyayoreva.

$$\frac{x^2 + 2x + 7}{x^2 + 3x + 5} = \frac{x + 2}{x + 3}$$

Consider

$$\frac{x^2 + 2x + 7}{x^2 + 3x + 5} = \frac{x + 2}{x + 3}$$

Observe that

$$\frac{x^2 + 2x}{x^2 + 3x} = \frac{x\,(x + 2)}{x\,(x + 3)} = \frac{x + 2}{x + 3}$$

This is according to the condition in the sutra. Hence from the sutra

$$\frac{x + 2}{x + 3} = \frac{7}{5}$$

$$5x + 10 = 7x + 21$$
$$7x - 5x = -21 + 10$$
$$2x = -11$$
$$x = -11 / 2$$

Algebraic Proof:

Consider the equation

$$\frac{AC + D}{BC + E} = \frac{A}{B} \quad \text{-------------} \ (i)$$

This satisfies the condition in the sutra since

$$\frac{AC}{BC} = \frac{A}{B}$$

Now cross–multiply the equation (i)

$$B\,(AC + D) = A\,(BC + E)$$
$$BAC + BD = ABC + AE$$
$$BD = AE \text{ which gives}$$

$$\frac{A}{B} = \frac{D}{E} \quad \text{--------}(ii)$$

i.e., the result obtained in solving equation **(i)** is same as the result obtained in solving equation **(ii)**.

Example 2: solve

$$\frac{2x^2 + 3x + 10}{3x^2 + 4x + 14} = \frac{2x + 3}{3x + 4}$$

Since

$$\frac{2x^2 + 3x}{3x^2 + 4x} = \frac{x\,(2x + 3)}{x\,(3x + 4)} = \frac{2x+3}{3x+4}$$

We can apply the sutra.

$$\frac{2x + 3}{3x+4} = \frac{10}{14}$$

Cross–multiplying

$$28x + 42 = 30x + 40$$
$$28x - 30x = 40 - 42$$
$$-2x = -2 \qquad \therefore x = -2 / -2 = 1.$$

Let us see the application of the sutra in another type of problem.

Example 3: (x + 1) (x + 2) (x + 9) = (x + 3) (x + 4) (x + 5)

Re–arranging the equation, we have

$$\frac{(x + 1)\,(x + 2)}{(x + 4)\,(x + 5)} = \frac{x + 3}{x + 9}$$

i.e.,

$$= \frac{x^2 + 3x + 2x + 3}{x^2 + 9x + 20x + 9}$$

Now

$$\frac{x^2 + 3x}{x^2 + 9x} = \frac{x\,(x + 3)}{x\,(x + 9)} = \frac{x + 3}{x + 9}$$ gives the solution by antyayoreva

Solution is obtained from

$$\frac{x + 3}{x + 9} = \frac{2}{20}$$

$20x + 60 = 2x + 18$
$20x – 2x = 18 – 60$
$18x = -42 \qquad \therefore x = -42 / 18 = -7 / 3.$

Once again look into the problem

$(x + 1)(x + 2)(x + 9) = (x + 3)(x+ 4)(x + 5)$

Sum of the binomials on each side

$x + 1 + x + 2 + x + 9 = 3x + 12$
$x + 3 + x + 4 + x + 5 = 3x + 12$

It is same. In such a case the equation can be adjusted into the form suitable for application of antyayoreva.

Example 4: $(x + 2)(x + 3)(x + 11) = (x + 4)(x + 5)(x + 7)$

Sum of the binomials on L.H.S. = $3x + 16$
Sum of the binomials on R.H.S. = $3x + 16$

They are same. Hence antyayoreva can be applied. Adjusting we get

$$\frac{(x + 2)(x + 3)}{(x + 4)(x + 7)} = \frac{x + 5}{x + 11} = \frac{2 \times 3}{4 \times 7} = \frac{6}{28}$$

$28x + 140 = 6x + 66$
$28x – 6x = 66 – 140$
$22x = -74$

$$x = \frac{-74}{22} = \frac{-37}{11}$$

Solve the following problems using 'antyayoreva'

1. $$\frac{3x^2 + 5x + 8}{5x^2 + 6x + 12} = \frac{3x + 5}{5x + 6}$$

2. $$\frac{4x^2 + 5x + 3}{3x^2 + 2x + 4} = \frac{4x + 5}{3x + 2}$$

3. $(x + 3)(x + 4)(x + 6) = (x + 5)(x + 1)(x + 7)$

4. $(x + 1)(x + 6)(x + 9) = (x + 4)(x + 5)(x + 7)$

5. $$\frac{2x^2 + 3x + 9}{4x^2 + 5x + 17} = \frac{2x + 3}{4x + 5}$$

16. Lopana Sthapanabhyam

Lopana sthapanabhyam means 'by alternate elimination and retention'.

Consider the case of factorization of quadratic equation of type $ax^2 + by^2 + cz^2 + dxy + eyz + fzx$ This is a homogeneous equation of second degree in three variables x, y, z. The sub-sutra removes the difficulty and makes the factorization simple. The steps are as follows:

i) Eliminate z by putting z = 0 and retain x and y and factorize thus obtained a quadratic in x and y by means of 'adyamadyena' sutra.;

ii) Similarly eliminate y and retain x and z and factorize the quadratic in x and z.

iii) With these two sets of factors, fill in the gaps caused by the elimination process of z and y respectively. This gives actual factors of the expression.

Example 1: $3x^2 + 7xy + 2y^2 + 11xz + 7yz + 6z^2$.

Step (i): Eliminate z and retain x, y; factorize
$3x^2 + 7xy + 2y^2 = (3x + y)(x + 2y)$

Step (ii): Eliminate y and retain x, z; factorize
$3x^2 + 11xz + 6z^2 = (3x + 2z)(x + 3z)$

Step (iii): Fill the gaps, the given expression
$= (3x + y + 2z)(x + 2y + 3z)$

Example 2: $12x^2 + 11xy + 2y^2 - 13xz - 7yz + 3z^2$.

Step (i): Eliminate z i.e., z = 0; factorize
$12x^2 + 11xy + 2y^2 = (3x + 2y)(4x + y)$

Step (ii): Eliminate y i.e., y = 0; factorize
$12x^2 - 13xz + 3z^2 = (4x - 3z)(3x - z)$

Step (iii): Fill in the gaps; the given expression
$= (4x + y - 3z)(3x + 2y - z)$

Example 3: $3x^2+6y^2+2z^2+11xy+7yz+6xz+19x+22y+13z+20$

Step (i): Eliminate y and z, retain x and independent term
i.e., y = 0, z = 0 in the expression (E).
Then $E = 3x^2 + 19x + 20 = (x + 5)(3x + 4)$

Step (ii): Eliminate z and x, retain y and independent term
i.e., z = 0, x = 0 in the expression.
Then E = $6y^2 + 22y + 20 = (2y + 4)(3y + 5)$

Step (iii): Eliminate x and y, retain z and independent term
i.e., x = 0, y = 0 in the expression.
Then E = $2z^2 + 13z + 20 = (z + 4)(2z + 5)$

Step (iv): The expression has the factors (think of independent terms)
= $(3x + 2y + z + 4)(x + 3y + 2z + 5)$.

In this way either homogeneous equations of second degree or general equations of second degree in three variables can be very easily solved by applying 'adyamadyena' and 'lopanasthapanabhyam' sutras.

Solve the following expressions into factors by using appropriate sutras:

1. $x^2 + 2y^2 + 3xy + 2xz + 3yz + z^2$.
2. $3x^2 + y^2 - 4xy - yz - 2z^2 - zx$.
3. $2p^2 + 2q^2 + 5pq + 2p - 5q - 12$.
4. $u^2 + v^2 - 4u + 6v - 12$.
5. $x^2 - 2y^2 + 3xy + 4x - y + 2$.
6. $3x^2 + 4y^2 + 7xy - 2xz - 3yz - z^2 + 17x + 21y - z + 20$.

Highest common factor:

To find the Highest Common Factor i.e. H.C.F. of algebraic expressions, the factorization method and process of continuous division are in practice in the conventional system. We now apply' Lopana - Sthapana' Sutra, the 'Sankalana vyavakalanakam' process and the 'Adyamadya' rule to find out the H.C.F in a more easy and elegant way.

Example 1: Find the H.C.F. of $x^2 + 5x + 4$ and $x^2 + 7x + 6$.

1. Factorization method:

$x^2 + 5x + 4 = (x + 4)(x + 1)$
$x^2 + 7x + 6 = (x + 6)(x + 1)$

H.C.F. is (x + 1).

2. Continuous division process.

$$
\begin{array}{l}
x^2 + 5x + 4 \;)\; x^2 + 7x + 6 \;(\; 1 \\
\qquad\qquad\quad\;\; x^2 + 5x + 4 \\
\hline
\qquad\qquad\qquad 2x + 2 \;)\; x^2 + 5x + 4 \;(\; \tfrac{1}{2}x \\
\qquad\qquad\qquad\qquad\qquad\; x^2 + x \\
\hline
\qquad\qquad\qquad\qquad\qquad 4x + 4 \;)\; 2x + 2 \;(\; \tfrac{1}{2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; 2x + 2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; 0
\end{array}
$$

Thus 4x + 4 i.e., (x + 1) is H.C.F.

3. Lopana - Sthapana process i.e. elimination and retention or alternate destruction of the highest and the lowest powers is as below:

$$
\text{Subtract} \left\{ \begin{array}{l}
x^2 + 5x + 4 \\
x^2 + 7x + 6 \\
\hline
-2)\; -2x - 2 \\
\hline
\quad\;\; x + 1
\end{array} \right.
$$

i.e.,, (x + 1) is H.C.F

Example 2: Find H.C.F. of $2x^2 - x - 3$ and $2x^2 + x - 6$

$$
\text{Subtract} \left\{ \begin{array}{l}
2x^2 - x - 3 \\
2x^2 + x - 6 \\
\hline
-1)\; -2x + 3 \\
\hline
\quad\;\; 2x + 3 \text{ is the H.C.F}
\end{array} \right.
$$

Example 3: $x^3 - 7x - 6$ and $x^3 + 8x^2 + 17x + 10$.

Now by Lopana - Sthapana and Sankalana – Vyavakalanabhyam

$$\text{Subtract}\left\{\begin{array}{l} x^3 - 7x - 6 \\ \underline{x^3 + 8x^2 + 17x + 10} \\ -8)\ \underline{-8x^2 - 24x - 16} \\ \quad x^2 + 3x + 2 \end{array}\right.$$

Example 4: $x^3 + 6x^2 + 5x - 12$ and $x^3 + 8x^2 + 19x + 12$.

$$\begin{array}{l} \text{Add} \\ \div\ 2x \end{array}\left\{\begin{array}{l} x^3 + 6x^2 + 5x - 12 \\ \underline{x^3 + 8x^2 + 19x + 12} \\ \underline{2x^3 + 14x^2 + 24x} \\ x^2 + 7x + 12 \end{array}\right.$$

(or)

$$\begin{array}{l} \text{Subtract} \\ \div\ -2 \end{array}\left\{\begin{array}{l} x^3 + 6x^2 + 5x - 12 \\ \underline{x^3 + 8x^2 + 19x + 12} \\ \underline{\quad 2x^2 - 14x - 24} \\ \quad x^2 + 7x + 12 \end{array}\right.$$

Example 5: $2x^3 + x^2 - 9$ and $x^4 + 2x^2 + 9$.

By Vedic sutras:

Add: $(2x^3 + x^2 - 9) + (x^4 + 2x^2 + 9)$
$= x^4 + 2x^3 + 3x^2$.

$\div\ x^2$ gives $x^2 + 2x + 3$ ------ (i)

Subtract after multiplying the first by x and the second by 2.

Thus $(2x^4 + x^3 - 9x) - (2x^4 + 4x^2 + 18)$
$= x^3 - 4x^2 - 9x - 18$ ------ (ii)

Multiply (i) by x and subtract from (ii)

$x^3 - 4x^2 - 9x - 18 - (x^3 + 2x^2 + 3x)$
$= - 6x^2 - 12x - 18$

$\div$ - 6 gives $x^2 + 2x + 3$.

Thus ($x^2 + 2x + 3$) is the H.C.F. of the given expressions.

Algebraic Proof:

Let P and Q be two expressions and H is their H.C.F. Let A and B the Quotients after their division by H.C.F.

i.e., $\frac{P}{H} = A$ and $\frac{Q}{H} = B$ which gives P = A.H and Q = B.H

$$P + Q = AH + BH = (A+B).H \quad \text{and} \quad P - Q = AH - BH = (A-B).H$$

Thus we can write $P \pm Q = (A \pm B).H$

Similarly MP = M.AH and NQ = N.BH gives $MP \pm NQ = H\,(MA \pm NB)$

This states that the H.C.F. of P and Q is also the H.C.F. of P±Q or MA±NB.

i.e. we have to select M and N in such a way that highest powers and lowest powers (or independent terms) are removed and H.C.F appears as we have seen in the examples.

Find the H.C.F. in each of the following cases using Vedic sutras:

1. $\mathbf{x^2 + 2x - 8, x^2 - 6x + 8}$

2. $\mathbf{x^3 - 3x^2 - 4x + 12, x^3 - 7x^2 + 16x - 12}$

3. $\mathbf{x^3 + 6x^2 + 11x + 6, x^3 - x^2 - 10x - 8}$

4. $\mathbf{6x^4 - 11x^3 + 16x^2 - 22x + 8,}$
   $\mathbf{6x^4 - 11x^3 - 8x^2 + 22x - 8.}$

17. Vilokanam

The Sutra 'Vilokanam' means 'Observation'. Generally we come across problems which can be solved by mere observation. But we follow the same conventional procedure and obtain the solution. But the hint behind the Sutra enables us to observe the problem completely and find the pattern and finally solve the problem by just observation.

Let us take the equation x + (1/x) = 5/2 Without noticing the logic in the problem, the conventional process tends us to solve the problem in the following way.

$$x + \frac{1}{x} = \frac{5}{2}$$

$$\frac{x^2 + 1}{x} = \frac{5}{2}$$

$$2x^2 + 2 = 5x$$
$$2x^2 - 5x + 2 = 0$$
$$2x^2 - 4x - x + 2 = 0$$
$$2x(x - 2) - (x - 2) = 0$$
$$(x - 2)(2x - 1) = 0$$
$$x - 2 = 0 \text{ gives } x = 2$$
$$2x - 1 = 0 \text{ gives } x = ½$$

But by Vilokanam i.e.,, observation

$$x + \frac{1}{x} = \frac{5}{2} \quad \text{can be viewed as}$$

$$x + \frac{1}{x} = 2 + \frac{1}{2} \quad \text{giving } x = 2 \text{ or } ½.$$

Consider some examples.

Example 1 :

$$\frac{x}{x + 2} + \frac{x + 2}{x} = \frac{34}{15}$$

In the conventional process, we have to take L.C.M, cross-multiplication. simplification and factorization. But Vilokanam gives

$$\frac{34}{15} = \frac{9 + 25}{5 \times 3} = \frac{3}{5} + \frac{5}{3}$$

$$\frac{x}{x + 2} + \frac{x + 2}{x} = \frac{3}{5} + \frac{5}{3}$$

gives

$$\frac{x}{x + 2} = \frac{3}{5} \quad \text{or} \quad \frac{5}{3}$$

$$\begin{array}{lcl} 5x = 3x + 6 & \text{or} & 3x = 5x + 10 \\ 2x = 6 & \text{or} & -2x = 10 \\ x = 3 & \text{or} & x = -5 \end{array}$$

Example 2 :

$$\frac{x + 5}{x + 6} + \frac{x + 6}{x + 5} = \frac{113}{56}$$

Now,

$$\frac{113}{56} = \frac{49 + 64}{7 \times 8} = \frac{7}{8} + \frac{8}{7}$$

$$\frac{x + 5}{x + 6} = \frac{7}{8} \quad \text{or} \quad \frac{x+5}{x+6} = \frac{8}{7}$$

$$\begin{array}{lcl} 8x + 40 = 7x + 42 & & 7x + 35 = 8x + 48 \\ & \text{or} & \\ x = 42 - 40 = 2 & & -x = 48 - 35 = 13 \\ x = 2 & \text{or} & x = -13. \end{array}$$

Example 3:

$$\frac{5x + 9}{5x - 9} + \frac{5x - 9}{5x + 9} = 2\frac{82}{319}$$

At first sight it seems to a difficult problem.

But careful observation gives

$$2\frac{82}{319} = \frac{720}{319} = \frac{841 - 121}{11 \times 29} = \frac{29}{11} - \frac{11}{29}$$

(Note: $29^2 = 841$, $11^2 = 121$)

$$\frac{5x + 9}{5x - 9} = \frac{29}{11} \text{ or } \frac{-11}{29}$$

(Note: 29 = 20 + 9 = 5 x 4 + 9 ; 11 = 20 – 9 = 5 x 4 – 9)

i.e.,

$x = 4$ or

$$\frac{5x + 9}{5x - 9} = \frac{-11}{29}$$

$$145x + 261 = -55x + 99$$
$$145x + 55x = 99 - 261$$
$$200x = -162$$

$$x = \frac{-162}{200} = \frac{-81}{100}$$

Simultaneous Quadratic Equations:

Example 1: $x + y = 9$ and $xy = 14$.

We follow in the conventional way that

$$(x - y)^2 = (x + y)^2 - 4xy = 9^2 - 4\,(14) = 81 - 56 = 25$$
$$x - y = \sqrt{25} = \pm 5$$

$$\left.\begin{aligned} x + y &= 9 \\ x - y &= 5 \end{aligned}\right\} \text{ gives } 2x = 14, \quad x = 7$$

$x + y = 9$ gives $7 + y = 9$

$y = 9 - 7 = 2$.

Thus the solution is $x = 7, y = 2$ or $x = 2, y = 7$.

But by Vilokanam, $xy = 14$ gives $x = 2, y = 7$ or $x = 7, y = 2$ and these two sets satisfy $x + y = 9$ since $2 + 7 = 9$ or $7 + 2 = 9$. Hence the solution.

Example 2: $5x – y = 7$ and $xy = 6$.

$xy = 6$ gives $x = 6, y = 1; x = 1, y = 6;$

$x = 2, y = 3; x = 3, y = 2$ and of course negatives of all these.

Observe that $x = 6, y = 1; x = 1, y = 6$: are not solutions because they do not satisfy the equation $5x – y = 7$.

But for $x = 2, y = 3$; $5x – y = 5 (2) – 3 = 10 – 3 = 7$ we have $5(3)–2\neq7$.

Hence $x = 2, y = 3$ is a solution.

For $x = 3, y = 2$ we get $5 (3) – 2 = 15 – 2 \neq 7$.

Hence it is not a solution.

Negative values of the above are also not the solutions. Thus one set of the solutions i.e., $x = 2, y = 3$ can be found. Of course the other will be obtained from solving $5x – y = 7$ and $5x + y = -13$.

i.e., $x = -3 / 5, y = -10$.

Partial Fractions:

Example 1: Resolve

$$\frac{2x + 7}{(x + 3)(x + 4)}$$ into partial fractions.

We write
$$\frac{2x + 7}{(x + 3)(x + 4)} = \frac{A}{(x + 3)} + \frac{B}{(x + 4)}$$

$$= \frac{A (x + 4) + B (x + 3)}{(x + 3) (x + 4)}$$

$$2x + 7 \equiv A(x + 4) + B(x + 3).$$

We proceed by comparing coefficients on either side

coefficient of x : $A + B = 2$ ....(i) X 3

Independent of x : $4A + 3B = 7$ ....(ii)

Solving (ii) – (i) x 3

$$\begin{array}{r} 4A + 3B = 7 \\ 3A + 3B = 6 \\ \hline A = 1 \end{array}$$

$A = 1$ in (i) gives, $1 + B = 2$ i.e., $B = 1$

Or we proceed as

$$2x + 7 \equiv A(x + 4) + B(x + 3).$$

Put $x = -3$, $2(-3) + 7 \equiv A(-3 + 4) + B(-3 + 3)$

$$1 = A(1) \quad \therefore A = 1.$$

$x = -4$, $2(-4) + 7 = A(-4 + 4) + B(-4 + 3)$

$$-1 = B(-1) \quad \therefore B = 1.$$

Thus $$\frac{2x + 7}{(x + 3)(x + 4)} = \frac{1}{(x + 3)} + \frac{1}{(x + 4)}$$

But by Vilokanam $\dfrac{2x + 7}{(x + 3)(x + 4)}$ can be resolved as

$(x + 3) + (x + 4) = 2x + 7$, directly we write the answer.

Example 2:

$$\frac{3x + 13}{(x + 1)(x + 2)}$$

from $(x + 1)$,$(x + 2)$ we can observe that

$10(x + 2) - 7(x + 1) = 10x + 20 - 7x - 7 = 3x + 13$

Thus $$\frac{3x + 13}{(x + 1)(x + 2)} = \frac{10}{x + 1} - \frac{7}{x + 2}$$

Example 3:

$$\frac{9}{x^2 + x - 2}$$

As $x^2 + x - 2 = (x - 1)(x + 2)$ and
$9 = 3(x + 2) - 3(x - 1)$
$(3x + 6 - 3x + 3 = 9)$

We get by Vilokanam, $$\frac{9}{x^2 + x - 2} = \frac{3}{x - 1} - \frac{3}{x + 2}$$

I. Solve the following by mere observation i.e. vilokanam

1. $$x + \frac{1}{x} = \frac{25}{12}$$

2. $$x - \frac{1}{x} = \frac{5}{6}$$

3. $$\frac{x}{x + 1} + \frac{x + 1}{x} = 9\frac{1}{9}$$

4. $$\frac{x + 7}{x + 9} - \frac{x + 9}{x + 7} = \frac{32}{63}$$

II. Solve the following simultaneous equations by vilokanam.

1. $x - y = 1, xy = 6$ 2. $x + y = 7, xy = 10$

3. $2x + 3y = 19, xy = 15$

4. $x + y = 4, x^2 + xy + 4x = 24.$

III. Resolve the following into partial fractions.

1. $$\frac{2x - 5}{(x - 2)(x - 3)}$$

2. $$\frac{9}{(x + 1)(x - 2)}$$

3. $$\frac{x - 13}{x^2 - 2x - 15}$$

4. $$\frac{3x + 4}{3x^2 + 3x + 2}$$

18. Gunita Samuccayah : Samuccaya Gunitah

In connection with factorization of quadratic expressions a sub-Sutra, viz. 'Gunita samuccayah-Samuccaya Gunitah' is useful. It is intended for the purpose of verifying the correctness of obtained answers in multiplications, divisions and factorizations. It means in this context:

'The product of the sum of the coefficients **sc** in the factors is equal to the sum of the coefficients **sc** in the product'

Symbolically we represent as **sc** of the product = product of the **sc** (in the factors)

Example 1: $(x + 3)(x + 2) = x2 + 5x + 6$

Now $(x + 3)(x + 2) = 4 \times 3 = 12$: Thus verified.

Example 2: $(x - 4)(2x + 5) = 2x^2 - 3x - 20$

Sc of the product $2 - 3 - 20 = -21$

Product of the **Sc** $= (1 - 4)(2 + 5) = (-3)(7) = -21$. Hence verified.

In case of cubics, biquadratics also the same rule applies.

We have $(x + 2)(x + 3)(x + 4) = x^3 + 9x^2 + 26x + 24$

Sc of the product $= 1 + 9 + 26 + 24 = 60$

Product of the **Sc** $= (1 + 2)(1 + 3)(1 + 4)$

$= 3 \times 4 \times 5 = 60$. Verified.

Example 3: $(x + 5)(x + 7)(x - 2) = x^3 + 10x^2 + 11x - 70$

$(1 + 5)(1 + 7)(1 - 2) = 1 + 10 + 11 - 70$

i.e., $6 \times 8 \times -1 = 22 - 70$
i.e., $-48 = -48$ Verified.

We apply and interpret **So** and **Sc** as sum of the coefficients of the odd powers and sum of the coefficients of the even powers and derive that **So** = **Sc** gives $(x + 1)$ is a factor for thee concerned expression in the variable x. **Sc** = 0 gives $(x - 1)$ is a factor.

Verify whether the following factorization of the expressions are correct or not by the Vedic check:
i.e. Gunita. Samuccayah-Samuccaya Gunitah:

1. $(2x + 3)(x - 2) = 2x^2 - x - 6$

2. $12x^2 - 23xy + 10y^2 = (3x - 2y)(4x - 5y)$

3. $12x^2 + 13x - 4 = (3x - 4)(4x + 1)$

4. $(x + 1)(x + 2)(x + 3) = x^3 + 6x^2 + 11x + 6$

5. $(x + 2)(x + 3)(x + 8) = x^3 + 13x^2 + 44x + 48$

So far we have considered a majority of the upa-sutras as mentioned in the Vedic mathematics book. Only a few Upa-Sutras are not dealt under a separate heading . They are

2) S'ISYATE S'ESASAMJ ÑAH

4) KEVALAIH SAPTAKAMGUNYAT

5) VESTANAM

6) YAVADŨNAM TAVADŨNAM and

10) SAMUCCAYAGUNITAH already find place in respective places.

Further in some other books developed on Vedic Mathematics DVANDAYOGA, SUDHA, DHVAJANKAM are also given as Sub-Sutras. They are mentioned in the Vedic Mathematics text also. But the list in the text (by the Editor) does not contain them. We shall also discuss them at appropriate places, with these three included, the total number of upa-Sutras comes to sixteen.

We now proceed to deal the Sutras with reference to their variety, applicability, speed, generality etc. Further we think how 'the element of choice in the Vedic system, even of innovation, together with mental approach, brings a new dimension to the study and practice of Mathematics. The variety and simplicity of the methods brings fun and amusement, the mental practice leads to a more agile, alert and intelligent mind and innovation naturally follow' (Prof. K.R.Williams, London).

III Vedic Mathematics - A briefing

In the previous chapters we have gone through the Vedic Mathematics Sutras and upa - Sutras: their application in solving problems. In this approach we have missed to note some points and merits of one method over the other methods at some instances.

Now we take a few steps in this direction. You may question why this book first gives examples and methods and then once again try to proceed as if an introduction to the Vedic Mathematics has been just started. This is because in this approach the reader first feels that it is easy to solve problems using Vedic Mathematics. This is clear from the examples given. But the reader may get doubt why we are doing this way or that way some times very close and almost akin to the conventional textual way; and some times very different from these procedures? why new representations and different meanings for the same Sutra (!) in different contexts? But observe that it is not uncommon to Mathematics.

Question some body showing the symbol Π.

Majority may say it is 22 / 7 (is it right?) some may say it is a radian measure. Very few may state it is a function or so.

What does the representation A X B mean?

A boy thinking about numbers may answer that is A multi plied by B and gives the product provided A and B are known. A girl thinking of set notation simply says that it is Cartesian product of the sets A and B. No sort of multiplication at all.

Another may conclude that it is a product of two matrices A and B . No doubt a multiplication but altogether different from above.

Some other may go deep in to elementary number theory and may take ' X ' to be the symbol ' X ' (does not divide) and conclude 'A does not divide B'

Now the question arises does a student fail to understand and apply contextual meaning and representation of symbols and such forms in mathematical writings? certainly not. In the same way the contextual meanings of the Sutras also can not bring any problem to the practitioners of Vedic Mathematics.

Again a careful observation brings all of us to a conclusion that even though the Sutras are not like mathematical formulae so as to fit in any context under consideration but they are intended to recognize the pattern in the problems and suggest procedures to solve. Now recall the terms, rules and methods once again to fill in some gaps that occur in the previous attempt.

Terms and Operations

a) Ekadhika means 'one more'

e.g: Ekadhika of 0 is 1
Ekadhika of 1 is 2

Ekadhika of 8 is 9

Ekadhika of 23 is 24

Ekadhika of 364 is 365------

b) Ekanyuna means 'one less'

e.g: Ekanyuna of 1 2 3 ..... 8 ..... 14 .....69 ......
is 0 1 2 ..... 7 ......13 .... 68 ......

c) Purak means ' complement'

e.g: purak of 1 2 3 ..... 8., 9 from 10
is 9 8 7 ..... 2 1

d) Rekhank means 'a digit with a bar on its top'. In other words it is a negative number.

e.g: A bar on 7 is $\bar{7}$. It is called rekhank 7 or bar 7. We treat purak as aRekhank.

e.g: $\bar{7}$ is 3 and $\bar{3}$ is 7

At some instances we write negative numbers also with a bar on the top of the numbers as

-4 can be shown as $\bar{4}$.

-21 can be shown as $\overline{21}$.

e) Addition and subtraction using Rekhank.

Adding a bar-digit i.e. Rekhank to a digit means the digit is subtracted.

e.g: $3 + \bar{1} = 2$, $5 + \bar{2} = 3$, $4 + \bar{4} = 0$

Subtracting a bar - digit i.e. Rekhank to a digit means the digit is added.

e.g: $4 - \bar{1} = 5$, $6 - \bar{2} = 8$, $3 - \bar{3} = 6$

1. Some more examples

e.g: $3 + 4 = 7$

$(-2) + (-5) = \bar{2} + \bar{5} = \bar{7}$ or -7

f) Multiplication and Division using rekhank.

1. Product of two positive digits or two negative digits (Rekhanks)

e.g: $2 \times 4 = 8$; $\bar{4} \times \bar{3} = 12$ i.e. always positive

2. Product of one positive digit and one Rekhank

e.g: $3 \times \bar{2} = \bar{6}$ or -6; $\bar{5} \times 3 = \overline{15}$ or -15 i.e. always Rekhank or negative.

3. Division of one positive by another or division of one Rekhank by anotherRekhank.

e.g: $8 \div 2 = 4$, $\bar{6} \div 3 = \bar{2}$ i.e. always positive

4. Division of a positive by a Rekhank or vice versa.

e.g: $\overline{10} \div \bar{5} = 2$, $\bar{6} \div \bar{2} = 3$ i.e. always negative or Rekhank.

g) Beejank: The Sum of the digits of a number is called Beejank. If the addition is a two digit number, Thenthese two digits are also to be added up to get a single digit.

e.g: Beejank of 27 is 2 + 7 = 9.

Beejank of 348 is 3 + 4 + 8 = 15

Further 1 + 5 = 6. i.e. 6 is Beejank.

Beejank of 1567 $\longrightarrow$ 1 + 5 + 6 + 7 $\longrightarrow$ 19 $\longrightarrow$ 1 + 9 $\longrightarrow$ 1

i.e. Beejank of 1567 is 1.

ii) Easy way of finding Beejank:

Beejank is unaffected if 9 is added to or subtracted from the number. This nature of 9 helps in finding Beejank very quickly, by cancelling 9 or the digits adding to 9 from the number.

eg 1: Find the Beejank of 632174.

As above we have to follow

632174 $\longrightarrow$ 6 + 3 + 2 + 1 + 7 + 4 $\longrightarrow$ 23 $\longrightarrow$ 2 + 3 $\longrightarrow$ 5

But a quick look gives 6 & 3 ; 2 & 7 are to be ignored because 6+3=9,2+7=9. Hence remaining 1 + 4 $\longrightarrow$ 5 is the beejank of 632174.

eg 2:

Beejank of 1256847 $\longrightarrow$ 1+2+5+6+8+4+7 $\longrightarrow$ 33 $\longrightarrow$ 3+3 $\longrightarrow$ 6.

But we can cancel 1& 8, 2& 7, 5 & 4 because in each case the sum is 9. Hence remaining 6 is the Beejank.

h) Check by Beejank method:

The Vedic sutra - Gunita Samuccayah gives 'the whole product is same'. We apply this sutra in this context as follows. It means that the operations carried out with the numbers have same effect when the same operations are carried out with their Beejanks.

Observe the following examples.

i) 42 + 39

Beejanks of 42 and 39 are respectively 4 + 2 = 6 and 3 + 9 = 12 and 1+2=3

Now 6 + 3 = 9 is the Beejank of the sum of the two numbers

Further 42 + 39 = 81. Its Beejank is 8+ 1 = 9.

we have checked the correctness.

ii) 64 + 125.

64 $\longrightarrow$ 6 + 4 $\longrightarrow$ 10 $\longrightarrow$ 1 + 0 $\longrightarrow$ 1

125 $\longrightarrow$ 1 + 2 + 5 $\longrightarrow$ 8

Sum of these Beejanks 8 + 1 = 9

Note that

64 + 125 = 189 → 1 + 8 + 9 → 18 → 1 + 8 → 9

we have checked the correctness.

iii) 134 - 49

134 → 1 + 3 + 4 → 8

49 → 4+9 → 13 → 1 + 3 → 4

Difference of Beejanks 8 -4 → 4, note that 134 – 49 = 85

Beejanks of 85 is 8 + 5 → 85 → 8 + 5 → 13 → 1 + 3 → 4 verified.

iv) 376 - 284

376 → 7 (∵ 6 + 3 → 9)

284 → 2 + 8 + 4 → 14 → 1 + 4 → 5

Difference of Beejanks = 7 – 5 = 2

376 – 284 = 92

Beejank of 92 → 9 + 2 → 11 → 1 + 1 → 2

Hence verified.

v) 24 X 16 = 384

Multiplication of Beejanks of

24 and 16 is 6 X 7 = 42 → 4 + 2 → 6

Beejank of 384 → 3 + 8 + 4 → 15 → 1 + 5 → 6

Hence verified.

vi) 237 X 18 = 4266

Beejank of 237 → 2 + 3 + 7 → 12 → 1 + 2 → 3

Beejank of 18 $\longrightarrow$ 1 + 8 $\longrightarrow$ 9

Product of the Beejanks = 3 X 9 $\longrightarrow$ 27 $\longrightarrow$ 2 + 7 $\longrightarrow$ 9

Beejank of 4266 $\longrightarrow$ 4 + 2 + 6 + 6 $\longrightarrow$ 18 $\longrightarrow$ 1 + 8 $\longrightarrow$ 9

Hence verified.

vii) $24^2 = 576$

Beejank of 24 $\longrightarrow$ 2 + 4 $\longrightarrow$ 6

square of it 6^2 $\longrightarrow$ 36 $\longrightarrow$ 9

Beejank of result = 576 $\longrightarrow$ 5 + 7 + 6 $\longrightarrow$ 18 $\longrightarrow$ 1 + 8 $\longrightarrow$ 9

Hence verified.

viii) $356^2 = 126736$

Beejank of 356 $\longrightarrow$ 3 + 5 + 6 $\longrightarrow$ 5

Square of it = $5^2 = 25$ $\longrightarrow$ 2+ 5 $\longrightarrow$ 7

Beejank of result 126736 $\longrightarrow$ 1 + 2 + 6 +7 + 3 + 6 $\longrightarrow$ 1 + 6 $\longrightarrow$ 7

($\because$ 7 + 2 = 9; 6 + 3 = 9) hence verified.

ix) Beejank in Division:

Let P, D, Q and R be respectively the dividend, the divisor, the quotient and the remainder.

Further the relationship between them is P = (Q X D) + R

eg 1: 187 ÷ 5

we know that 187 = (37 X 5) + 2 now the Beejank check.

We know that 187 = (37 X 5) +2 now the Beejank check.

187 $\longrightarrow$ 1 + 8 + 7 $\longrightarrow$ 7($\because$ 1 + 8 = 9)

(37 X 5) + 2 $\longrightarrow$ Beejank [(3 + 7) X 5] + 2

$\longrightarrow$ 5 + 2 $\longrightarrow$ 7

Hence verified.

eg 2: 7986 ÷ 143

7896 = (143 X 55) + 121

Beejank of 7986 $\longrightarrow$ 7 + 9 + 8 + 6 $\longrightarrow$ 21

($\because$ 9 is omitted) $\longrightarrow$ 2 + 1 $\longrightarrow$ 3

Beejank of 143 X 55 $\longrightarrow$ (1 + 4 + 3) (5 + 5)

$\longrightarrow$ 8 X 10 $\longrightarrow$ 80 $\longrightarrow$ (8 + 0) $\longrightarrow$ 8

Beejank of (143 X 55) + 121 $\longrightarrow$ 8 + (1 + 2 + 1)

$\longrightarrow$ 8 + 4 $\longrightarrow$ 12 $\longrightarrow$ 1 + 2 $\longrightarrow$ 3

hence verified.

Check the following results by Beejank method

1. **67 + 34 + 82 = 183**
2. **4381 - 3216 = 1165**
3. **63^2 = 3969**
4. **$(1234)^2$ = 1522756**
5. **(86X17) + 34 = 1496**
6. **2556 ÷ 127 gives Q =20, R = 16**

---

i) Vinculum : The numbers which by presentation contains both positive and negative digits are called vinculum numbers.

ii) Conversion of general numbers into vinculum numbers.

We obtain them by converting the digits which are 5 and above 5 or less than 5 without changing the value of that number.

Consider a number say 8. (Note it is greater than 5). Use it complement (purak - rekhank) from 10. It is 2 in this case and add 1 to the left (i.e. tens place) of 8.

Thus 8 = 08 = $1\bar{2}$.

The number 1 contains both positive and negative digits i.e. 1 and $\bar{2}$. Here $\bar{2}$ is in unit place hence it is -2 and value of 1 at tens place is 10.

Thus $1\bar{2}$ = 10 - 2 = 8

Conveniently we can think and write in the following way

General Number	Conversion	Vinculum number
6	10 - 4	$1\bar{4}$
97	100 - 3	$10\bar{3}$
289	300 - 11	$3\overline{11}$ etc.,,

The sutras 'Nikhilam Navatascharamam Dasatah' and 'Ekadhikena purvena' are useful for conversion.

eg 1: 289, Edadhika of 2 is 3

Nikhilam from 9 : 8 - 9 = -1 or $\bar{1}$

Charmam from 10 :9 -10 = -1 or $\bar{1}$

i.e. 289 in vinculum form$3\overline{11}$

eg 2: 47768

'Ekadhika' of 4 is 5

'Nikhilam' from 9 (of 776) $\overline{223}$

'Charmam from 10 (of 8) $\bar{2}$

Vinculum of 47168 is 5 $\overline{2232}$

eg 3: 11276.

Here digits 11 are smaller. We need not convert. Now apply for 276 the two sutras Ekadhika of 2 is 3

'Nikhilam Navata' for 76 is $\overline{24}$

$11276 = 113\overline{24}$

i.e. $113\overline{24} = 11300 - 24 = 11276$.

The conversion can also be done by the sutra sankalana vyavakalanabhyam as follows.

eg 4: 315.

sankalanam (addition) = 315+315 = 630.

Vyvakalanam (subtraction) = 630 - 315 = $32\overline{5}$

Working steps :

$0 - 5 = \overline{5}$

$3 - 1 = 2$

$6 - 3 = 3$

Let's apply this sutra in the already taken example 47768.

Samkalanam = 47768 + 47768 = 95536

Vyavakalanam = 95536 - 47768.

Steps :

$$\left.\begin{array}{l} 6 - 8 = \overline{2} \\ 3 - 6 = \overline{3} \\ 5 - 7 = \overline{2} \\ 5 - 7 = \overline{2} \\ 9 - 4 = 5 \end{array}\right\} = 5\overline{2232}$$

Consider the convertion by sankalanavyavakalanabhyam and check it by Ekadhika and Nikhilam.

eg 5: 12637

1. Sankalana .... gives, 12637 + 12637 = 25274

$25274 – 12637 = (2 – 1) / (5 – 2) / (2 – 6) / (7 – 3) / (4 – 7) = 1344\overline{3}$

2. Ekadhika and Nikhilam gives the following.

As in the number 1 2 6 3 7, the smaller and bigger digits (i.e. less than 5 and; 5, greater than 5) are mixed up, we split up in to groups and conversion is made up as given below.

Split 1 2 6 and 3 7

Now the sutra gives 1 2 6 as $13\overline{4}$ and 37 as $4\overline{3}$

Thus 12637 = $13\overline{4}4\overline{3}$

Now for the number 315 we have already obtained vinculum as $32\overline{5}$ by "sankalana ... " Now by 'Ekadhika and Nikhilam ...' we also get the same answer.

315 Since digits of 31 are less than 5,

We apply the sutras on 15 only as

Ekadhika of 1 is 2 and Charman of 5 is 5 .

Consider another number which comes under the split process.

eg 6: 24173

As both bigger and smaller numbers are mixed up we split the number 24173 as 24 and 173 and write their vinculums by Ekadhika and Nikhilam sutras as

24 = $3\overline{6}$ and 173 = $2\overline{27}$

Thus 24173 = $3\overline{6}2\overline{27}$

Convert the following numbers into viniculum number by
i. Ekadhika and Nikhilam sutras ii. Sankalana vyavakalana sutra.
Observe whether in any case they give the same answer or not.

1. 64 **2. 289** **3. 791**

4. 2879 **5. 19182** **6. 823672**

7. 123456799 **8. 65384738**

---

ii) Conversion of vinculum number into general numbers.

The process of conversion is exactly reverse to the already done. Rekhanks are converted by Nikhilam where as other digits by 'Ekanyunena' sutra. thus:

i) $1\bar{2}$ = (1 – 1) / (10 – 2) Ekanyunena 1 – 1

= 08 = 8 Nikhilam. $\bar{2}$ = 10 – 2

ii) $3\overline{26}$ = (3 – 1) / (9 – 2) / (10 – 6)

= 274

iii) $3\bar{3}4\bar{4}$ = (3 – 1) / (10 – 3) / (4 – 1) / (10 – 4)

= 2736 (note the split)

iv) $20\overline{34}\,\overline{01}21$ = 2/(0–1)/(9–3)/(10–4)/(0–1)/(9–1)/(10–2)/1

= $2\bar{1}66\bar{1}881$

= $2\bar{1}$ / 6 / $6\bar{1}$ / 881. once again split

= (2 – 1) / (10 –1) / 6 / (6 –1) / (10 –1) / 881

= 19659881

v) $30\overline{321}2$ = 3 / 0321 / 2

= 3 / (0-1) / (9-3) / (9-2) / (10-1) / 2

3 / $\bar{1}$ / 6792

(3 –1) / (10 –1) / 6792

= 296792.

iii) Single to many conversions.

It is interesting to observe that the conversions can be shown in many ways.

eg 1: 86 can be expressed in following many ways

86 = 90 - 4 =$\overline{94}$

= 100 – 14 = $1\overline{14}$

$= 1000 - 914 = 1\overline{914}$

Thus $86 = 9\overline{4} = 1\overline{14} = 1\overline{914} = 1\overline{9914} = \ldots\ldots\ldots\ldots$

eg 2 :

$0\overline{7} = -10 + 3 = 1\overline{3}$

$\overline{36} = -100 + 64 = 1\overline{6}4$

$\overline{978} = -1000 + 22 = \overline{1}022$. etc.,

*** Convert by Vedic process the following numbers into vinculum numbers.**

1) 274 2) 4898 3) 60725 4) 876129.

*** Convert the following vinculum numbers into general form of numbers (normalized form)**

1) 283 2) 3619 3) 27216

4) 364718 5) 60391874

(iv) Vedic check for conversion:

The vedic sutra "Gunita Samuctayah" can be applied for verification of the conversion by Beejank method.

Consider a number and find its Beejank. Find the vinculum number by the conversion and find its Beejank. If both are same the conversion is correct.

eg.

$196 = \overline{2}16$. Now Beejank of 196 $\longrightarrow$ 1 + 6 $\longrightarrow$ 7

Beejank of 216 $\longrightarrow$ 2 + (-1) + 6 $\longrightarrow$ 7. Thus verified.

But there are instances at which, if beejank of vinculum number is rekhank i.e. negative. Then it is to be converted to +ve number by adding 9 to Rekhank (

already we have practised) and hence 9 is taken as zero, or vice versa in finding Beejank.

eg:

$2\overline{13} = 200 - 13 = 187$

Now Beejank of $2\overline{1}3 = 2 + (-1) + (-3) = -2$

Beejank of $187 = 1 + 8 + 7 = 16 \longrightarrow 1 + 6 = 7$

The variation in answers can be easily understood as

$\bar{2} = \bar{2} + 9 \longrightarrow - 2 + 9 = 7$ Hence verified.

Use Vedic check method of the verification of the following result.

1. $\mathbf{24 = 3\bar{6}}$
2. $\mathbf{2736 = 3\bar{3}4\bar{4}}$.
3. $\mathbf{3\overline{26} = 274}$
4. $\mathbf{\bar{2}3\bar{2}13 = 17187}$**</B< p>**

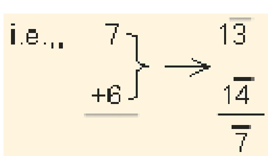
Addition and subtraction using vinculum numbers.

eg 1: Add 7 and 6 i.e., 7+6.

i) Change the numbers as vinculum numbers as per rules already discussed.

$\{ 7 = 1\bar{3}$ and $6 = 1\bar{4} \}$

ii) Carry out the addition column by column in the normal process, moving from top to bottom or vice versa.



iii) add the digits of the next higher level i.e.,, 1 + 1 = 2

$1\bar{3}$

$1\bar{4}$

$$\frac{\quad}{2\bar{7}}$$

iv) the obtained answer is to be normalized as per rules already explained. rules already explained.

i.e., $\bar{2}7$ = (2 - 1) (10- 7) = 13 Thus we get 7 + 6 = 13.

eg 2 : Add 973 and 866.

$$973 = 1\ 0\ \bar{3}\ 3 \qquad\qquad 1\ 0\ \bar{3}\ 3$$

$$866 = 1\ \bar{1}\ \overline{3\ 4} \qquad\qquad 1\ \bar{1}\ \bar{3}\ \bar{4}$$

$$\frac{\qquad\qquad}{2\ \bar{1}\ \bar{6}\ \bar{1}}$$

But $\overline{2161}$ = 2000 - 161 = 1839.

Thus 973+866 by vinculum method gives 1839 which is correct.

Observe that in this representation the need to carry over from the previous digit to the next higher level is almost not required.

eg 3 : Subtract 1828 from 4247.

i.e.,, 4247
-1828

Step (i) : write –1828 in Bar form i.e.,, $\overline{1828}$

(ii) : Now we can add 4247 and $\overline{1828}$ i.e.,,

$$\begin{array}{r} 4247 \\ +\overline{1828} \\ \hline 3\bar{6}2\bar{1} \end{array}$$

since 7 + $\bar{8}$ = $\bar{1}$, 4 + $\bar{2}$ = 2, 2 + $\bar{8}$ =$\bar{6}$, 4 + $\bar{1}$ = 3

(iii) Changing the answer $3\bar{6}2\bar{1}$ into normal form using Nikhilam, we get

3621 = 36 / 21 split

= (3 –1) / (10 – 6) / (2 – 1) / (10 – 1) = 2419

∴ 4247 – 1828 = 2419

Find the following results using Vedic methods and check them

1) 284 + 257 **2) 5224 + 6127**

3) 582 - 464 **4) 3804 - 2612**

---

2. Addition and Subtraction

ADDITION:

In the convention process we perform the process as follows.

234 + 403 + 564 + 721

write as 234
403
564
721

Step (i): 4 + 3 + 4 + 1 = 12   2 retained and 1 is carried over to left.

Step (ii): 3 + 0 + 6 + 2 = 11   the carried '1' is added

i.e., Now 2 retained as digit in the second place (from right to left) of the answer and 1 is carried over to left.

step (iii): 2 + 4 + 5 + 7 = 18   carried over '1' is added

i.e., 18 + 1 = 19. As the addition process ends, the same 19 is retained in the left had most part of the answer.

thus

$$\begin{array}{r} 234 \\ 403 \\ 564 \\ +721 \\ \hline 1922 \end{array}$$

is the answer

we follow sudhikaran process Recall 'sudha' i.e., dot (.) is taken as an upa-sutra (No: 15)

consider the same example

$$\begin{array}{r} 234 \\ 403 \\ 564 \\ +721 \\ \hline \end{array} \uparrow$$

i) Carry out the addition column by column in the usual fashion, moving from bottom to top.

(a) 1 + 4 = 5, 5 + 3 = 8, 8 + 4 = 12 The final result is more than 9. The tenth place '1' is dropped once number in the unit place i.e., 2 retained. We say at this stage sudha and a dot is above the top 4. Thus column (1) of addition (right to left)

.
4
3
4
1
___
2

b) Before coming to column (2) addition, the number of dots are to be counted, This shall be added to the bottom number of column (2) and we proceed as above.

Thus second column becomes

```
 .
 3      dot=1,   1 + 2 = 3
 0               3 + 6 = 9
 6               9 + 0 = 9
 2               9 + 3 = 12
___
 2
```

2 retained and '.' is placed on top number 3

c) proceed as above for column (3)

```
 2    i)dot = 1        ii) 1 + 7 = 8
 4    iii)8 + 5 = 13    iv) Sudha is said.
 .
 5    A dot is placed on 5 and proceed
 7    with retained unit place 3.
___
 9    v) 3+4=7,7+2=9 Retain 9 in 3rd digit i.e.,in 100th place.
```

d) Now the number of dots is counted. Here it is 1 only and the number is carried out left side ie. 1000^{th} place

```
         ..
Thus     234
        403
         .
        564
       +721
       _____
       1922   is the answer.
```

Though it appears to follow the conventional procedure, a careful observation and practice gives its special use.

eg (1):

$$
\begin{array}{r}
43\dot{7} \\
\dot{6}2\dot{4} \\
5\dot{8}6 \\
+162 \\
\hline
1809
\end{array}
$$

Steps 1:

i) 2 + 6 = 8, 8 + 4 = 12 so a dot on 4 and 2 + 7 = 9 the answer retained under column (i)

ii) One dot from column (i) treated as 1, is carried over to column (ii),

thus 1 + 6 = 7, 7 + 8 = 15 A' dot'; is placed on 8 for the 1 in 15 and the 5 in 15 is added to 2 above.

5 + 2 = 7, 7 + 3 = 10 i.e. 0 is written under column (ii) and a dot for the carried over 1 of 10 is placed on the top of 3.

(iii) The number of dots counted in column (iii) are 2.

Hence the number 2 is carried over to column (ii) Now in column (iii)

2 + 1 = 3, 3 + 5 = 8, 8 + 6 = 14 A dot for 1 on the number 6 and 4 is retained to be added 4 above to give 8. Thus 8 is placed under column (iii).

iv) Finally the number of dots in column (iii) are counted. It is '1' only. So it carried over to 1000th place. As there is no fourth column 1 is the answer for 4th column. Thus the answer is 1809.

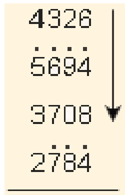
Example 3:

$$
\begin{array}{r}
3\dot{6}4\dot{7} \\
\dot{4}0\dot{8}6 \\
5\dot{7}1\dot{8} \\
+\,16\dot{9}2 \\
\hline
15143
\end{array}
$$

Check the result verify these steps with the procedure mentioned above.

The process of addition can also be done in the down-ward direction i.e., addition of numbers column wise from top to bottom

Example 1:



Step 1: 6 + 4 = 10, 1 dot ; 0 + 8 = 8; 8 + 4 = 12;

1 dot and 2 answer under first column - total 2 dots.

Step 2: 2+2 (∵ 2 dots) = 4; 4+9 = 13: 1 dot and 3+0= 3; 3+8 = 11;

1 dot and 1 answer under second column - total 2 dots.

Step 3: 3+2 (∵ 2 dots) = 5; 5+6 = 11:1 dot and 1+7 = 8; 8+7 = 15;

1 dot and 5 under third column as answer - total 2 dots.

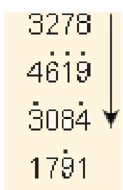
Step 4: 4 + 2 (∵ 2 dots) = 6; 6 + 5 =11:

1 dot and 1+3 = 4; 4+2 = 6. - total 1 dot in the fourth 6 column as answer.

Step 5: 1 dot in the fourth column carried over to 5th column (No digits in it) as 1

Thus answer is from **Step5** to **Step1**;16512

Example 2:



Steps

(i): 8 + 9 = 17; 7 + 4 = 11; 1 + 1 = (2) → (2dots)

(ii): 7 + 2 = 9; 9 + 1 = 10; 0 + 8 = 8, 8 + 9 = 17, (7) → (2dots)

(iii): 2 + 2 = 4; 4 + 6 = 10; 0 + 0 = 0; 0 + 7 = (7) → (1 dot)

(iv): 3 + 1 = 4; 4 + 4 = 8; 8 + 3 = 11; 1 + 1 = (2) → (1 dot)

(v): 1

Thus answer is 12772.

Add the following numbers use 'Sudhikaran' whereever applicable.

1.	2.	3.
486	5432	968763
395	3691	476509
721	4808	+584376
+609	+6787	

Check up whether 'Sudhkaran' is done correctly. If not write the correct process. In either case find the sums.

1.
$$\begin{array}{r} 4\ \dot{3} \\ \dot{7}\ \dot{8} \\ +\ 6\ 9 \\ \hline \\ \hline \end{array} \uparrow$$

2.
$$\begin{array}{r} 3\ \dot{7}\ \dot{9} \\ 8\ \dot{5}\ 4 \\ \dot{7}\ \dot{6}\ \dot{7} \\ 4\ 2\ 6 \\ \hline \\ \hline \end{array}$$

3.
$$\begin{array}{r} 7\ 8\ \dot{9}\ 2\ 4 \\ \dot{2}\ \dot{7}\ 2\ \dot{7}\ 2 \\ +\ 7\ 2\ 6\ \dot{8}\ 4 \\ \hline \\ \hline \end{array} \downarrow$$

SUBTRACTION:

The 'Sudha' Sutra is applicable where the larger digit is to be subtracted from the smaller digit. Let us go to the process through the examples.

Procedure:

i) If the digit to be subtracted is larger, a dot (sudha) is given to its left.

ii) The purak of this lower digit is added to the upper digit or purak-rekhank of this lower digit is subtracted.

Example (i): 34 - 18

$$\begin{array}{r} 34 \\ -\dot{1}8 \\ \hline \end{array}$$

Steps: (i): Since 8>4, a dot is put on its left i.e.$\dot{1}$

(ii) Purak of 8 i.e. 2 is added to the upper digit i.e. 4

2 + 4 = 6. or Purak-rekhank of 8 i.e. $\bar{2}$ is

Subtracted from i.e. 4 - $\bar{2}$ =6.

Now at the tens place a dot (means1) makes the '1' in the number into 1+1=2.This has to be subtracted from above digit. i.e. 3 - 2 = 1. Thus

$$\begin{array}{r} 34 \\ -\dot{1}8 \\ \hline 16 \end{array}$$

Example 2:

$$\begin{array}{r} 63 \\ -\dot{3}7 \\ \hline \end{array}$$

Steps: (i) 7>3. Hence a dot on left of 7 i.e.,$\dot{3}$

(ii) Purak of 7 i.e. 3 is added to upper digit 3 i.e. 3+3 = 6.

This is unit place of the answer.

Thus answer is 26.

Example (3) :

$$\begin{array}{r} 3274 \\ -\dot{1}\dot{8}92 \\ \hline \end{array}$$

Steps:

(i) 2 < 4. No sudha . 4-2 = 2 first digit (form right to left)

(ii) 9 > 7 sudha required. Hence a dot on left of 9 i.e. $\dot{8}$

(iii) purak of 9 i.e. 1, added to upper 7 gives1 + 7 = 8 second digit

(iv) Now means $\dot{8}$ + 1 = 9.

(v) As 9 > 2, once again the same process: dot on left of i.e.,$\dot{1}$

(vi) purak of 9 i.e. 1, added to upper 2 gives1 + 2 = 3, the third digit.

(vii) Now $\dot{1}$ means 1+1 = 2

(viii) As 2 < 3, we have 3-2 = 1, the fourth digit

Thus answer is 1382

Vedic Check :

Eg (i) in addition : 437 + 624 + 586 + 162 = 1809.

By beejank method, the Beejanks are

437 → 4 +3 + 7 → 14 → 1 + 4 → 5

624 → 6 +2 + 4 → 12 → 1 + 2 → 3

586 → 5 +8 + 6 → 19 → 1 + 9 → 10 → 1 + 0 → 1

162 → 1 +6 + 2 → 9

Now

437 + 624 + 586 + 162 → 5 + 3 + 1 + 9 → 18 → 1+ 8 → 9

Beejank of 1809 → 1 + 8 + 0 + 9 → 18 → 1 + 8 → 9 verified

Eg.(3) in subtraction :

3274 – 1892 = 1382

now beejanks

3274 $\longrightarrow$ 3 + 2 + 7 + 4 $\longrightarrow$ 3 + 4 $\longrightarrow$ 7

1892 $\longrightarrow$ 1 + 8 + 9 + 2 $\longrightarrow$ 2

3292-1892 $\longrightarrow$ 7-2 $\longrightarrow$ 5

1382 $\longrightarrow$ 1 + 3 + 8 + 2 $\longrightarrow$ 5 Hence verified.

Mixed addition and subtraction using Rekhanks:

Example 1 : 423 - 654 + 847 - 126 + 204.

In the conventional method we first add all the +ve terms

423 + 847 + 204 = 1474

Next we add all negative terms

- 654 - 126 = -780

At the end their difference is taken

1474 - 780 = 694

Thus in 3 steps we complete the problem

But in Vedic method using Rekhank we write and directly find the answer.

$$
\begin{array}{l}
4\ 2\ 3 \\
\bar{6}\ \bar{5}\ \bar{4} \\
8\ 4\ 7 \\
\bar{1}\ \bar{2}\ \bar{6} \\
2\ 0\ 4 \\
\hline
7\ \bar{1}\ 4
\end{array}
$$

This gives (7 -1) / (10 - 1) / 4 = 694.

Example (2):

6371 – $\underline{2647}$ + 8096 – $\underline{7381}$ + 1234

= 6371 + 2647 + 8096 + 7381 + 1234

$=(6+\bar{2}+8+\bar{7}+1)/(\bar{3}+6+\bar{0}+3+2)/(\bar{7}+4+\bar{9}+8+3)\overline{/}(1+\bar{7}+6+1+4)$

$= 6 / \bar{4} / 7 / 3$

= (6 – 1) / (10 – 4)/ 73

= 5673

*** Find the results in the following cases using Vedic methods.**

1) 57 -39 **3) 384 -127 + 696 -549 +150**

2) 1286 -968 **4) 7084 +1232 - 6907 - 3852 + 4286**

*** Apply Vedic check for the above four problems and verify the results.**

---

3. Multiplication

We have already observed the application of Vedic sutras in multiplication. Let us recall them.

It enables us to have a comparative study of the applicability of these methods, to assess advantage of one method over the other method and so-on.

Example (i) : Find the square of 195.

The Conventional method :

$$
195^2 = \begin{array}{r} 195 \\ \times\ 195 \\ \hline 975 \\ 1755 \\ 195 \\ \hline 38025 \\ \hline \end{array}
$$

(ii) By Ekadhikena purvena, since the number ends up in 5 we write the answer split up into two parts.

The right side part is 5^2 where as the left side part 19 X (19+1) (Ekhadhikena)

Thus $195^2 = 19 \text{ X } 20/5^2 = 380/25 = 38025$

(iii) By Nikhilam Navatascaramam Dasatah; as the number is far from base 100, we combine the sutra with the upa-sutra 'anurupyena' and proceed by taking working base 200.

a) Working Base = 200 = 2 X 100.

Now $195^2 = 195 \text{ X } 195$

195	-5
195	-5
(195 - 5) or = 190	-5 X -5 = 25
380 / 025	= 38025

iv) By the sutras "yavadunam tavadunikritya vargamca yojayet" and "anurupyena"

195^2, base 200 treated as 2 X 100 deficit is 5.

$$
\begin{aligned}
195^2 &= (195 - 5) \,/\, 5^2 \\
&= 190 \,/\, 25 \\
&\quad \times 2 \qquad \text{operating on } 100 \\
&= 38025
\end{aligned}
$$

v) By ‘antyayor dasakepi’ and ‘Ekadhikena’ sutras

Since in 195 x 195, 5 + 5 = 10 gives

195^2 = 19 x 20 / 5 x 5 = 380 / 25 = 38025.

vi) Now "urdhva-tiryagbhyam" gives

$$
\begin{array}{ccccccccc}
 & & & & 195 & & & & \\
 & & & & \times\ 195 & & & & \\
\hline
(1\times1) & / & (1\times9)+(1\times9) & / & (1\times5)+(9\times9)+(5\times1) & / & (9\times5)+(5\times9) & / & (5\times5) \\
1 & / & 9+9 & / & 5+81+5 & / & 45+45 & / & 25 \\
= 1 & / & 8 & / & 1 & / & 0 & / & 5 \\
 & & 1 & & 9 & & 9 & & 2
\end{array}
$$

By the carryovers the answer is 38025

Example 2 : 98 X 92

i) ‘Nikhilam’ sutra

$$
\begin{array}{rr}
98 & -2 \\
\times\ 92 & -8 \\
\hline
\multicolumn{2}{c}{90 \,/\, 16 = 9016}
\end{array}
$$

ii) ‘Antyayordasakepi’ and ‘Ekadhikena’ sutras.

98 X 92 Last digit sum= 8+2 =10 remaining digit (s) = 9 samesutras work.

$\therefore$98 X 92 = 9 X (9 + 1) / 8X2 = 90/16 = 9016.

iii) urdhava-tiryak sutra

$$
\begin{array}{r}
98 \\
\times\ 92
\end{array}
$$

$$\begin{array}{r} 106 \\ 891 \\ \hline 9016 \end{array}$$

vi) by vinculum method

$98 = 100 - 2 = 10\bar{2}$

$92 = 100 - 8 = 10\bar{8}$

now

$$\begin{array}{l} 10\bar{2} \\ 10\bar{8} \\ \hline 10\bar{0}06 \\ \bar{1}\ \ 1 \\ \hline 1\overline{10}16 \quad = \quad 9016 \end{array}$$

Example 3: 493 X 497.

1) ‘Nikhilam’ Method and ‘Anurupyena’:

a) Working base is 500, treated as 5 X 100

$$\begin{array}{ll} 493 & -7 \\ 497 & -3 \\ \hline 490 \,/ & 21 \\ \times 5 \,/ & \qquad \text{since base is } 5 \times 100 \\ \hline 2450 \,/ & 21 \quad = 245021 \end{array}$$

b) Working base is 500, treated as 1000 / 2

493 -7

$$\begin{array}{l} \quad 497 \quad -3 \\ \hline 2)\ 490 \ / \ 021 \\ \hline \quad 245 \ / \ 021 \quad = \quad 245021 \end{array}$$

2) ‘Urdhva tiryak’ sutra.

$$\begin{array}{c} 493 \\ 497 \\ \hline (4\text{x}4) \ / \ (4\text{x}9) + (9\text{x}4) \ / \ (4\text{x}7) + (9\text{x}9) + (4\text{x}3) \ / \ (9\text{x}7) + (3\text{x}9) \ / \ (3\text{x}7) \\ 16 \ / \ 72 \ / \ 121 \ / \ 90 \ / \ 21 \end{array}$$

$= {}_{1}6 \ / \ {}_{7}2 \ / \ {}_{12}1 \ / \ {}_{9}0 \ / \ {}_{2}1 = 245021$ (adjusting carry overs)

3) Since end digits sum is 3+7 = 10 and remaining part 49 is same in both the numbers, ‘antyayordasakepi’ is applicable. Further Ekadhikena Sutra is also applicable.

Thus

493 x 497 =49 x 50 / 3x7
= 2450 / 21
= 245021

4) With the use of vinculum.

$493 = 500 - 07 = 50\bar{7}$

$497 = 500 - 03 = 50\bar{3}$.

Now $\bar{4}97$ x $\bar{4}97$ can be taken as $50\bar{7}$ x $50\bar{3}$

$$\begin{array}{r} 50\bar{7} \\ \times\ 50\bar{3} \\ \hline 50\bar{0}01 \\ 2\bar{5}2 \\ \hline 25\bar{5}\bar{0}21 \end{array} \quad = 245021$$

Example 4: 99 X 99

1) Now by urdhva - tiryak sutra.

$$\begin{array}{r} 99 \\ \times\ 99 \\ \hline 8121 \\ 168\ \ \\ \hline 9801 \end{array}$$

2) By vinculum method

$99 = 100 - 1 = 10\bar{1}$

Now 99 X 99 is

$$\begin{array}{r} 10\bar{1} \\ \times\ 10\bar{1} \\ \hline 10\bar{2}01 \end{array} = 9801$$

3) By Nikhilam method

$$\begin{array}{rr} 99 & -1 \\ 99 & -1 \\ \hline 98\ / & 01 \end{array} = 9801.$$

4) 'Yadunam' sutra : 99^2 Base = 100

Deficiency is 1 : It indicates $99^2 = (99 - 1) / 1^2 = 98 / 01 = 9801$.

In the above examples we have observed how in more than one way problems can be solved and also the variety. You can have your own choice for doing multiplication. Not only that which method suits well for easier and quicker calculations. Thus the element of choice, divergent thinking, insight into properties and patterns in numbers, natural way of developing an idea, resourcefulness play major role in Vedic Mathematics methods.

4. Division

In the conventional procedure for division, the process is of the following form.

```
                Quotient
              ________
Divisor ) Dividend       or    Divisor ) Dividend ( Quotient
          ----------                     ----------
          ----------                     ----------
          __________                     __________
          Remainder                       Remainder
```

But in the Vedic process, the format is

```
        Divisor ) Dividend
                  --------
                  --------
          ____________________
          Quotient / Remainder
```

The conventional method is always the same irrespective of the divisor. But Vedic methods are different depending on the nature of the divisor.

Example 1: Consider the division 1235 ÷ 89.

i) Conventional method:

```
89 ) 1235 ( 13
      89
     ______
      345
      267     Thus Q = 13 and R = 78.
     ______
       78
```

ii) Nikhilam method:

This method is useful when the divisor is nearer and less than the base. Since for 89, the base is 100 we can apply the method. Let us recall the nikhilam division already dealt.

Step (i):

Write the dividend and divisor as in the conventional method. Obtain the modified divisor (M.D.) applying the Nikhilam formula. Write M.D. just below the actual divisor.

Thus for the divisor 89, the M.D. obtained by using Nikhilam is 11 in the last from 10 and the rest from 9. Now Step 1 gives

```
89 ) 1235
__
11
```

Step (ii):

Bifurcate the dividend by by a slash so that R.H.S of dividend contains the number of digits equal to that of M.D. Here M.D. contains 2 digits hence

89) 12 / 35

$\overline{11}$

Step (iii): Multiply the M.D. with first column digit of the dividend. Here it is 1. i.e. 11 x 1 = 11. Write this product place wise under the 2nd and 3rd columns of the dividend.

89) 12 / 35

$\overline{11}$ 1 1

Step (iv):

Add the digits in the 2nd column and multiply the M.D. with that result i.e. 2+1=3 and 11x3=33. Write the digits of this result column wise as shown below, under 3^{rd} and 4^{th} columns. i.e.

89) 12 / 35

$\overline{11}$ 1 1
33

13 / 78

Now the division process is complete, giving Q = 13 and R = 78.

Example 2: Find Q and R for 121134 ÷ 8988.

Steps (1+2):

8988) 12 / 1134

$\overline{1012}$

Step (3):

8988) 12 / 1134

$\overline{1012}$ 1 012

Step(4):

$$\begin{array}{l} 8988\ )\ 12\ /\ 1134 \\ \overline{1012}\quad 1\ \ 012 \quad [\ 2+1=3 \text{ and } 3\text{x}1012=3036\ ] \\ \qquad\qquad 3036 \end{array}$$

Now **final Step**

$$\begin{array}{l} 8988\ )\ 12\ /\ 1134 \\ \overline{1012}\quad 1\quad 012 \\ \qquad\qquad 3036 \text{(Column wise addition)} \\ \hline \qquad 13\ /\ 4290 \end{array}$$

Thus 121134¸ 8988 gives Q = 13 and R = 4290.

iii) Paravartya method: Recall that this method is suitable when the divisor is nearer but more than the base.

Example 3: 32894 ÷ 1028.

The divisor has 4 digits. So the last 3 digits of the dividend are set apart for the remainder and the procedure follows.

1 0 2 8	32	8	9	4	i) 3 x (0, -2, -8)
0 -2 -8	0	-6	-24		= 0, -6, -24
		0	-4	-16	ii) 2 – 0 = 2 and
					2 x (0, -2, -8)
	32	2	-19	-12	= 0, -4, -16
					iii) 8 – 6 + 0 = 2
					9 – 24 – 4 = - 19
					4 – 16 = - 12

Now the remainder contains -19, -12 i.e. negative quantities. Observe that 32 is quotient. Take 1 over from the quotient column i.e. 1x1028 = 1028 over to the right side and proceed thus: 32 - 1 = 31 becomes the Q and R = 1028+200 - 190 - 12 =1028-2 =1026.

Thus 3289 ÷ 1028 gives Q = 31 and R = 1026.

The same problem can be presented or thought of in any one of the following forms.

$$
\begin{array}{llllll}
{}^{*}\ 1028\ ) & 32 & / & 8 & 9 & 4 \\
\overline{028} & 0 & & \bar{6} & \overline{24} & \\
 & & & 0 & \bar{4} & \overline{16} \\
\hline
 & 32 & & 2 & \overline{19} & \overline{12} \\
\end{array}
$$

$$= 31 \;/\; 1028 + 200 - 190 - 12$$

$$= 31 \;/\; 1026.$$

*Converting the divisor 1028 into vinculum number we get 1028 = $103\bar{2}$ Now

$$
\begin{array}{llllll}
\underline{103\bar{2}}\ ) & 32 & / & 8 & 9 & 4 \\
0\bar{3}2 & 0 & & \bar{9} & 6 & \\
 & & & 0 & \bar{6} & 4 \\
\hline
 & 32 & / & \bar{1} & 9 & 8 \\
\end{array}
$$

$$= 31 \;/\; 103\bar{2} + \bar{1}98$$

$$= 31 \;/\; 1026.$$

*Converting dividend into vinculum number 32894 = 33114 and proceeding we get

$$
\begin{array}{llllll}
1028\ ) & 3\ 3 & / & \bar{1} & \bar{1} & 4 \\
0\overline{28} & 0 & & \bar{6} & \overline{24} & \\
 & & & \bar{0} & \bar{6} & \overline{24} \\
\hline
 & 3\ 3 & / & \bar{7} & \overline{31} & \overline{20} \\
 & = 3\ 3 & / & \overline{10} & \bar{3} & 0 \\
 & = 3\ 3+\bar{1} & / & 0 & \bar{3} & 0 \\
 & & & 0 & 2 & 8 \\
 & = 32 & / & 0 & \bar{1} & 8 \\
\end{array}
$$

$$= 31 \;/\; 1028 + \bar{1}8$$

$$= 31 \;/\; 1026$$

Now we take another process of division based on the combination of Vedic sutras **urdhva-tiryak** and **Dhvjanka**. The word Dhvjanka means " on the top of the flag"

Example 4: 43852 ÷ 54.

Step1: Put down the first digit (5) of the divisor (54) in the divisor column as operator and the other digit (4) as flag digit. Separate the dividend into two parts where the right part has one digit. This is because the falg digit is single digit. The representation is as follows.

$$
\begin{array}{l}
\ \ 4 : 4\ 3\ 8\ 5 : 2 \\
5
\end{array}
$$

Step2: i) Divide 43 by the operator 5. Now Q= 8 and R = 3. Write this Q=8 as the 1st Quotient - digit and prefix R=3, before the next digit i.e. 8 of the dividend, as shown below. Now 38 becomes the gross-dividend (G.D.) for the next step.

```
  4 : 4 3  8 5 : 2
 5  :    3
________________
    :  8
```

ii) Subtract the product of falg digit (4) and first quotient digit (8) from the G.D. (38) i.e. 38-(4X8)=38-32=6. This is the net - dividend (N.D) for the next step.

Step3: Now N.D Operator gives Q and R as follows. 6 ÷ 5, Q = 1, R = 1. So Q = 1, the second quotient-digit and R - 1, the prefix for the next digit (5) of the dividend.

```
  4 : 4 3  8  5 : 2
 5  :    3  1
________________
    :  8  1
```

Step4: Now G.D = 15; product of flag-digit (4) and 2nd quotient - digit (1) is 4X1=4 Hence N.D=15-4=11 divide N.D by 5 to get 11 ÷ 5, Q = 2, R= 1. The representation is

```
  4 : 4 3  8  5 : 2
 5  :    3   1  :1
________________
    :  8  1  2 :
```

Step5: Now the R.H.S part has to be considered. The final remainder is obtained by subtracting the product of falg-digit (4)and third quotient digit (2) form 1^2 i.e., 12:

Final remainder = 12 - (4 X 2) = 12 - 8 = 4. Thus the division ends into

```
  4 : 4 3  8  5 : 2
 5  :    3  1   :1
________________
    :  8  1  2 : 4
```

Thus 43852 ÷ 54 gives Q = 812 and R = 4.

Consider the algebraic proof for the above problem. The divisor 54 can be represented by 5x+4, where x=10

The dividend 43852 can be written algebraically as $43x^3 + 8x^2 + 5x + 2$

since $x^3 = 10^3 = 1000$, $x^2 = 10^2 = 100$.

Now the division is as follows.

$$
\begin{array}{l}
5x + 4 \;)\; 43x^3 + 8x^2 + 5x + 2 \;(\; 8x^2 + x + 2 \\
\qquad\qquad 43x^3 + 32x^2 \\
\hline
\qquad\qquad 3x^3 - 24x^2 \\
\qquad\quad = 6x^2 + 5x \qquad (\because 3x^3 = 3 \,.\, x \,.\, x^2 \\
\qquad\qquad 5x^2 + 4x \qquad\qquad\quad = 3 \,.\, 10x^2 = 30\, x^2) \\
\hline
\qquad\qquad x^2 + x \\
\qquad\quad = 11x + 2 \qquad (\because x^2 = x \,.\, x = 10x\,) \\
\qquad\qquad 10x + 8 \\
\hline
\qquad\qquad x - 6 \\
\qquad\qquad = 10 - 6 \\
\qquad\qquad = 4.
\end{array}
$$

Observe the following steps:

1. $43x^3 \div 5x$ gives first quotient term $8x^2$, remainder $=3x^3 - 24x^2$ which really mean $30x^2 + 8x^2 -32x^2 = 6x^2$.

Thus in step 2 of the problem 43852 ÷ 54, we get Q= 8 and N.D = 6.

2. $6x^2 \div 5x$ gives second quotient term x, remainder = $x^2 + x$ which really mean $10x + x = 11x$.

Thus in step 3 & Step 4, we get Q=1and N.D =11.

3. $11x \div 5x$ gives third quotient term 2, remainder = x - 6 , which really mean the final remainder 10-6=4.

Example 5: Divide 237963 ÷ 524

Step1: We take the divisor 524 as 5, the operator and 24, the flag-digit and proceed as in the above example. We now seperate the dividend into two parts where the RHS part contains two digits for Remainder.

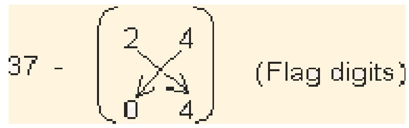
Thus

24 : 2 3 7 9: 63
5

Step2:

i) 23÷5 gives Q = 4 and R = 3, G.D = 37.

ii) N.D is obtained as



= 37 – (8 + 0)
= 29.

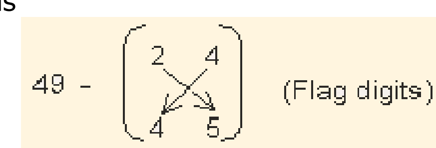
Representation

```
 24 : 2 3  7 9 : 63
5         3
______________________
      : 4
```

Step3:

i) N.D ÷ Operator = 29 ÷ 5 gives Q = 5, R = 4 and G.D = 49.

ii) N.D is obtained as



= 49 – (10 + 16)
= 49 – 26
= 23.

i.e.,

```
 24 : 2 3  7  9 : 63
5   :    3  4   :
______________________
     :  4   5     :
```

Step 4:

i) N.D ÷ Operator = 23 ÷ 5 gives Q = 4, R = 3 and G.D = 363.

Note that we have reached the remainder part thus 363 is total sub–remainder.

```
 24 : 2 3  7  9 : 63
5   :    3  4   :3
______________________
     :  4  5  4  :
```

Step 5: We find the final remainder as follows. Subtract the cross-product of the two, falg-digits and two last quotient-digits and then vertical product of last flag-digit with last quotient-digit from the total sub-remainder.

i.e.,,

$$363 - \begin{pmatrix} 2 & 4 \\ 4 & 5 \end{pmatrix} \Big/ \begin{matrix} 4 \\ \downarrow \\ 4 \end{matrix} \begin{aligned} &= 363 - (8 + 20)/_{1}6 \\ &= 363 - 28\ /_{1}6 \\ &= 363 - 296 = 67. \end{aligned}$$

Note that 2, 4 are two falg digits: 5, 4 are two last quotient digits:

$$\begin{pmatrix} 4 \\ \\ 4 \end{pmatrix}$$

represents the last flag - digit and last quotient digit.

Thus the division 237963 ÷ 524 gives Q = 454 and R = 67.

Thus the Vedic process of division which is also called as **Straight division** is a simple application of *urdhva-tiryak* together with *dhvajanka*. This process has many uses along with the one-line presentation of the answer.

5. Miscellaneous Items

1. Straight Squaring:

We have already noticed methods useful to find out squares of numbers. But the methods are useful under some situations and conditions only. Now we go to a more general formula.

The sutra Dwandwa-yoga (Duplex combination process) is used in two different meanings. They are i) by squaring ii) by cross-multiplying.

We use both the meanings of Dwandwa-yoga in the context of finding squares of numbers as follows:

We denote the Duplex of a number by the symbol D. We define for a single digit 'a', D $=a^2$. and for a two digit number of the form 'ab', D=2(a x b). If it is a 3 digit number like 'abc', D =2(a x c) + b^2.

For a 4 digit number 'abcd', D = 2(a x d) + 2(b x c) and so on. i.e. if the digit is single central digit, D represents 'square': and for the case of an even number of digits equidistant from the two ends D represent the double of the cross- product.

Consider the examples:

Number	DuplexD
3	$3^2 = 9$
6	$6^2 = 36$
23	2 (2 x 3) = 12
64	2 (6 x 4) = 48
128	2 (1 x 8) + 2^2 = 16 + 4 = 20
305	2 (3 x 5) + 0^2 = 30 + 0 = 30
4231	2 (4 x 1) + 2 (2 x 3) = 8 + 12 = 20
7346	2 (7 x 6) + 2 (3 x 4) = 84 + 24 = 108

Further observe that for a n- digit number, the square of the number contains 2n or 2n-1 digits. Thus in this process, we take extra dots to the left one less than the number of digits in the given numbers.

Examples:1 62^2 Since number of digits = 2, we take one extra dot to the left. Thus

```
 .62        for 2, D = 2² = 4
 ____
 644        for 62, D = 2 x 6 x 2 = 24
 32         for 62, D = 2(0 x2) + 6²
 ___                   = 36
 3844
```

∴62^2 = 3844.

Examples:2 234^2 Number of digits = 3. extradots =2 Thus

```
 ..234    for 4, D = 4² = 16
 _____
 42546    for 34, D = 2 x 3 x 4 = 24
 1221     for 234, D = 2 x 2 x 4 + 3² = 25
 _____
 54756    for  .234, D = 2.0.4 + 2.2.3 = 12
          for ..234, D = 2.0.4 + 2.0.3 + 2² = 4
```

Examples:3 1426^2. Number of digits = 4, extra dots = 3

```
i.e
  ...1426        6,  D = 36
  _______
  1808246        26,  D= 2.2.6 = 24
   22523        426,  D =2.4.6 + 2² = 52
  _______
  2033476      1426,  D = 2.1.6 + 2.4.2= 28
```

.1426, D = 2.0.6 + 2.1.2 + 4^2 = 20
..1426, D = 2.0.6 + 2.0.2 + 2.1.4 = 8
...1426, D = 1^2 = 1

Thus $1426^2 = 2033476$.

With a little bit of practice the results can be obtained mentally as a single line answer.

Algebraic Proof:

Consider the first example 62^2

$$\begin{aligned} \text{Now } 62^2 = (6 \times 10 + 2)^2 &= (10a + b)^2 \text{ where } a = 6,\ b = 2 \\ &= 100a^2 + 2.10a.b + b^2 \\ &= a^2\,(100) + 2ab\,(10) + b^2 \end{aligned}$$

i.e. b^2 in the unit place, 2ab in the 10^{th} place and a^2 in the 100^{th} place i.e. $2^2 = 4$ in units place, 2.6.2 = 24 in the 10^{th} place (4 in the 10^{th} place and with carried over to 100^{th} place). 6^2=36 in the 100^{th} place and with carried over 2 the 100^{th} place becomes 36+2=38.

Thus the answer 3844.

Find the squares of the numbers 54, 123, 2051, 3146.
Applying the Vedic sutra Dwanda yoga.

2.CUBING

Take a two digit number say 14.

i) Find the ratio of the two digits i.e. 1:4

ii) Now write the cube of the first digit of the number i.e. 1^3

iii) Now write numbers in a row of 4 terms in such a way that the first one is the cube of the first digit and remaining three are obtained in a geometric progression with common ratio as the ratio of the original two digits (i.e. 1:4) i.e. the row is

1 4 16 64.

iv) Write twice the values of 2^{nd} and 3^{rd} terms under the terms respectively in second row.

i.e.,

1	4	16	64
	8	32	

(∵ 2 x 4 = 8, 2 x 16 = 32)

v) Add the numbers column wise and follow carry over process.

1	4	16	64	Since 16 + 32 + 6 (carryover) = 54
	8	32		4 written and 5 (carryover) + 4 + 8 = 17
2	7	4	4	7 written and 1 (carryover) + 1 = 2.

This 2744 is nothing but the cube of the number 14

Example 1: Find 18^3

i)	1:8				
i)& iii)	1	8	64	512	
iv)	1	8	64	512	(2X8 =16, 2X64 = 128)
		16	128		
v)	1	8	64	512	51 2
		16	128		64 + 128 + 51 = 24 3
	5	8	3	2	8+16+ 24 = 4 8
					1 + 4 = 5

i.e.,183 = 5832.

Example 2: Find 33^3

I)	3 : 3 = 1 : 1				$3^3 = 27$ ratio is 1 : 1
ii) & iii)	27	27	27	27	
iv)& (v)	27	27	27	27	(2x27=54)
		54	54		
	35	9	3	7	

2 7

27 + 54 + 2 = 8 3

Thus 33^3 = 35937   27 + 54 + 8 = 8 9

27 + 8 = 35

Algebraic Proof:

Let a and b be two digits.

Consider the row a^3 a^2b ab^2 b^3

the first is a^3 and the numbers are in the ratio a:b

since $a^3:a^2b=a^2b:b^3=a:b$

Now twice of a^2b, ab^2 are $2a^2b$, $2ab^2$

$$\begin{array}{r} a^3 + a^2b + ab^2 + b^3 \\ 2a^2b + 2ab^2 \\ \hline a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3. \end{array}$$

Thus cubes of two digit numbers can be obtained very easily by using the vedic sutra 'anurupyena'. Now cubing can be done by using the vedic sutra 'Yavadunam'.

Example 3: Consider 106^3.

i) The base is 100 and excess is 6. In this context we double the excess and then add.

i.e. 106 + 12 = 118. ($\because 2 \times 6 = 12$)

This becomes the left - hand - most portion of the cube.

i.e. 106^3 = 118 / - - - -

ii) Multiply the new excess by the initial excess

i.e. 18 x 6 = 108 (excess of 118 is 18)

Now this forms the middle portion of the product of course 1 is carried over, 08 in the middle.

i.e. $106^3 = 118 \,/\, \underset{1}{08} \,/$ - - - - -

iii) The last portion of the product is cube of the initial excess.

i.e. $6^3 = 216$.

16 in the last portion and 2 carried over.

i.e. $106^3 = 118 \,/\, \underset{1}{0}8\underset{2}{1} \,/ 16 = 1191016$

Example 4: Find 1002^3.

i) Base = 1000. Excess = 2. Left-hand-most portion of the cube becomes 1002+(2x2)=1006.

ii) New excess x initial excess = 6 x 2 = 12.

Thus 012 forms the middle portion of the cube.

iii) Cube of initial excess = $2^3 = 8$.

So the last portion is 008.

Thus 1002^3 = 1006 / 012 / 008 = 1006012008.

Example 5: Find 94^3.

i) Base = 100, deficit = -6. Left-hand-most portion of the cube becomes94+(2x-6)=94-12=82.

ii) New deficit x initial deficit = -(100-82)x(-6)=-18x-6=108

Thus middle potion of the cube = 08 and 1 is carried over.

iii) Cube of initial deficit = $(-6)^3 = -216$

$$94^3 = 82 \,/\, \underset{1}{08} \,/\, \underset{\overline{2}}{\overline{16}} = 83 \,/\, 06 \,/\, \overline{16}$$

$$= 83 \,/\, 05 \,/\, (100 - 16)$$
$$= 830584.$$

Find the cubes of the following numbers using Vedic sutras.

103, 112, 91, 89, 998, 9992, 1014.

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3. Equation of Straight line passing through two given points:

To find the equation of straight line passing through the points (x_1,y_1) and (x_2, y_2) , we generally consider one of the following methods.

1. General equation $y = mx + c$.

It is passing through (x_1, y_1) then$y_1 = mx_1 + c$.

It is passing through (x_2, y_2) also, then$y_2 = mx_2 + c$.

Solving these two simultaneous equations, we get 'm' and 'c' and so the equation.

2. The formula

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)} (x - x_1)$$ and substitution.

Some sequence of steps gives the equation. But the paravartya sutra enables us to arrive at the conclusion in a more easy way and convenient to work mentally.

Example1: Find the equation of the line passing through the points (9,7) and (5,2).

Step1: Put the difference of the y - coordinates as the x - coefficient and vice - versa.

i.e. x coefficient = 7- 2 = 5

y coefficient = 9 - 5 = 4.

Thus L.H.S of equation is 5x - 4y.

Step 2: The constant term (R.H.S) is obtained by substituting the co-ordinates of either of the given points in

L.H.S (obtained through step-1)

i.e. R.H.S of the equation is

5(9) -4(7) = 45 - 28 = 17

or 5(5) - 4(2) = 25 - 8 = 17.

Thus the equation is 5x - 4y = 17.

Example 2: Find the equation of the line passing through (2, -3) and (4,-7).

Step 1 : x[-3-(-7)] –y[2-4] = 4x + 2y.

Step 2 : 4(2) + 2(-3) = 8 –6 = 2.

Step 3 : Equation is 4x + 2y =2 or 2x +y = 1.

Example 3 : Equation of the line passing through the points (7,9) and (3,-7).

Step 1 : x[9 - (-7)] – y(7 - 3) = 16x - 4y.

Step 2 : 16(7) - 4(9) = 112 – 36 = 76

Step 3 : 16x- 4y = 76 or 4x – y = 19

Find the equation of the line passing through the points using Vedic methods.

1. (1, 2), (4,-3) **2. (5,-2), (5,-4)**

3. (-5, -7), (13,2) **4. (a, o) , (o,b)**

IV Conclusion

After going through the content presented in this book, you may, perhaps, have noted a number of applications of methods of Vedic Mathematics. We are aware that this attempt is only to make you familiar with a few special methods. The methods discussed, and organization of the content here are intended for any reader with some basic mathematical background. That is why the serious mathematical issues, higher level mathematical problems are not taken up in this volume, even though many aspects like four fundamental operations, squaring, cubing, linear equations, simultaneous equations. factorization, H.C.F, recurring decimals, etc are dealt with. Many more concepts and aspects are omitted unavoidably, keeping in view the scope and limitations of the present volume.

Thus the present volume serves as only an 'introduction'. More has to be presented to cover all the issues in Swamiji's 'Vedic Mathematics'. Still more steps are needed to touch the latest developments in Vedic Mathematics. As a result, serious and sincere work by scholars and research workers continues in this field both in our country and abroad. Sri Sathya Sai Veda Pratisthan intends to bring about more volumes covering the aspects now left over, and also elaborating the content of Vedic Mathematics.

The present volume, even though introductory, has touched almost all the Sutras and sub-Sutras as mentioned in Swamiji's 'Vedic Mathematics'. Further it has given rationale and proof for the methods. As there is a general opinion that the 'so called Vedic Mathematics is only rude, rote, non mathematical and none other than some sort of tricks', the logic, proof and Mathematics behind the 'the so called tricks' has been explained. An impartial reader can easily experience the beauty, charm and resourcefulness in Vedic Mathematics systems. We feel that the reader can enjoy the diversity and simplicity in Vedic Mathematics while applying the methods against the conventional textbook methods. The reader can also compare and contrast both the methods.

The Vedic Methods enable the practitioner improve mental abilities to solve difficult problems with high speed and accuracy.